Pivot selection strategies in Jacobi joint block-diagonalization

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Abstract. A common problem in independent component analysis after prewhitening is to optimize some contrast on the orthogonal or unitary group. A popular approach is to optimize the contrast only with respect to a single angle (Givens rotation) and to iterate this procedure. In this paper we discuss the choice of the sequence of rotations for such so-called Jacobi-based techniques, in the context of joint block-diagonalization (JBD). Indeed, extensive simulations with synthetic data, reported in the paper, illustrates the sensitiveness of this choice, as standard cyclic sweeps appear to often lead to non-optimal solutions. While not being able to guarantee convergence to an optimal solution, we propose a new schedule which, from empirical testing, considerably increases the chances to achieve global minimization of the criterion. We also point out the interest of initializing JBD with the output of joint diagonalization (JD), corroborating the idea that JD could in fact perform JBD up to permutations, as conjectured in previous works.

1 Introduction

Joint diagonalization techniques have received much attention in the last fifteen years within the field of signal processing, and more specifically within the fields of independent component analysis (ICA) and blind source separation (BSS). JADE, one of the most popular ICA algorithms developed by Cardoso and Souloumiac [1], is based on orthonormal joint diagonalization (JD) of a set of cumulant matrices. To this purpose the authors designed a Jacobi algorithm for approximate joint diagonalization of a set of matrices [2]. In a BSS parlance, JADE allows for separation of determined linear instantaneous mixtures of mutually independent sources, exploiting fourth-order statistics. Other standard BSS techniques involving joint diagonalization include the SOBI algorithm [3], TDSEP [4], stBSS [5] and TFBSS [6], which all rely on second-order statistics of the sources, namely covariance matrices in the first through third case and spatial Wigner-Ville spectra in the fourth case; see [7] for a review.

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Joint block-diagonalization (JBD) came into play in BSS when Abed-Meraim, Belouchrani and co-authors extended the SOBI algorithm to overdetermined convolutive mixtures [8]. Their idea was to turn the convolutive mixture into an overdetermined linear instantaneous mixture of block-dependent sources, the second-order statistics matrices of the source vector thus becoming block-diagonal instead of diagonal. Hence, the joint diagonalization step in SOBI needed to be replaced by a JBD step. Another area of application can be found in the context of multidimensional ICA or independent subspace analysis [9,10]. Its goal is to linearly transform an observed multivariate random vector such that its image is decomposed into groups of stochastically independent vectors. It has been shown that by using fourth-order cumulants to measure the independence, JADE now translates into a JBD problem [11]; similarly also SOBI and other JD-based criteria can be extended to this group ICA setting [12,13].

Abed-Meraim et al. have sketched several Jacobi strategies in [14–16]: the JBD problem is turned into a minimization problem, where the matrix parameter (the joint block-diagonalizer) is constrained to be unitary (because of spatial prewhitening). The minimizer is searched for iteratively, as a product of Givens rotations, each rotation minimizing a block-diagonality criterion around a fixed axis, which we refer to as 'pivot'. Convergence of the algorithm is easily shown, but convergence to an optimal solution (which minimizes the chosen JBD criterion) is not guaranteed. In fact, we observed that results vary widely according to the choice of the successive pivots (which we refer to as 'schedule') and the initialization of the algorithm, which is not discussed in previous works [14–16]. The main contributions of this paper are 1) to point out that the choice of the rotation schedule is a sensitive issue which greatly influences the convergence properties of the Jacobi algorithm, as illustrated on extensive simulations with synthetic data, 2) to propose a new schedule, which, from empirical testing, offers better chances to converge to the optimal solution (while still not guaranteeing it), as compared to the standard cyclic Jacobi technique. We also point out the interest of initializing JBD with the output of JD, corroborating the idea that JD could in fact perform JBD up to permutations, as suggested by Cardoso in [10], more recently conjectured by Abed-Meraim and Belouchrani in [16] and partially proved in [11,17].

