Imprecise specification of ill-known functions using gradual rules

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Abstract. Functional laws may be known only at a finite number of points, and then the function is completed by interpolation techniques obeying some smoothness conditions. We rather propose here to specify constraints by means of gradual rules for delimiting areas where the function may lie between known points. The more general case where the known points of the function are imprecisely located is also dealt with.
The use of gradual rules for expressing constraints on the closeness with respect to reference points leads to interpolation graphs that are imprecise but still crisp. We thus propose a refinement of the rule-based representation that enables the handling of fuzzy interpolation graphs.

Keywords: Fuzzy rules, Gradual rules, Imprecise interpolation, Fuzzy interpolation, Fuzzy data.

1 Introduction

Nowadays, most automated applications are based on models of the systems under consideration. In this framework, a precise representation is often used, even when this representation is based on a fuzzy rule base. Indeed, in Mamdani-like fuzzy systems precision is artificially introduced by the defuzzification step, while Sugeno-like rules do not allow the handling of imprecise conclusions. When uncertainties are explicitly dealt with, they affect parameters of analytical laws, leading to probabilistic or interval-based processing. The main objective of this paper is to propose an alternative to these analytical approaches by investigating the interest of gradual rules [6] for defining imprecise representations of ill-known functions. Actually, the proposed strategy relies on set-valued interpolative reasoning. When specifying a representation with gradual rules, there is no need to choose a parameterized function for the interpolator. The proposed rule-based approach should thus be distinguished from works on fuzzy polyno-

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mial [14] or fuzzy spline interpolation [12], [1], which rely on fuzzy-valued functions. Using such techniques, fuzzy interpolation still depends on the analytical form of the interpolant. Here, using a rule-based formalism, the imprecise model is directly obtained from the constraints expressed by the rules.

What is supposed to be known, in a precise or in an imprecise way, is the behaviour of the system at some reference points, the problem being to interpolate between these points. Figure 1 illustrates our view of an imprecise interpolation in a case where the points on which interpolation is based are imprecise. In order to avoid the choice of an analytical model (linear, polynomial, piecewise linear, cubic spline, ...), an imprecise representation which includes different possible precise interpolators is preferred. So, we are no longer looking for a function, only for a relation linking input variables to output variables. In the one-dimensional input case (as in figure 1) considered in this paper, this relation is represented by its graph Γ defined on the Cartesian product $X \times Z$ (where X is the input domain, and Z the output domain). A similar approach, recently proposed in [15], also considers the design of uncertain fuzzy models in the setting of the approximation of multi-valued mappings called "ambiguous functions".

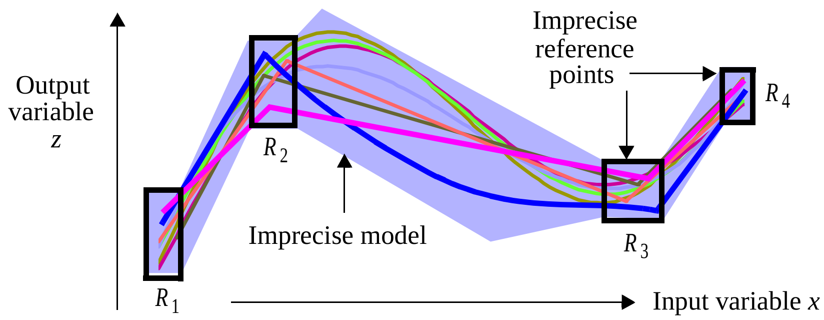

Fig. 1. : Imprecise interpolation

Actually, using gradual rules, the fuzziness introduced for modeling closeness with respect to reference points is not present in the interpolation graph, which is imprecise but still crisp. We thus propose a refinement of the rule-based representation that enables the handling of fuzzy interpolation graphs. The method consists in implementing level 2 gradual rules, which are fuzzy sets of nested gradual rules.

The paper, after some brief background on gradual rules, discusses the process of building an imprecise model, constrained by precise reference points. The case of imprecise reference points with interval-valued coordinates is then developed. Finally, the building of nested interpolation graphs is addressed. Their weighting allows the definition of fuzzy graphs as fuzzy sets of crisp graphs.

2 Interpolation and gradual rules

The idea of imprecise modelling suggested above is based on constraints to be satisfied, namely that the results of the interpolation should agree with the reference points. These constraints should be expressed in order to define the graph Γ of the relation on $X \times Z$.

We first consider the case of precise reference points P_i with coordinates (x_i, z_i), $i = 1, ..., n$. Then the relation Γ should satisfy $\Gamma(x_i, z_i) = 1$ and $\forall z \neq z_i \in Z$, $\Gamma(x_i, z) = 0$ for $i = 1, ..., n$. Without any further constraint on the nature of the interpolation, we only have $\forall x \neq x_i \in X$, $\forall z \in Z$, $\Gamma(x, z) = 1$. So each interpolation point induces the constraint "If $x = x_i$ then $z = z_i$", represented by $(x = x_i) \rightarrow (z = z_i)$ where $\rightarrow$ is the material implication. The relation Γ is thus obtained as the conjunction:

$$\Gamma(x, z) = \wedge_{i = 1, .., n} \ (x = x_i) \rightarrow (z = z_i). \tag{1}$$

This relation is extremely imprecise since there is no constraint at all outside the interpolation points. The absence of a choice of a precise type of interpolation function should be alleviated by the use of fuzzy rules in order to express constraints in the vicinity of the interpolation points. The idea is to use rules of the form "the closer x is to x_i, the closer z is to z_i" [6]. The extension to gradual rules of equation (1) provides the following expression for the graph Γ:

$$\Gamma(x, z) = \min_{i = 1, .., n} \ \mu_{\text{close to } x_i}(x) \rightarrow \mu_{\text{close to } z_i}(z) \tag{2}$$

where $\rightarrow$ represents the Rescher-Gaines implication ($a \rightarrow b = 1$ if $a \leq b$ and $a \rightarrow b = 0$ if $a > b$), and $\mu_{\text{close to } x_i}(x)$ is the degree of truth of the proposition "x is close to x_i".

