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FROM CLASSICAL TO NORMAL MODAL LOGICS

1 INTRODUCTION

Classical modal logics (Segerberg [27], Chellas [2]) are weaker than the well-known normal modal logics: The only rule that is common to all classical modal logics is

$$RE: \frac{F \leftrightarrow G}{\Box F \leftrightarrow \Box G}$$

(We nevertheless note that this principle raises problems in systems containing equality (Hughes and Cresswell [14]).)

Classical modal logics do not necessarily validate

$$RN: \frac{F}{\Box F}$$

$$RM: \frac{F \rightarrow G}{\Box F \rightarrow \Box G}$$

$$C: (\Box F \wedge \Box G) \rightarrow \Box(F \wedge G)$$

$$K: (\Box F \wedge \Box(F \rightarrow G)) \rightarrow \Box G$$

which are valid in any normal modal logic. [1]

In an epistemic reading, adopting one of the above formulas as an axiom means to close knowledge under a particular principle. Precisely, RE corresponds to the principle of knowledge closure under logical equivalences, RN under logical truth,

[1] Under RE, the principles RN, RM, C, and K are respectively equivalent to

$$N: \Box\top$$

$$M: \Box(F \wedge G) \rightarrow (\Box F \wedge \Box G)$$

$$C': (\Box F \wedge \Diamond(F \rightarrow G)) \rightarrow \Diamond G$$

$$K': (\Box F \wedge \Diamond G) \rightarrow \Diamond(F \wedge G)$$

H. Wansing (ed.), Proof Theory of Modal Logic, 293–311.

RM under logical consequence, C under conjunction, and K under material implication.

These principles, in an epistemic reading, as well as in a deontic one, are not always desirable (see e.g. (Fagin and Halpern [3]), (Jones and Pörn [16, 17])). In fact, each closure principle expresses an aspect of what has been called the omniscience problem.

Each of these principles can be adopted independently of the others. Nevertheless, under RE and RM, C is equivalent to K, and under RE and RN, K is equivalent to RM and C.

Classical modal logics have possible worlds semantics in terms of minimal models (cf. Scott-Montague structures in (Montague [20])), involving a neighbourhood function mapping worlds to sets of sets of worlds. Now each of the above closure principles identifies a particular class of minimal models. For several of these principles, specialisations of minimal models have been given such as augmented minimal models, models with queer worlds or models with inaccessible worlds. They are more tailored to these principles and give a better account of the logic associated to the principle.

While standard Kripke semantics can be translated straightforwardly into classical first-order logic, this is *a priori* not the case of the above modal logics. The reason is that the neighbourhood function of minimal models cannot be represented directly by a first-order formula.

In this paper (section 4) we show that nevertheless semantics in terms of minimal models can be expressed in first-order logic. We show this indirectly: What we prove is that minimal models can be translated into standard Kripke models (which on their turn can be translated into first-order logic).

A major advantage of such translations - and in fact our main motivation - is that they allow to reuse the proof systems that have been developed for normal modal logics.

In the rest of this paper we first briefly present several classes of minimal models and their simplified semantics (section 3). Then for each class we present a translation into normal multimodal logics (section 4).

2 GENERAL POINTS

2.1 *Language*

The *language* of modal logic is built on a set of propositional variables, classical connectives and a modal operator $\Box$. F, G, and H denote formulas, and $\top$ and $\bot$ respectively stand for logical truth and falsehood. $\Diamond F$ is an abreviation of $\neg\Box\neg F$. $\mathcal{FOR}$ denotes the set of formulas.

2.2 Semantics

Semantics is stated in terms of frames and models. Generally, a *frame* is composed of a set W (whose elements are called worlds) and some structure $\mathcal{S}$ on W. In the well-known case of K-frames (or Kripke frames), the structure is just some binary relation r on W: $r \subseteq W \times W$. [2] Then a *model* is composed of a frame and a meaning function m mapping propositional variables to sets of worlds.

Given a model, it is the truth conditions which uniquely determine a ternary forcing relation between models, worlds, and formulas. In the case of propositional variables and classical connectives, the truth conditions are the usual ones:

- $M, w \models F$ iff $w \in m(F)$ if F is a propositional variable.
- $M, w \models F \wedge G$ iff $M, w \models F$ and $M, w \models G$.
- $M, w \models \neg F$ iff $not(M, w \models F)$.