The paper is organized as follows. Section 2 briefly describes the Jacobi approach to approximate JBD, with fixed equal block sizes. Section 3 compares the convergence results obtained with three choices of initialization/schedule on generated sets of matrices exactly joint block-diagonalizable, with various size, block size and set dimension. Section 4 reports conclusions.

2 Jacobi approximate joint block-diagonalization

2.1 Approximate joint block-diagonalization

Let $\mathcal{A} = \{\mathbf{A}_1, \ldots, \mathbf{A}_K\}$ be a set of K complex matrices of size $n \times n$. The problem of approximate JBD consists of finding a unitary matrix $\mathbf{U} \in \mathbb{C}^{n \times n}$ such that

$\forall k \in [\![1, K]\!] := \{1, \ldots, K\}$, the matrices

$$\mathbf{U}\,\mathbf{A}_k\,\mathbf{U}^H = \mathbf{B}_k$$

are as block-diagonal as possible. More precisely, let us denote L the (fixed) length of the diagonal blocks and $m = n/L$ the number of blocks. Writing for $k \in [\![1, K]\!]$

$$\mathbf{A}_k = \begin{bmatrix} \mathbf{A}_{k11} & \ldots & \mathbf{A}_{k1m} \\ \vdots & & \vdots \\ \mathbf{A}_{km1} & \ldots & \mathbf{A}_{kmm} \end{bmatrix}$$

where $\mathbf{A}_{kij}$ is a subblock of dimensions $L \times L$, $\forall (i, j) \in [\![1, m]\!]^2$, our block-diagonality criterion is chosen as

$$\text{boff}\,(\mathbf{A}_k) := \sum_{1 \leq i \neq j \leq m} \|\mathbf{A}_{kij}\|_F^2. \tag{1}$$

Here $\|\mathbf{B}\|_F^2 = \sum_{ij} |b_{ij}|^2$ denotes the Frobenius norm. We look for $\mathbf{U}$ by minimizing the cost function

$$C_{\text{jbd}}(\mathbf{V}; \mathcal{A}) := \sum_{i=1}^{K} \text{boff}\left(\mathbf{V}\,\mathbf{A}_i\,\mathbf{V}^H\right)$$

with respect to $\mathbf{V} \in U(n)$, where $U(n)$ is the set of unitary $n \times n$-matrices.

2.2 The Jacobi approach

Jacobi approaches rely on the fact that any unitary matrix $\mathbf{V} \in U(n)$ can be written as a product of complex Givens matrices $\mathbf{G}(p, q, c, s) \in U(n)$, $1 \leq p < q \leq n$, defined as everywhere equal to the identity $\mathbf{I}_n$ except for $[\mathbf{G}(p, q, c, s)]_{pp} = [\mathbf{G}(p, q, c, s)]_{qq} = c$, $[\mathbf{G}(p, q, c, s)]_{pq} = \bar{s}$, $[\mathbf{G}(p, q, c, s)]_{qp} = -s$, with $(c, s) \in \mathbb{R} \times \mathbb{C}$ such that $c^2 + |s|^2 = 1$. The Jacobi approach consists of iteratively applying the same Givens rotation to all the matrices in set $\mathcal{A}$, with (p, q) chosen as to minimize criterion C_{jbd}. In other words, for fixed p and q, one iteration of the method consists of the following two steps:

- compute $(c^\star, s^\star) = \text{argmin}_{c,s}\; C_{\text{jbd}}(\mathbf{G}(p, q, c, s); \mathcal{A})$
- $\forall k \in [\![1, K]\!]$, $\mathbf{A}_k \leftarrow \mathbf{G}(p, q, c^\star, s^\star)\,\mathbf{A}_k\,\mathbf{G}(p, q, c^\star, s^\star)^H$