Two comments on equation (2) are worth stating. First, the principle underlying the rules is the one at work in analogical or case-based reasoning and (2) is interpreting this principle as a constraint (as opposed to a weaker interpretation leading to Mamdani-like fuzzy systems, see [4]). Moreover, (2) embeds interpolation in a purely logical setting (see [3]) which does not require a defuzzification step.

We have just to define what is meant by "close to". Let A_i denote the fuzzy set of values close to x_i. It is natural to set $\mu_{A_i}(x) = 1$ if $x = x_i$ and to assume that the membership degree to A_i decreases on each side of x_i with the distance to x_i. The simplest solution consists in choosing triangular fuzzy sets with a support denoted by $[x_i^-, x_i^+]$. In a similar way, the closeness to z_i will be modelled by a triangular fuzzy set B_i with modal value z_i and support $[z_i^-, z_i^+]$. Then the interpolation relation only depends on $4n$ parameters x_i^-, x_i^+ z_i^-, z_i^+ for n interpolation points. The purpose of the next sections is to study criteria for choosing these parameters.

3 Interpolation between precise reference points

3.1 Coverage and consistency of the rule base

In order to make the analysis of the interpolation relation simpler, we further assume that at most two rules can be simultaneously fired at each point of the input domain (i.e., $x_i \leq x_{i+1}^-$ and $x_i^+ \leq x_{i+1}$, $i = 1, ..., n$-1, which implies for triangular membership functions that the coverage degree $cov(x) = \Sigma_{i = 1, .., n} \ \mu_{A_i}(x)$ is such that $\forall x \in X$, $cov(x) \leq 1$).

It is generally expected that a set of parallel fuzzy rules covers any possible imput, which means that each possible input value x should fire at least one rule. In other words, the A_i 's should lead to a nonzero degree everywhere in X, i.e.:

$$x_{i+1}^{-} < x_i^{+}. \tag{3}$$

Then, we might think of using a strong fuzzy partition both for X and Z (i.e. with a coverage degree *cov* exactly equal to 1 for any $x \in X$ and $z \in Z$), with triangular fuzzy sets A_i and B_i. However, in this case, as shown in [7] and [8], gradual rules lead to a precise and linear interpolation, as pictured in figure 2 with 3 interpolation points. This is not what we are looking for.

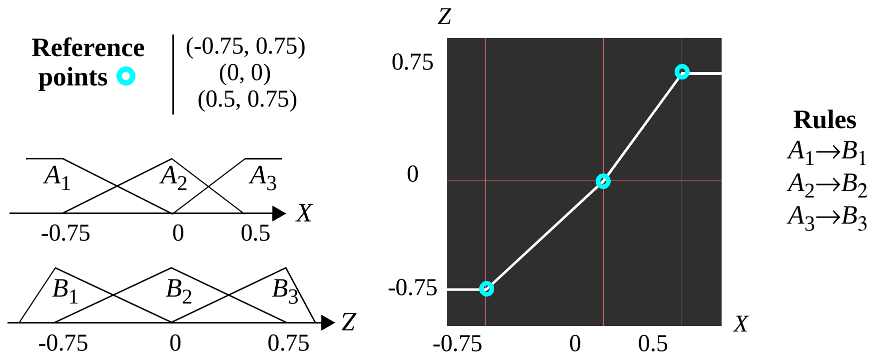

Fig. 2. : Linear interpolation

In order to tune the fuzzy set parameters, let us first restrict ourselves to *increasing* reference points (i.e. $x_i < x_{i+1}$ and $z_i < z_{i+1}$) and study the generic case of figure 3 corresponding to a pair of gradual rules $A_i \rightarrow B_i$ and $A_{i+1} \rightarrow B_{i+1}$. The grey area delimited by points M_k, $k = 1, ..., 6$ corresponds to the interpolation graph obtained from both gradual rules.

Figure 3 makes it clear that the interpolation graph should be connected in order to guarantee that any feasible input is associated with at least one possible output value. This means that there should be no conflict when two rules are simultaneously fired, i.e. when $x \in [x_{i+1}^{-}, x_i^{+}]$. From a geometric point of view, this amounts to locating the point M_5 in figure 3 under the segment M_1M_2, which is analytically expressed by the inequality:

$$(z_{i+1}^{-} - z_i)(x_i^{+} - x_i) \leq (z_i^{+} - z_i)(x_{i+1}^{-} - x_i). \tag{4}$$

In a similar way, the point M_2 should be above the segment M_5M_4, i.e.:

$$(z_i^{+} - z_{i+1})(x_{i+1}^{-} - x_{i+1}) \leq (z_{i+1}^{-} - z_{i+1})(x_i^{+} - x_{i+1}). \tag{5}$$

Conditions (4) and (5) are necessary for obtaining a connected interpolation graph. They are also sufficient as far as the point M_6 is located under the segment M_1M_2 (i.e., $z_i^{-} < z_i < z_i^{+}$) and M_3 is above M_4M_5 (i.e., $z_{i+1}^{-} < z_{i+1} < z_{i+1}^{+}$). In fact, conditions (4) and (5) are a particular case of coherence conditions for a set of fuzzy rules, which were previously established in [9].