The structure of W is exploited when it comes to the truth condition for the modal operator. In the case of standard Kripke semantics, the forcing relation must satisfy the following one:

- $M, w \models \Box F$ iff for all $v \in r(w)$, $M, v \models F$.

Structures that are richer have more complex truth conditions. In order to avoid confusions we shall add a superscript to $\models$ designating the type of the non-normal semantics.

Generally, in a given semantics, we say that a formula F is *true in a model* $M = (W, \mathcal{S}, m)$ if $M, w \models F$ for every $w \in W$. A formula F is *true in a frame* $(W, \mathcal{S})$ if for every every meaning function m, F is true in $(W, \mathcal{S}, m)$. A formula F is *valid in a class of frames* $\mathcal{F}$ (noted $\models_{\mathcal{F}} F$) if F is true in every frame of $\mathcal{F}$.

A particular semantics will always be identified by a condition on the structure type together with the truth condition for the modal operator. An example in standard Kripke semantics is condition $t : w \in r(w)$ for every $w \in W$. (As in (Chellas [2]), conditions on structures are denoted by small letters.)

We confuse conditions and the class of frames satisfying them: k being the case of standard Kripke models, the class of standard Kripke frames satisfying condition t is called $k + t$ (or kt for short).

2.3 Axiomatics

When a class of frames can be characterized by some axiom system we denote the latter by the corresending capital letters. E.g. the basic normal modal logic is called K, and $\vdash_K F$ expresses that F is a theorem of K. The class of frames kt being

[2] We shall often write $w' \in r(w)$ instead of $(w, w') \in r$.

characterized by the T-axiom $\Box F \to F$, the corresponding logic is called $K + T$ (or KT for short).

2.4 Multimodal Logics

Multimodal languages generalize modal languages by allowing indexed modal operators. The target logics of our translations being bi- or trimodal logics, what we need is a language containing three modal operators $[1], [2], [3]$.

The multimodal logics we need are normal ones having a standard Kripke semantics. We index every accessibility relation and condition on it by the number of the corresponding modal operator. E.g. $t[1]$ expresses that the accessibility relation r_1 is reflexive. As before, classes of multimodal frames are refered to by sums of conditions. $k[1] + k[2]$ denotes the class of bimodal frames, and $k[1] + k[2] + k[3]$ the class of trimodal frames. (Sometimes we write $k[1,2]$ and $k[1,2,3]$ for short.)

On the axiomatical side, we index axioms and inference rules by the corresponding number. E.g. $K[1]$ is the axiom $([1]F \wedge [1](F \to G)) \to [1]G$, and $D[2]$ is $[2]F \to \langle 2\rangle F$. As before, multimodal systems are referred to by sums of axiom or inference rule names. Our basic normal bimodal logic is $K[1]+K[2]$, and the basic trimodal one is $K[1] + K[2] + K[3]$ ($K[1,2]$ and $K[1,2,3]$ for short).

3 MINIMAL FRAMES AND THEIR CHILDREN

In this section we present the class of minimal frames as well as three subclasses of it: supplemented minimal frames, quasi-filters, and singleton minimal frames. (The last subclass corresponds to Humberstone's inaccessible worlds logic.)

3.1 Minimal Frames

Minimal frames are the basic semantical tool for classical modal logics.

A *minimal frame* (Chellas [2]) is a couple (W, N), where

- W is a set of worlds, and
- $N : W \to 2^{2^W}$ maps worlds to sets of sets of worlds.

Sets of worlds being usually called propositions, we may also say that N associates a set of propositions to every world. $N(w)$ is sometimes called the neighbourhood of w. We recall that then a *minimal model* is a triple (W, N, m) where (W, N) is a minimal frame, and m is a meaning function.

The forcing relation $\models^{min}$ results from the following truth condition for the modal connective:

- $M, w \models^{min} \Box F$ iff there is $V \in N(w)$ such that ($v \in V$ **iff** $M, v \models^{min} F$)

Hence $\Box F$ is true in a world if its neighbourhood contains the extension of F.

The class of minimal frames is noted e. Subclasses of e can be obtained by adding conditions on the mapping N. An example of a condition is t: $w \in V$ for all $V \in N(w)$ and $w \in W$. The class of minimal frames satisfying condition t is called $e + t$, or et for short.

The class of minimal frames e can be characterized by some axiom system for classical propositional logic plus the inference rule RE (Chellas [2]). This basic classical modal logic is called E.