Let $I_1, \ldots, I_m$ be the partition of $[\![1, n]\!]$ defined by $I_i = [\![(i-1)\,L + 1, i\,L]\!]$, and let $i(k) = \lceil i/L \rceil$ give the index i of the interval I_i to which k belongs. Let $\mathbf{B}_k = \mathbf{G}(p, q, c, s)\,\mathbf{A}_k\,\mathbf{G}(p, q, c, s)^H$, $k \in [\![1, K]\!]$. $\mathbf{B}_k$ is everywhere equal to $\mathbf{A}_k$, except for its p^{th} and q^{th} rows and columns, which depend on c and s, such

that [17, 18]

$$
\begin{aligned}
b_{kpp} &= c^2\, a_{kpp} + |s|^2\, a_{kqq} + c\, s\, a_{kpq} + c\, \bar{s}\, a_{kqp} \\
b_{kqq} &= c^2\, a_{kqq} + |s|^2\, a_{kpp} - c\, s\, a_{kpq} - c\, \bar{s}\, a_{kqp} \\
b_{kpj} &= c\, a_{kpj} + \bar{s}\, a_{kqj} \quad (j \in I_{i(p)}, j \neq p) \\
b_{kjp} &= c\, a_{kjp} + s\, a_{kjq} \quad (j \in I_{i(p)}, j \neq p) \\
b_{kqj} &= -s\, a_{kpj} + c\, a_{kqj} \quad (j \in I_{i(q)}, j \neq q) \\
b_{kjq} &= -\bar{s}\, a_{kjp} + c\, a_{kjq} \quad (j \in I_{i(q)}, j \neq q)
\end{aligned}
$$

Using the fact that the Frobenius norm is invariant to rotations, minimization of criterion $C_{\text{jbd}}(\mathbf{G}(p,q,c,s);\mathcal{A})$ with respect to (c,s) can be shown to be equivalent to the maximization of

$$
C'_{\text{jbd}}(c,s) :=
$$

$$
\sum_{k=1}^{K} \left\{ |b_{kpp}|^2 + |b_{kqq}|^2 + \sum_{j \in I_{i(p)}, j \neq p} |b_{kpj}|^2 + |b_{kjp}|^2 + \sum_{j \in I_{i(q)}, j \neq q} |b_{kqj}|^2 + |b_{kjq}|^2 \right\}
$$

However, the latter criterion is constant if p and q belong to the same interval $I_i(p)$ (i.e, $i(p) = i(q)$). Details of above derivations can be found in [17, 18].

It may be shown [15, 16] that the maximization of $C'_{\text{jbd}}(c,s)$ boils down to the constrained maximization of a linear quadratic form. This optimization can be achieved using Lagrange multipliers. The computation of the latter requires solving a polynomial of degree 6 in the complex case (i.e, $\mathbf{U} \in \mathbb{C}^{n \times n}$), and of degree 4 in the real case (i.e, $\mathbf{U} \in \mathbb{R}^{n \times n}$). First order approximations of the criterion are also considered in [15, 16] to simplify its maximization. A tensorial rank-1 approximation is also found in [19]. For real matrices, when both $\mathbf{A}_k$ and $\mathbf{U}$ belong to $\mathbb{R}^{n \times n}$, maximization of $C'_{\text{jbd}}(c,s)$ directly amounts to rooting a polynomial of order 4 (without requiring a Lagragian parametrization), as sketched in [19] and developed in [17, 18].

So far, the indices p and q have been fixed. However, the important issue appears not to be how to maximize $C'_{\text{jbd}}(c,s)$, which can be done exactly in a way or another, but how to choose these pivots (p,q). Similarly to JD, the convergence of the proposed (joint) block-diagonalization scheme is guaranteed by construction, whatever the chosen rotation schedule [17, 18]. If convergence to the *global* minimum was in practice usually observed with joint diagonalization schemes, this is certainly not the case for joint block-diagonalization, where we found convergence to be very sensitive to initialization and rotation schedule, as illustrated in the next section.