The counterpart to inequalities (4) and (5) for handling *decreasing* reference points (i.e. $x_i < x_{i+1}$ and $z_{i+1} < z_i$) using gradual rules $A_i \rightarrow B_i$ can be easily derived by taking the mirror image of figure 3 with respect to the Z axis. For keeping an understandable notation of the support of the B_is, i.e. $[z_i^- \ z_i^+]$ with $z_i^- < z_i < z_i^+$, exponents + and – attached to the z variable should be exchanged in (4) and (5).

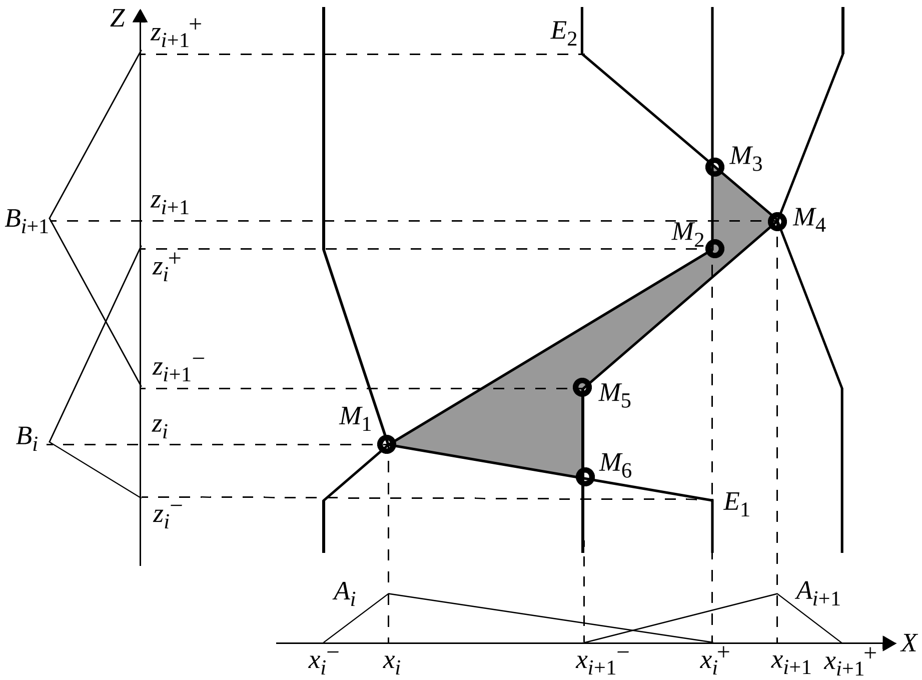

Fig. 3. : Graph corresponding to a pair of gradual rules

Figure 4 illustrates the importance of the coherence of a rule base with respect to the interpolation relation. The two rules $A_2 \rightarrow B_2$ and $A_3 \rightarrow B_3$ are incoherent (constraint (5) is violated), which results in a gap in the interpolation graph between the reference points P_2 and P_3.

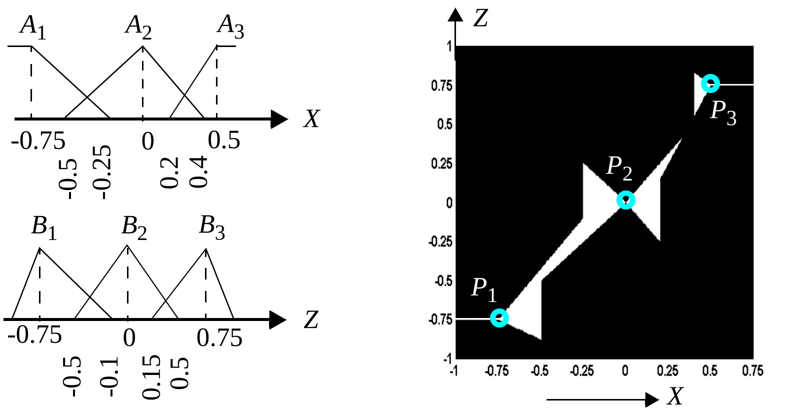

Fig. 4. : Effect of incoherent rules on the interpolation relation

3.2 Shaping the interpolation areas

Coherent rules ensure the non-emptiness of the image of each input via the relation Γ. However viewing figure 3, one may be disappointed by the shape of the graph which is complicated and hard to justify on the basis of practical needs. Comparing figures 3 and 5, it can be stated that a slight modification of the parameters z_i^+ and z_{i+1}^- yields an interesting change in the shape of the interpolation graph which should then be controlled.

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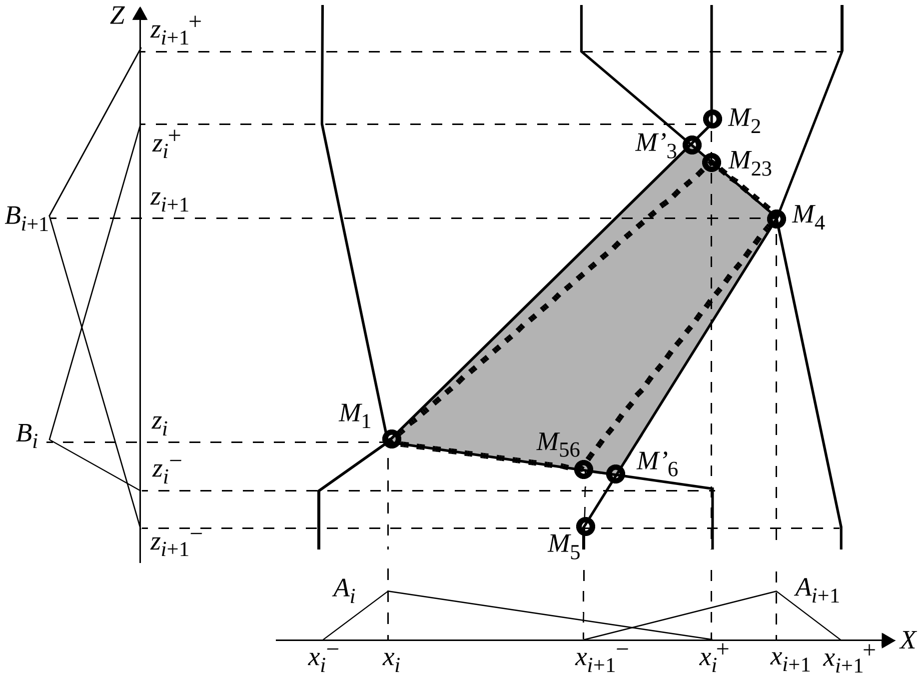