The class of frames et is characterized by the T-axiom $\Box F \rightarrow F$. As expected, the corresponding logic is called $E + T$, or ET for short.

REMARK 1 *There is an isomorphic form of minimal frames (see e.g. (Fitting [5])), where there is a set of accessibility relations instead of the neighbourhood function N. In order to account for worlds with empty neighbourhoods, just as in regular modal logics the concept of a queer world is used.* [3]

Formally, Fitting's frames are triples (W, Q, R), where W and Q are sets of worlds such that $Q \subseteq W$ and R is a set of relations on W. Q is called the set of queer worlds. Here, the forcing relation $\models^{min}$ must fulfil the following truth condition for the modal operator:

- $M, w \models \Box F$ *iff* $w \notin Q$, *and there is* $r \in R$ *such that* $v \in r(w)$ **iff** $M, v \models F$

Hence in a queer world all formulas or the form $\Box F$ are false.

3.2 Singleton Minimal Frames

Singleton minimal frames correspond to inaccessible world frames that were introduced and axiomatized in (Humberstone ([15]).

A *singleton minimal frame* is a minimal frame, where for every world w, the set $N(w)$ is a singleton. In other words, there is exactly one propostion that is necessary in a given world.

The only principle of our list that supplemented minimal frames validate is that of closure under conjunction:

$C: (\Box F \wedge \Box G) \rightarrow \Box(F \wedge G)$

There are countermodels for RN, RM, and K.

Clearly, singleton minimal frames are isomorphic to standard Kripke frames of the form (W, r). Hence the only difference from the standard Kripke semantics is the modal truth condition:

- $M, w \models^{iw} \Box F$ iff ($v \in r(w)$ **iff** $M, v \models^{iw} F$)

[3]Note that although Fitting uses these frames only for monotonic modal logics, they can be used as well to give semantics to classical modal logics.

Note that RE and C are not enough to completely characterize singleton minimal frames, and that Humberstone's ([15]) axiomatization is infinitary. This logic has also been studied formally in (Goranko [11], Goranko and Passy [13]). The concept of inaccessible worlds is fundamental in the logics of knowledge and belief of Levesque ([19]). There, $\Box F$ is read "I only know F".

3.3 *Supplemented Minimal Frames*

Supplemented minimal frames are at the base of monotonic modal logics.

A *supplemented minimal frame* is a minimal frame where for every world w, the set $N(w)$ is closed under supersets. [4]

Supplemented minimal frames can be characterized by the principle of closure under logical consequences:

$$RM: \ \frac{F \rightarrow G}{\Box F \rightarrow \Box G}$$

(from which RE can be derived). There are countermodels for RN, C, and K. Modal logics containing RE and RM are called *monotonic*, and the basic monotonic logic is called EM.

There is a slightly different semantics where the closure under supersets is implicit in the truth condition. It has been used e.g. in Fitting's ([5]) underlying logic U. There, frames are just the minimal ones. It is in the forcing relation (that we note $\models^{sup}$) where the truth condition for the modal operator is different from that of classical modal logics:

- $M, w \models^{sup} \Box F$ iff there is $V \in N(w)$ such that ($v \in V$ **implies** $M, v \models^{sup} F$)

In other words, we can simulate superset closure by replacing "iff" by "implies" in the truth conditions.

For the same semantics, the terms *local reasoning frames* and *logic of local reasoning* have been employed by Fagin and Halpern ([3]). They have employed these logics to model implicit and explicit belief: Explicit implies implicit belief, and the operator of explicit belief has a monotonic modal logic, whereas that of implicit belief has a normal modal logic.

3.4 *Quasi-Filters*

Quasi-filters are the base of regular modal logics.

A supplemented minimal frame is a *quasi-filter* if $N(w)$ is closed under supersets and under finite intersection, for every world w. Closure under finite intersection can be strengthened to closure under arbitrary intersections (v. (Chellas [2]), p.

[4] An equivalent condition is: If $U \cap V \in N(w)$ then $U \in N(w)$ and $V \in N(w)$.

255). Hence quasi-filters are isomorphic to local reasoning frames where for every world w, the set $N(w)$ is either a singleton or the empty set.

Such a presentation can be specialised to the following one in terms of augmented Kripke frames (Fitting [5], [6]). An *augmented Kripke frame*[5] is a quadruple (W, Q, r, m) where

- W is a set of worlds,
- $Q \subseteq W$ is a set of worlds (called queer worlds), and
- $r \subseteq W^2$ is a relation on W.