3 Simulations

The employed algorithms as well as some of the following examples are freely available for download at [20]. The programs have been realized in MATLAB,

and sufficient documentation is given to reproduce the results and extend the algorithms. We propose to test the following initialization/schedule strategies.

(M1) The first method is inspired from the standard cyclic Jacobi approach [2,21], which consists of systematically sweeping the pivots one after the other, except for the fact that the couples (p, q) are chosen not to include the diagonal blocks. The algorithm is initialized with the identity matrix, i.e $\mathbf{U} = \mathbf{I}_n$. The algorithm is stopped when all the values of $s^\star$ are lower than 10^{-4} within a sweep.

(M2) The second method is identical to (M1) except for the fact that the algorithm is initialized with the matrix $\mathbf{U}_{\rm jdr}$ provided by joint diagonalization of $\mathcal{A}$, as obtained from [2].

(M3) The third method is inspired from the classical Jacobi method for the diagonalization of a normal matrix [21] and consists, after initialization as in (M2), of choosing at each iteration the pivot (p, q) ensuring a maximum decrease of criterion $C_{\rm jbd}$. This requires computing all the differences $|\sum_{k=1}^{K} \text{boff}\,(\mathbf{B}_k) - \text{boff}\,(\mathbf{A}_k)\,|$ for all couples (p, q) and to pick up the couple which yields the largest difference value. The algorithm stops when 20 successive values of $s^\star$ are all lower than 10^{-4}.

For simplicity, the three methods are tested in the real case. The three methods are applied to 100 random draws of K real matrices *exactly* joint block-diagonalizable in a real common orthogonal basis (optimal rotation angles are thus computed by rooting a polynomial of order 4 like in [17,18]). Various values of L (size of the blocks), m (number of blocks) and K (number of matrices) are considered. The number of failures over the 100 realizations (i.e, the number of times the methods do not converge to a solution such that $C_{\rm jbd} = 0$) is reported in Table 1. The results emphasize the importance of the initialization and the choice of the schedule. Failure rates of (M1) are very high, in particular when m and L increase. (M2) and (M3), which are both initialized by joint diagonalization, give much better results, with (M3) being in nearly every case more reliable than (M2). However, none of the two methods systematically converge to a global minimum of $C_{\rm jbd}$ when $m \geq 3$, and, interestingly, the methods do not usually fail on the same data sets. Also, Fig. 1 and Fig. 2 show that (M3) only need a few iterations after JD to minimize $C_{\rm jbd}$. This indicates the validity of the claim from [16], that JD minimizes the joint block-diagonality $C_{\rm jbd}$, however only up to a permutation. In the above simulation, the permutation is then discovered by application of the JBD algorithm — this also explains why in Figures 1 and 2, when (M2) is used, the cost function after JD only decreases in discrete steps, corresponding to identified permutations.

Audio results of the separation of a convolutive mixture with 3 observations and 2 sources, obtained with the generalization of SOBI using our pivot selection scheme and followed by a SIMO identification step removing filtering ambiguities are found at [22], following the approach described in [23].

Table 1. Number of failures of methods M1, M2 and M3 over 100 random realizations of K matrices exactly block-diagonalizable in a common orthonormal basis.

m	2														
L	2					4					6				
K	1	3	6	12	24	1	3	6	12	24	1	3	6	12	24
M1	1	4	4	1	2	32	33	25	10	11	55	33	21	24	16
M2	0	0	0	0	0	11	1	0	0	0	43	2	0	0	0
M3	0	0	0	0	0	5	0	0	0	0	14	0	0	0	0

m	3														
L	2					4					6				
K	1	3	6	12	24	1	3	6	12	24	1	3	6	12	24
M1	3	14	11	18	8	68	54	38	33	32	84	60	48	51	52
M2	0	0	0	0	0	29	5	1	2	0	53	10	8	7	8
M3	0	0	0	0	0	15	1	0	3	1	44	0	0	2	8

m	4														
L	2					4					6				
K	1	3	6	12	24	1	3	6	12	24	1	3	6	12	24
M1	5	30	21	19	16	87	75	68	60	59	99	83	77	77	75
M2	0	0	0	0	0	47	7	6	4	2	88	15	8	4	10
M3	0	0	0	0	0	21	5	4	2	3	65	8	2	0	5