Fig. 5. : Graph corresponding to a pair of gradual rules

It remains to clarify the conditions which determine the 4-sided area between two reference points. From a geometric point of view, the deletion of the vertical segment M_2M_3 in figure 3 requires that the point M_2 be located above M_3, which after computation of the z coordinate of M_3 leads to:

$$(z_i^+ - z_{i+1})(x_{i+1}^- - x_{i+1}) \leq (z_{i+1}^+ - z_{i+1})(x_i^+ - x_{i+1}). \tag{6}$$

In a similar way the constraint that M_5 be under M_6 is expressed by:

$$(z_{i+1}^- - z_i)(x_i^+ - x_i) \leq (z_i^- - z_i\ )(x_{i+1}^- - x_i). \tag{7}$$

Keeping the hypothesis of the coverage of X by at most two rules everywhere in the input domain, the satisfaction of (6) and (7) requires that:

$$z_{i+1} \leq z_i^+ \quad \text{and} \quad z_{i+1}^- \leq z_i \tag{8}$$

and thus the coverage of Z by the B_i's should be sufficiently high (more precisely $\forall z \in Z$, $\Sigma_{i=1,..,n}\ \mu_{B_i}(z) \geq 1$), which also ensures the consistency of the rules, since (4) and (5) then hold.

When inequalities (6) and (7) hold, the interpolation graph between the reference points is the 4-sided area delimited by M_1, M'_3, M_4 and M'_6 (see figure 5). Although the coordinates of points M'_3 and M'_6 can be expressed in terms of the considered parameters, it appears simpler to delimit the interpolation area by the points M_1, M_{23}, M_4 and M_{56}, which corresponds to changing (6) and (7) into equalities. In other words, points M_2 and M_3 (respectively M_5 and M_6) in figure 3 are positioned at the same location denoted M_{23} (respectively M_{56}) in figure 5. Actually, the graph obtained using such a tuning is the convex hull of the shaded region in figure 3, which corresponds to the smallest 4-sided interpolation area that can be defined from gradual rules. In such a case, the x-coordinates of M_{23} and M_{56}, namely x_i^+ and x_{i+1}^-, can be explicitly chosen. Then, for a given choice of the x-parameters, the definition of the rules requires the determination of $2n$ z-parameters, which are linked together by $2(n\text{-}1)$ equations. Thus, two parameters remain free. A simple way of building the partition of Z is then to prescribe the values on the boundaries, namely z_1^- and z_n^+, and to solve the system of equations obtained by the equality constraints associated with (6) and (7).

Figure 6 pictures the interpolation graph which is obtained with the interpolation points of figure 2. The partitioning of X is obtained by cutting the intervals $[x_i, x_{i+1}]$ into three equal parts, i.e. $x_{i+1}^- - x_i = x_i^+ - x_{i+1}^- = x_{i+1} - x_i^+$, as illustrated by the parameters which define the fuzzy sets A_i. The extreme values z_1^- and z_3^+ are also predefined (see figure 6). Lastly, the other parameters have been obtained by solving the system of equations derived from (6) and (7).

Interpolation points

	x_i^-	x_i	x_i^+
A_1		-0.75	-0.25
A_2	-0.5	0	0.3333
A_3	0.1667	0.5	

Partitioning of X

Interpolation points

	z_i^-	z_i	z_i^+
B_1	-1	-0.75	0.4375
B_2	-0.875	0	0.875
B_3	-0.4375	0.75	1

Solution of the system of equations

Extreme values

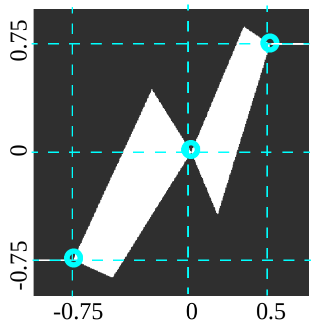


Fig. 6. : Construction of a piecewise 4-sided interpolation graph

3.3 Improving the interpolation areas

In particular applications, supplementary constraints such as "If $x \in [x_i, x_{i+1}]$ then $z \in [z_i, z_{i+1}]$" can be available. In this case, the interpolation graph can be further refined by limiting the imprecision area between two interpolation points to a quadrangle with horizontal edges as suggested in [7].

The first intuitive idea for coping with the desired graph form consists in translating the constraints related to the considered variables into constraints concerning the parameters that define A_i and B_i. According to figure 3, the horizontal alignment of M_2, M_3 and M_4 leads to the following conditions:

$$z_i^+ = z_{i+1}\ , \tag{9}$$

$$\frac{z_{i+1}^+ - z_{i+1}}{x_{i+1}^- - x_{i+1}}\ (x_i^+ - x_{i+1}) = 0\cdot \tag{10}$$

In the same way, the horizontal alignment *of* M_1, M_5 and M_6 induces that:

$$z_{i+1}^- = z_i\ , \tag{11}$$

$$\frac{z_i^- - z_i}{x_i^+ - x_i}\ (x_{i+1}^- - x_i) = 0\cdot \tag{12}$$

Applying conditions (9) and (11) to the entire rule base results in a strict partitioning of Z. Furthermore, satisfying equations (10) and (12) requires either a strict partitioning of X which produces linear interpolation (see figure 2), or the guarantee that $z_{i+1}^+ = z_{i+1}$ and $z_i^- = z_i$, which induces conflicts with (9) and (11) when applied to any interpolation point. In other words, there is no parameter configuration that defines the desired shape with horizontal edges.