Thus queer worlds correspond to worlds in minimal frames having an empty neighbourhood. In this semantics the truth condition for the modal connective is:

- $M, w \models^{aug} \Box F$ iff ($w \notin Q$ and for every $v \in r(w) : M, v \models^{aug} F$)

We are quite close to standard Kripke semantics here, except that there are some worlds that do not satisfy any formula of the form $\Box F$.

Augmented Kripke frames can be characterized (Fitting [5]) by the principles of closure under logical consequences and material implication:

$$RM : \frac{F \to G}{\Box F \to \Box G}$$

$$K : (\Box F \land \Box(F \to G)) \to \Box G$$

They can also be characterized by RM and C (closure under conjunction). There are countermodels for RN (closure under logical truth).

Modal logics containing RM and C are called *regular*, and the basic monotonic modal logic is called EMC.

4 TRANSLATIONS

In this section, for each class of classical modal logics that we have presented we give a translation into normal multimodal logics.

Monotonic, regular, and inaccessible world modal logics are translated into bimodal logics, whereas classical modal logics are translated into a trimodal modal logic. As the latter translation combines the principles of the previous ones, we give it at the end.

[5]Augmented Kripke frames should not be confused with augmented minimal frames (Chellas [2]) which are minimal frames that are isomorphic to standard Kripke frames.

4.1 Monotonic Modal Logics

The Translation

The idea of the translation is to give a world status to those propositions that are associated to worlds via the neighbourhood function N: Every element of $N(w)$ is viewed as a world accessible from w. Thus "there is a $V \in N(w)$" can be expressed via some existential modal operator $\langle 1 \rangle$. The truth of a formula in every element of V can then be expressed via a second universal modal operator [2]. In this way we can transform a minimal model into a standard Kripke model, and the logic of $\langle 1 \rangle$ and [2] is a normal one.

Formally we define a translation τ from monotonic modal logics into normal multi-modal logics as follows:

- $\tau(F) = F$ if F is a propositional variable
- $\tau(\Box F) = \langle 1 \rangle [2] \tau(F)$

and homomorphic for the cases of the classical connectives.

The same translation has been given independently by F. Wolter ([28]) in order to prove the completeness of a large class of monotonic modal logics.

THEOREM 2 $\models_{em} F$ *iff* $\models_{k[1,2]} \tau(F)$.

Hence by soundness and completeness of EM and $K[1,2]$ we also have that a formula F is a theorem of basic monotonic modal logic EM if and only if $\tau(F)$ is a theorem of normal bimodal logic $K[1,2]$.

REMARK 3 *We can also translate the operator $\mathcal{I}$ of implicit belief of (Fagin and Halpern [3]) by adding the supplementary case $\tau(\mathcal{I}F) = [1][2]\tau(F)$.*

REMARK 4 *At least for the basic monotonic modal logic EM we are able to strengthen our translation to $\tau(\Box F) = \langle 1 \rangle [1] \tau(F)$ and thus to translate into mono-modal logic.*

In the next theorem we give a more general result for those monotonic modal logics that are characterized by axioms among the following standard ones:

$D: \Box F \to \Diamond F$

$T: \Box F \to F$

$4: \Box F \to \Box\Box F$

The systems $EMD, EMT, EM4, EMD4, EMT4$ are complete (Chellas [2]). We denote by *emd*, *emt*, *em4*, *emd4*, *emt4* the corresponding classes of frames. Completeness of these logics is used in the following theorem:

THEOREM 5 *Let κ be any sum of conditions on minimal frames among $d, t, 4$. Let $\mathcal{A}$ be the corresponding combination of axioms $D, T, 4$. Then $\models_{em+\kappa} F$ iff $\vdash_{K[1,2]+\tau(\mathcal{A})} \tau(F)$.*

Hence using the completeness of $EMD, EMT, EM4, EMD4, EMT4$ we can prove now theorems of these monotonic modal logics via our translation into particular multi-modal logics.

Note that the translations $\tau(D), \tau(T), \tau(4)$ of the standard modal axioms D, T, and 4 become multi-modal axioms in the style of Sahlqvist [26] for which completeness results (Catach [1], Kracht [18]) and automated deduction methods (Ohlbach [23], [24], Gasquet [7], [8], Nonnengart [21], Fariñas and Herzig [4]) are known. Note also that the translations of the modal axioms 5 and B would become axioms which have not been studied yet in the literature.