4 Conclusions

The main algorithmic conclusion of this paper is: Jacobi algorithms for joint block-diagonalization bring up convergence problems that do not occur in joint diagonalization and that still need to be properly addressed. However we proposed a strategy (method (M3)) which considerably reduces the failure rates of the straightforward approach (M1). The fact that lower failure rates are obtained with (M2) and (M3), which are initialized with joint diagonalization, tend to corroborate the conjecture that JBD diagonalization could be achieved up to an arbitrary permutation of columns via JD [10, 16], but it still does not explain why this permutation cannot be solved by minimization of C_{jbd}. This is a question we are currently working on, and for which partial results exist already [11, 17]. Moreover, extensions to the case of varying, possibly unknown block sizes are interesting [11], with respect to both the optimization and the application in the field of ICA.

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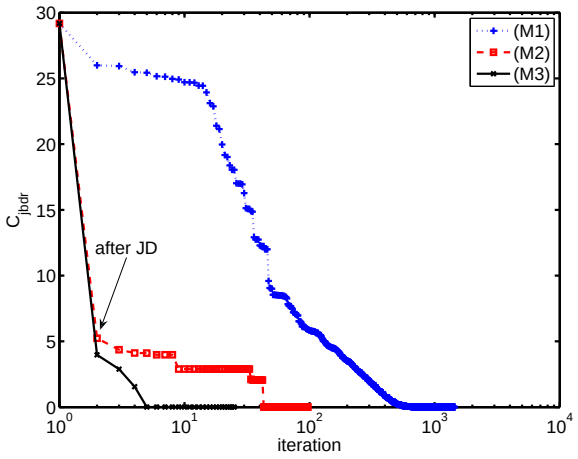

Fig. 1. Evolution of criterion C_{jbd} for a random set $\mathcal{A}$ such that $m = 3$, $L = 4$, $K = 3$. Using a 2.60 GHz Pentium 4 with 1 Go RAM, the computation times for this particular dataset are: (M1 - 1.2 s), (M2 - 0.3 s), (M3 - 1.2 s). The three methods succeed in minimizing the criterion.

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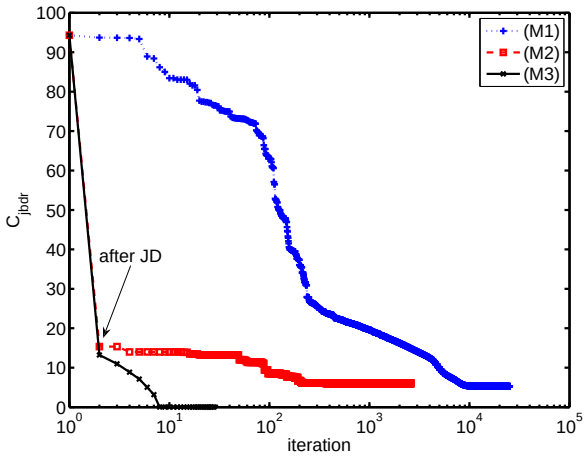

Fig. 2. Evolution of criterion C_{jbd} for a random set $\mathcal{A}$ such that $m = 4$, $L = 6$, $K = 3$. Using a 2.60 GHz Pentium 4 with 1 Go RAM, the computation times for this particular dataset are: (M1 - 28.4 s), (M2 - 4.1 s), (M3 - 6.9 s). Only (M3) succeeds in minimizing the criterion.

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