A simple strategy for obtaining the desired graph form consists in adding new rules to the initial rule base that directly translate the interval-based constraints "If $x \in [x_i, x_{i+1}]$ then $z \in [z_i, z_{i+1}]$". Two additional symbols, denoted $A_{i/i+1}$ and $B_{i/i+1}$, i = 1, ..., n-1, with rectangular membership functions are thus defined for representing the intervals between two consecutive interpolation points. Then, by implementing the rules "If x is $A_{i/i+1}$ then z is $B_{i/i+1}$", the upper and lower parts of the piecewise original graph are suppressed (see figure 7). Such an approach is successful only when the rectangular restriction of the initial graph is composed of 4-sided areas, which means that conditions (6) and (7) must be satisfied.

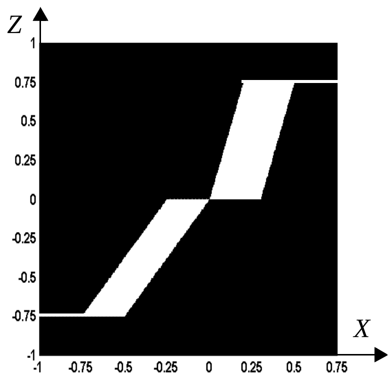


Fig. 7. : Design of piecewise truncated graphs

4 Interpolation between imprecise reference points

An imprecise point can be viewed as a precise point whose coordinates are not exactly known. Choosing an interval-based representation, an imprecise point is then modelled by a rectangular area to which the ill-known point necessarily belongs. In this way, four parameters, respectively denoted x_i^l, x_i^r, z_i^l and z_i^r, define the i^{th} imprecise point. Using gradual rules, trapezoidal membership functions are defined to handle closeness to imprecise points. Let A_i be the trapezoidal fuzzy set defined by parameters x_i^-, x_i^l, x_i^r and x_i^+ as illustrated in figure 8. Then the support of A_i, i.e. $[x_i^l, x_i^r]$, is associated with the x-coordinate of the imprecise point and the membership degree to A_i decreases on each side of $[x_i^l, x_i^r]$ with the distance to the nearest interval bound. Using a similar representation for the z-coordinate of imprecise points, each interpolation point is associated with a gradual rule of the form "If x is A_i then z is B_i" where symbols A_i (resp. B_i) represent trapezoidal fuzzy sets with support $[x_i^-, x_i^+]$ (resp. $[z_i^-, z_i^+]$) and core $[x_i^l, x_i^r]$ (resp. $[z_i^l, z_i^r]$).

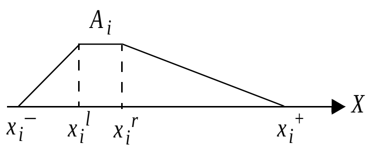


Fig. 8. : Fuzzy representation of imprecise points

When n imprecise reference points are known, the design of the graph relies on parameters x_i^-, x_i^+, z_i^- and z_i^+, $i = 1, ..., n$. In fact, the problem is similar to the one previously addressed in the case of precise interpolation points and an identical solving procedure can be developed.

Once again, coherence conditions can be determined from geometric constraints on the position of points M_2 and M_5 in figure 9. The counterparts to inequalities (4) and (5) are respectively given by:

$$(z_{i+1}^- - z_i^r)(x_i^+ - x_i^r) \le (z_i^+ - z_i^r)(x_{i+1}^- - x_i^r). \tag{13}$$

$$(z_i^+ - z_{i+1}^l)(x_{i+1}^- - x_{i+1}^l) \le (z_{i+1}^- - z_{i+1}^l)(x_i^+ - x_{i+1}^l). \tag{14}$$

In the same way, it is again possible to control the shape of the imprecision area for obtaining a 4-sided-like interpolation graph between two imprecise points. Constraints (6) and (7) are then transformed into:

$$(z_i^+ - z_{i+1}^r)(x_{i+1}^- - x_{i+1}^l) \le (z_{i+1}^+ - z_{i+1}^r)(x_i^+ - x_{i+1}^l). \tag{15}$$

$$(z_{i+1}^- - z_i^l)(x_i^+ - x_i^r) \le (z_i^- - z_i^l)(x_{i+1}^- - x_i^r). \tag{16}$$