Examples

EXAMPLE 6 The formula $(\Box p \wedge \Box q) \rightarrow \Box(p \wedge q)$ is translated into
$(\langle 1\rangle[2]p \wedge \langle 1\rangle[2]q) \rightarrow \langle 1\rangle[2](p \wedge q)$,
which can be proved neither in EM nor in $EMD, EMT, EM4, EMD4$, $EMT4$.

Proofs

We do not give the proofs, because the technique is a particular case of that for classical modal logics (cf. subsection 4.4).

4.2 Regular Modal Logics

The Translation

Having at our disposal the preceding translation for monotonic modal logics, an immediate way to translate regular modal logics is to prove that $\models_{emc} F$ iff $\vdash_{K[1,2]+\tau(C)} \tau(F)$, where C is the axiom of closure under conjunction, and τ is the same as in subsection 4.1. Nevertheless,

$\tau(C) = (\langle 1\rangle[2]\tau(F) \wedge \langle 1\rangle[2]\tau(G)) \rightarrow \langle 1\rangle[2](\tau(F) \wedge \tau(G))$is a rather complex axiom. Here we give an optimization by associating a special axiom just to the first modal operator [1]. This will also permit to state a more general exactness theorem.

What we shall prove is that the translation is exact when the modal logics for [1] and [2] are normal, and moreover that for [1] satisfies an axiom that we call T_c (the converse of the standard T-axiom):

$T_c : F \rightarrow [1]F$

Semantically, this corresponds to the condition t_c that the accessibility relation r_1 is a subset of the diagonal of W: $r_1 \subseteq \delta_W$. (In other words, for every world w,

$r_1(w)$ is either a singleton w or the empty set). The idea behind this axiom T_c is that, in our proof, queer worlds will not be in r_1, while $\Box A$ will be translated into $\langle 1\rangle[2]A$; queer worlds will correspond to worlds without r_1-successors and hence will invalidate $\langle 1\rangle[2]A$.

Clearly, $\tau(C)$ is a theorem of $K[1,2]+T_c[1]$. In the following theorem we give a general result for systems of regular modal logics:

THEOREM 7 *Let κ be any semantical condition on accessibility relations r of augmented Kripke frames. Let $\kappa(1,2)$ be the translation of κ on accessibility relations r_1 and r_2 of standard Kripke frames ($u \notin Q$ being translated into $(u,v) \in r_1$ and $(u,v) \in r$ into $(u,v) \in r_2$.*

Then $\models_{emc+\kappa} F$ iff $\models_{k[1,2]+t_c[1]+\kappa(2)} \tau(F)$.

Hence if κ can be characterized by an axiom $\mathcal{A}$ and $\kappa(1,2)$ by a bimodal axiom $\mathcal{A}^\tau$, then we can prove theorems of regular modal logic $EMC + \mathcal{A}$ by proving theorems of the normal bimodal logic $K[1,2] + T_c[1] + \mathcal{A}^\tau$.

THEOREM 8 *Let κ be any semantical condition on accessibility relations r of augmented Kripke frames that is characterized by some axiom $\mathcal{A}$. Let $\kappa(1,2)$ be as in the previous theorem. Then $\models_{emc+\kappa} F$ iff $\vdash_{K[1,2]+T_c[1]+\tau(\mathcal{A})} \tau(F)$.*

Examples

EXAMPLE 9 The formula $\tau(C) = (\Box p \wedge \Box q) \to \Box(p \wedge q)$ is translated into $\tau(C) = ((\langle 1\rangle[2]p \wedge \langle 1\rangle[2]q) \to \langle 1\rangle[2](p \wedge q)$.
As $((\langle 1\rangle F \wedge \langle 1\rangle G) \to \langle 1\rangle(F \wedge G)$ can be proved in $K[1,2] + T_c[1]$, the formula $\tau(C)$ is a theorem.

Proofs

LEMMA 10 *For augmented Kripke models of the form $M_{aug} = (W, Q, r, m)$ let f be a mapping such that $f(M_{aug}) = (W, R_1, R_2, m)$ where*

- $R_1 = \delta_Q = \{(w,w) : w \notin Q\}$,
- $R_2 = R$.

Then f is an isomorphism between augmented Kripke models and standard Kripke $K[1,2] + T_c[1]$-models.