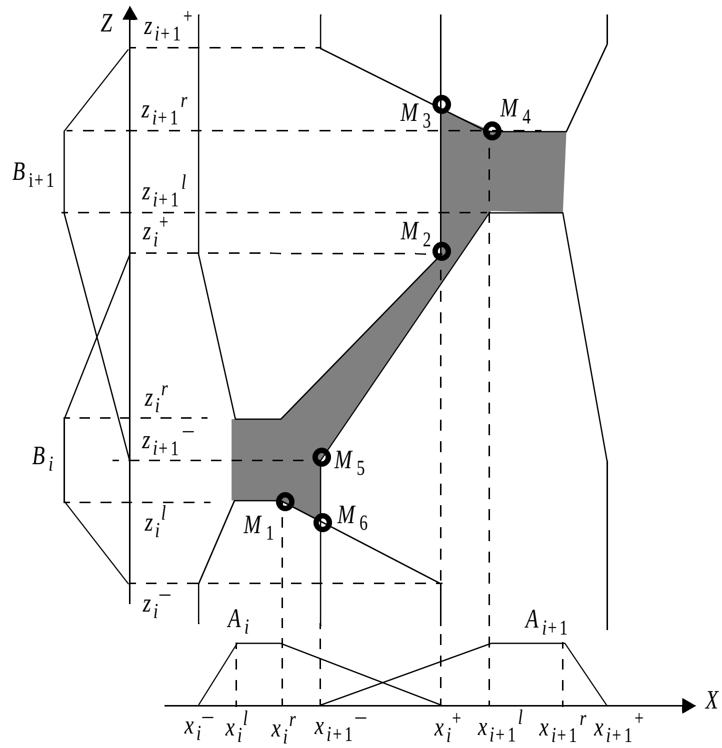
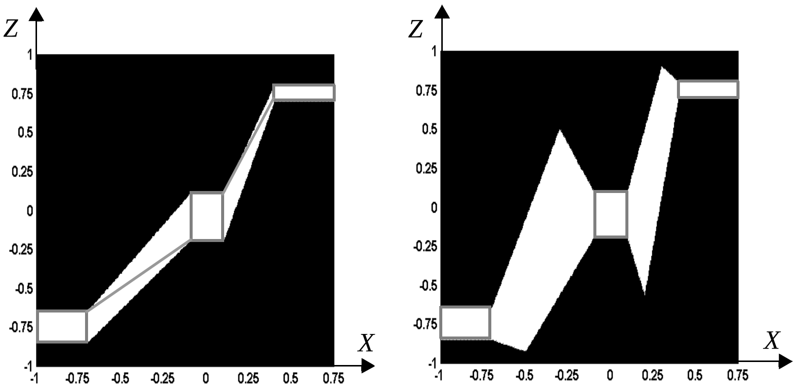


Fig. 9. : Interpolation graph between imprecise points

Figure 10 shows two different examples of interpolation graphs between imprecise points marked with grey rectangles. Each situation can be linked to its counterpart that was previously dealt with when considering precise interpolation points. Figure 10(a) coincides with linear interpolation of figure 2 and figure 10(b) is the extension of figure 6. In the linear case, it should be noticed that a strong partitioning of *X* and Z with trapezoidal fuzzy sets, i.e. $x_{i+1}^{-} = x_i^{r}$, $x_i^{+} = x_{i+1}^{l}$ and $z_{i+1}^{-} = z_i^{r}$, $z_i^{+} = z_{i+1}^{l}$, leads to coherent rules since constraints (4) and (5) hold but the interpolation simply links the reference points by means of a segment joining their nearest corners (see grey lines in figure 10(a)). Keeping a strong partition of *X* but modifying the *Z* partition such that $z_{i+1}^{-} = z_i^{l}$ and $z_i^{+} = z_{i+1}^{r}$ results in the graph pictured in figure 10(a) which includes all possible linear interpolants that link the vertical edges of the reference points.



(a) Linear interpolation (b) Quadrangle-based interpolation

Fig. 10. : Different interpolation graphs with imprecise points

5 Interpolative fuzzy graph

One may be disappointed that the fuzziness introduced in the closeness relations is no more present in the interpolation graph. This section is devoted to the issue of introducing membership degrees in the 4-sided areas while keeping their support unchanged. Such an approach is motivated by the need to evaluate the relative merits of different possible paths between the reference points.
According to equation (2), it is obvious that using a crisp implication for defining the graph necessarily results into a crisp graph. Then, the most intuitive strategy for keeping memberhip degrees in the interpolative graph consists in replacing Rescher-Gaines implication by another one. In order to be in accordance with the semantics of gradual rules, only residuated implications can be used, which means that:

$$a \rightarrow b = 1 \text{ if } a \leq b \text{ and } a \rightarrow b = \alpha \text{ if } a > b \tag{17}$$

where α depends on the chosen implication. From equations (2) and (17), it is obvious that the core of the graph Γ (white areas in figures 6 and 10) does not depend on the chosen implication. Actually, only values outside the core area (black areas with zero membership grade in figures 6 and 10) are affected by the choice of an implication. Thus, our aim when introducing membership degrees in the 4-sided areas cannot be achieved simply by choosing a suitable fuzzy implication. As an alternative, an approach based on level 2 gradual rules (still implemented using Rescher Gaines implication) is proposed. In the following developments, we restrict ourselves to precise reference points, that is to triangular fuzzy sets, but similar results can be obtained with imprecise interpolation points.

5.1 Nested graph family

According to sections 3, it is clear that given a set of rules, i.e. a set of reference points, a collection of crisp graphs is obtained by varying the support parameters of the A_i's and/or the B_i's. Moreover, inclusion properties between the built graphs can be exhibited for controlling the variation of the supports as expressed by the following statements.

> P1: If $A_i \subseteq A_i^*$, $i=1, ..., n$, then $\Gamma^* \subseteq \Gamma$,
> where Γ and Γ^* are the graphs associated with rules $A_i \to B_i$ and $A_i^* \to B_i$ respectively.

Indeed, $(x, z) \in \Gamma^*$ means that $\forall i, A_i^*(x) \leq B_i(z)$. According to the assumption that $\forall i$, $A_i \subseteq A_i^*$, it follows that $\forall i, A_i(x) \leq B_i(z)$ which results in $(x, z) \in \Gamma$.

> P2: If $B_i^* \subseteq B_i$, $i=1, ..., n$, then $\Gamma^* \subseteq \Gamma$,
> where Γ and Γ^* are now the graphs associated with rules $A_i \to B_i$ and $A_i \to B_i^*$ respectively.

Indeed, $(x, z) \in \Gamma^*$ if and only if $\forall i, A_i(x) \leq B_i^*(z)$. Since $\forall i$, $B_i^* \subseteq B_i$, it follows that $\forall i, A_i(x) \leq B_i(z)$, i.e. $(x, z) \in \Gamma$.

The combination of P1 and P2 leads to:

> P3: If $A_i \subseteq A_i^*$ and $B_i^* \subseteq B_i$, $i=1, ..., n$, then $\Gamma^* \subseteq \Gamma$,
> where Γ and Γ^* are now the graphs associated with rules $A_i \to B_i$ and $A_i^* \to B_i^*$ respectively.

Thus, according to the above inclusion properties, it is possible to design a family of nested graphs simply by building collections of nested fuzzy subsets on X and Z. Indeed, denote $\{A_i^\lambda, \lambda \in [0,1]\}$ a family of fuzzy subsets on X such that $A_i^{\lambda'} \subseteq A_i^\lambda$ if $\lambda \geq \lambda'$ and $\{B_i^\lambda, \lambda \in [0,1]\}$ a family of fuzzy subsets on Z such that $B_i^\lambda \subseteq B_i^{\lambda'}$ if $\lambda \geq \lambda'$. The graph family associated with rules $A_i^\lambda \to B_i^\lambda$, $\lambda \in [0,1]$, guarantees that $\Gamma^\lambda \subseteq \Gamma^{\lambda'}$ if $\lambda \geq \lambda'$. Actually, such a construction of nested graphs simply expresses that implicative graphs increase in the sense of inclusion when underlying constraints become more permissive. Indeed, more permissive rules are obtained either by restricting input conditions further, or by enlarging output fuzzy sets.

Using a convex linear combination of fuzzy intervals enables the automatic construction of a collection of nested fuzzy subsets ranging from the lower bound of the family to the upper one. Applying such a technique results in the following equation:

$$A_i^\lambda = (1-\lambda)A_i^0 \oplus \lambda A_i^1,\ \lambda \in [0,1],\ i=1, .. ,n \tag{18}$$

where A_i^0 and A_i^1, such that $A_i^0 \subseteq A_i^1$, are the lower and upper bounds of the family and $\oplus$ denotes the extended sum of fuzzy numbers. In the same way, nested output fuzzy subsets can be built according to:

$$B_i^\lambda = (1-\lambda)B_i^0 \oplus \lambda B_i^1,\ \lambda \in [0,1],\ i=1, .. ,n \tag{19}$$

where B_i^0 and B_i^1, such that $B_i^1 \subseteq B_i^0$, are the upper and lower bounds of the family. It should be noted that the inclusion ordering of the B_i^λ for increasing λ is the converse of the one of the A_i^λ, due to opposite behaviors with respect to graph inclusion.

Using such fuzzy subset families (see figure 11) results in the following graph inclusions:

$$\Gamma^1 \subseteq \Gamma^\lambda \subseteq \Gamma^{\lambda'} \subseteq \Gamma^0,\ \lambda,\lambda' \in [0,1] \text{ and } \lambda \geq \lambda' \tag{20}$$

One interesting point is now to study whether the 4-sided shape introduced in section 3 can be shared by nested graphs. In other words, do inequalities (6) and (7) hold for any Γ^λ when they hold for Γ^0 and Γ^1 ? A positive answer can be given to this question according to property P4 whose proof is established in the appendix.

> P4: If both (A_i^0, B_i^0) and (A_i^1, B_i^1), i=1, .. ,n, satisfy equations (6) and (7), then (6) and (7) are still valid for any (A_i^λ, B_i^λ), $\lambda \in [0,1]$, i=1, .. ,n, when A_i^λ and B_i^λ are built according to (18) and (19).

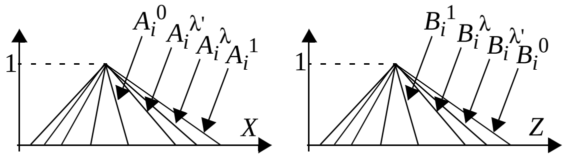


Fig. 11. : Nested fuzzy subsets (λ>λ')

5.2 Fuzzy graph

According to the previous section, the construction of indexed nested graphs can be easily handled from the knowledge of lower and upper graphs. Now, given the above family $\{\Gamma^\lambda, \lambda \in [0,1]\}$, there is a unique fuzzy set F whose λ-cuts F_λ are precisely Γ^λ for each $\lambda \in [0,1]$. This fuzzy set is built using the standard representation theorem [16], that is:

$$\mu_F(x, z) = \sup_{\lambda \in [0,1]} \min (\lambda, \Gamma^\lambda(x, z)) \tag{21}$$

According to formulation (21), the reconstructed F is finally a classical fuzzy graph defined on $X \times Z$. Another interpretation consists in viewing F as a fuzzy set of crisp graphs, that is as a level 2 fuzzy set [17]. In this case, F is represented as:

$$F = \int_{\lambda \in [0,1]} \lambda / \Gamma^\lambda \tag{22}$$

according to Zadeh's notation where the integral sign stands for the union of the fuzzy singletons λ / Γ^λ.

Figure 12 plots the fuzzy graph obtained when the lower graph Γ^1 is precise and piecewise linear (as in figure 2) and the upper graph Γ^0 has the 4-sided shape of figure 6.

According to figure 11, families $\{A_i^\lambda, \lambda \in [0,1]\}$ and $\{B_i^\lambda, \lambda \in [0,1]\}$, i=1, ...,n, can also be viewed as type 2 fuzzy subsets, i.e. fuzzy sets with fuzzy membership grades [17]. In this framework, one may wonder if extending the Rescher-Gaines implication to fuzzy set-valued arguments would be compatible with equation (21). This approach is studied in [11] where it is shown that the fuzzy graph F can be also obtained from a type 2 fuzzy set-based view.