Proof. It is easy to see that for every augmented Kripke model M_{aug}, $f(M_{aug})$ is a $K[1,2]+T_c[1]$-model, and $M_{aug}, w \models^{aug} F$ iff $M, w \models \tau(F)$, for every $w \in W$.

In the other sense, for every $K[1,2]+T_c[1]$-model M, we have that $f^{-1}(M)$ is defined and is an augmented Kripke model, and $M, w \models \tau(F)$ iff M_{aug}, $w \models^{aug} F$, for every $w \in W$. ■

Thus we also have an isomorphism between $EM + \mathcal{A}$-models and $K[1,2] + T_c[1] + \tau(\mathcal{A})$-models. Theorem 7 follows immediately from that.

4.3 Humberstone's Inaccessible Worlds Logic

The Translation

The idea of the translation has been given in (Goranko [12]). It is the following: The truth condition is equivalent to

- $M_{iw}, w \models^{iw} \Box F$ iff for all $v \in W$
  - if $v \in R(w)$ then $M_{iw}, v \models^{iw} F$, **and**
  - if $v \in \bar{R}(w)$ then $M_{iw}, v \models^{iw} \neg F$.

where $\bar{R}$ denote the complement of R. What we do is to introduce two modal connectives, [1] to access R-successors, and [2] to access $\bar{R}$-successors. This leads to the following translation τ from logics with inaccessible worlds into normal bimodal logics:

- $\tau(F) = F$ if F is a propositional variable
- $\tau(\Box F) = [1]\tau(F) \wedge [2]\neg\tau(F)$

and homomorphic for the cases of the classical connectives.

In order to get an exact translation, we must choose $KT5[1 \cup 2]$ [6] as our target logic:

THEOREM 11 $\models^{iw} F$ *iff* $\vdash_{KT5[1\cup 2]} \tau(F)$.

Hence F is valid in Humberstone's inaccessible worlds logic iff $\tau(F)$ is valid in normal bimodal logic $KT5[1 \cup 2]$.

REMARK 12 *Contrarily to what could be expected, our target logic is not required to have accessibility relations for* [1] *and* [2] *with an empty intersection. In fact, it is sufficient to prove that* $KT5[1 \cup 2]$ *is characterized by the frames where:* $R_1 \cup R_2$ *is an equivalence relation* **and** $R_1 \cap R_2 = \varnothing$. *(This key lemma is given in the next section.)*

[6]By $KT5[1 \cup 2]$ we mean a normal multi-modal logic where both modal connectives [1] and [2] have the axioms of modal logic K, plus the axioms T and 5, stated for the modal operator $[1 \cup 2]$. ($[1 \cup 2]F$ is an abreviation for $[1]F \wedge [2]F$.) In other words, $KT5[1 \cup 2]$ is axiomatized by some axiomatization of classical logic, necessitation rules for [1] and [2], plus the axioms $[1 \cup 2]F \rightarrow F$ and $\neg[1 \cup 2]F \rightarrow [1 \cup 2]\neg[1 \cup 2]F$. It is well-known that $KT5[1 \cup 2]$ is characterized by the class of Kripke frames (W, R_1, R_2) where $R_1 \cup R_2$ is an equivalence relation over W (see e.g. (Catach [1])).

REMARK 13 *The axiomatization given in (Humberstone [15]) is infinitary. On the contrary, the above translation provides an indirect (because the modal connectives are not the original ones) but still modal axiomatization of this logic. Most of all, this axiomatization is finitary.*

Examples

EXAMPLE 14 The formula $\Box\top$ is translated into $[1]\top \wedge [2]\neg\top$, which is not valid in $KT5[1 \cup 2]$. Hence $\Box\top$ is not valid in inaccessible worlds logic.

EXAMPLE 15 The formula $\Box(p \wedge q) \rightarrow \Box p$ is translated into

$$[1](p \wedge q) \wedge [2]\neg(p \wedge q) \rightarrow [1]p \wedge [2]\neg p.$$

In $KT5[1 \cup 2]$, this formula is equivalent to the conjunction of $(([1](p \wedge q) \wedge [2]\neg(p \wedge q)) \rightarrow [1]p)$ and $(([1](p \wedge q) \wedge [2]\neg(p \wedge q)) \rightarrow [2]\neg p)$. Now the first conjunct is a theorem of $KT5[1 \cup 2]$ (because of $[1](p \wedge q) \rightarrow [1]p$), but the second is not.