The proposed approach based on (21) provides a constructive method for deriving a genuine fuzzy implication from a set of gradual rules. That it is a genuine implication can be checked by verifying the following properties:

if $\mu_{A1}(x) \geq \mu_{A1}(x^*)$ then $\mu_F(x, z) \leq \mu_F(x^*, z)$
if $\mu_{B0}(z) \geq \mu_{B0}(z^*)$ then $\mu_F(x, z) \geq \mu_F(x, z^*)$

Moreover,

if $\mu_{A1}(x) = 1$ then $\mu_F(x, z) = 0$ when $\mu_{B0}(z) \neq 1$
if $\mu_{A1}(x) = 0$ then $\mu_F(x, z) = 1$
if $\mu_{B0}(z) = 1$ then $\mu_F(x, z) = 1$ (identity principle).

It is interesting to compare our construction to the one in [5]. This paper establishes results under which fuzzy implications can be decomposed as convex sums of crisp rules. It assumes a finite number of membership grades. Under certain mild conditions, this decomposition involves a nested family of gradual rules of the form $m_i(A) \rightarrow B$ where $\{m_i\}$ is a family of modifiers affecting the condition part only. In the present paper, both conditions and conclusions are varied.

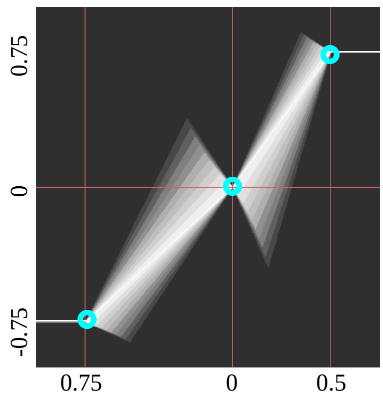


Fig. 12. : Fuzzy graph obtained from a set of gradual rules

6 Conclusion

This paper has proposed a representation framework which is faithful to the imprecision of available data. In the intervals between interpolation points where it is difficult to specify an analytical law, imprecision is captured by means of 4-sided areas. In practice, the specification of these areas is rather simple and can take into account qualitative knowledge about the system behavior such as the absence of overshoot, or some information on the magnitude order of the derivative at the reference points. The modelling methodology, which has been applied for categorizing time series [10], has exhibited interesting performance. Moreover, we can hope to improve the proposed imprecise representation by using fuzzy gradual rules as introduced in the last section. Indeed, it enables us to distinguish between typical members of a class which remain in subareas with high membership degrees from borderline members which go through subareas with smaller membership degrees.

The interest of using gradual rules for specifying imprecise interpolation is potentially manifold: easy interfacing with the user, embedding of the representation into an inference or fusion system, possibility of extending the approach to fuzzy interpolation points and to fuzzy inputs, specifying queries in data mining applications (see [13] for an example of such possible use), ... These advantages should be more patent in the case of multi-input systems. So, the practical use of the suggested strategy now requires that the multi-input case be developed. Perspectives also include comparison between interpolation graphs obtained from gradual rules and those that can be derived from interpolation between rules in sparse rule bases [2].

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Appendix

The proofs concerning inequalities (6) and (7) are distinct but can be handled using exactly the same approach. For the sake of conciseness, we restrict this appendix to the proof of (6).

According to fuzzy arithmetics, the convex combination of two triangular fuzzy subsets produces a new triangular fuzzy subset whose parameters are obtained using the convex combination under consideration. The three A_i^λ parameters are thus derived from equation (18) as follows:

$$x_i^{-\lambda} = (1-\lambda)\, x_i^{-0} + \lambda\, x_i^{-1},$$
$$x_i^{\lambda} = (1-\lambda)\, x_i^{0} + \lambda x_i^{1},$$
$$x_i^{+\lambda} = (1-\lambda) x_i^{+0} + \lambda x_i^{+1}.$$

According to equation (19), the same mechanism is applied for the B_i^λ parameters.
Then, substituting λ-indexed parameters in equation (6), the sign of the following f polynomial has to be established:

$$f(\lambda) = a^\lambda\, b^\lambda - c^\lambda\, d^\lambda, \tag{23}$$

where $a^\lambda = z_i^{+\lambda} - z_{i+1}^{\lambda}$, $b^\lambda = x_{i+1}^{\lambda} - x_{i+1}^{-\lambda}$,
$c^\lambda = z_{i+1}^{+\lambda} - z_{i+1}^{\lambda}$, $d^\lambda = x_{i+1}^{\lambda} - x_i^{+\lambda}$.

The development of f with respect to λ leads to:

$$f(\lambda) = (1-\lambda)^2 f(0) + \lambda^2 f(1) + \lambda(1-\lambda)\,(a^0 b^1 + a^1 b^0 - c^0 d^1 - c^1 d^0).$$

The assumption that both (A_i^0, B_i^0) and (A_i^1, B_i^1), i=1, .. ,n, satisfy equation (6) can be reformulated as:

$$f(0) \geq 0 \quad \text{and} \quad f(1) \geq 0. \tag{24}$$

Furthermore, the inclusion properties $A_i^0 \subseteq A_i^1$ and $B_i^1 \subseteq B_i^0$, i=1, .. ,n, induce that $a^1 \leq a^0$, $b^0 \leq b^1$, $c^1 \leq c^0$and $d^1 \leq d^0$, which expresses that:

$$(a^0 - a^1)(b^1 - b^0) + (c^0 - c^1)(d^0 - d^1) \geq 0, \tag{25}$$

Then, the development of (25) leads to:

$$a^0 b^1 + a^1 b^0 - c^0 d^1 - c^1 d^0 \geq f(0) + f(1). \tag{26}$$

Finally, using (24) and (26) it follows that:

$$f(\lambda) \geq 0,\ \lambda \in [0,1], \tag{27}$$

which means that inequality (6) is valid for any (A_i^λ, B_i^λ), $\lambda \in [0,1]$, i=1, .. ,n, and concludes the proof.