EXAMPLE 16 The formula $(\Box p \wedge \Box q) \rightarrow (p \leftrightarrow q)$ is translated into

$$([1]p \wedge [2]\neg p \wedge [1]q \wedge [2]\neg q) \rightarrow (p \leftrightarrow q).$$

In $KT5[1 \cup 2]$, the antecedent is equivalent to $([1](p \leftrightarrow q) \wedge [2](p \leftrightarrow q)$, and now

$$([1](p \leftrightarrow q) \wedge [2](p \leftrightarrow q) \rightarrow (p \leftrightarrow q)$$

is an instance of the $T[1 \cup 2]$-axiom.

Proofs

The proof has been first given in (Goranko [12]).

LEMMA 17 *Let $M_{iw} = (W, R, m)$ be an inaccessible worlds model wherein a formula F is satisfied at w_0. Let $R_1 = R$ and $R_2 = W^2 \setminus R$. Then (W, R_1, R_2, m) is a $KT5[1 \cup 2]$-model satisfying $\tau(F)$ at w_0.*

Proof. First note that the $KT5[1 \cup 2]$ axioms are true in (W, R_1, R_2) because $R_1 \cup R_2$ is an equivalence relation. Let $M = (W, R_1, R_2, m)$. We prove that for every formula G and $w \in W, M_{iw}, w \models^{iw} G$ iff $M, w \models \tau(G)$ by induction on the structure of G. The only non-trivial case is

- $M_{iw}, w \models^{iw} \Box H$
- iff for all $v \in W$, ($v \in R(w)$ iff $M_{iw}, v \models^{iw} H$)
- iff for all $v \in W$, ($v \in R(w)$ iff $M, v \models \tau(H)$), by induction hypothesis
- iff for all v in W,
  - if $v \in R_1(w)$ then $M, v \models \tau(H)$ by construction of R_1, and
  - if $v \in R_2(w)$ then $M, v \models \neg\tau(H)$, by construction of R_2

- iff $M, w \models [1]\tau(H) \wedge [2]\neg\tau(H)$
- iff $M, w \models \tau(\Box H)$

■

LEMMA 18 *$KT5[1\cup 2]$ is characterized by the class of frames (W, R_1, R_2) where $R_1 \cup R_2$ is universal (i.e. $R_1 \cup R_2 = W^2$) and $R_1 \cap R_2 = \varnothing$.*

Proof. As the converse is trivial, we only have to prove that if F is $KT5[1 \cup 2]$-satisfiable then it is also satisfiable in a frame (W, R_1, R_2) where $R_1 \cup R_2$ is an equivalence relation and $R_1 \cap R_2 = \varnothing$. Suppose given a standard Kripke model $M = (W, R_1, R_2, m)$ such that $M, w_0 \models F$ for some $w_0 \in W$. First, let W' be the connected part of W that contains w_0, i.e. the set of worlds that can be reached from w_0 via $R_1 \cup R_2$. Let R'_1, R'_2, and m' be the respective restrictions of R_1, R_2, and m to W'. It is well-known that $M' = (W', R'_1, R'_2, m')$ is still a model of $KT5[1\cup 2]$, and $M', w_0 \models F$.
Now, we build a new model which will simulate (W', R'_1, R'_2, m'), but where the two relations will have an empty intersection. Let V_1 and V_2 be two sets such that V_1, V_2 and W' are pairwise disjoint, V_1 is isomorphic to W' via an isomorphism f_1, and V_2 is isomorphic to W' via an isomorphism f_2.(For example, V_i could be $W \times \{i\}$.)Let V denote $V_1 \cup V_2$, and let R''_1 and R''_2 be relations on V as follows:

1. $R''_1 = \{(f_i(v), f_i(w)) : w \in W,\ i = 1,2 \text{ and } (v,w) \in R_1\} \cup$
   $\{(f_i(v), f_{3-i}(w)) : w \in W,\ i = 1,2,\ (v,w) \in R_1 \text{ and } (v,w) \notin R_2\}$
2. $R''_2 = \{(f_i(v), f_i(w)) : w \in W,\ i = 1,2 \text{ and } (v,w) \in R_2\} \cup$
   $\{(f_i(v), f_{3-i}(w)) : w \in W,\ i = 1,2,\ (v,w) \in R_2 \text{ and } (v,w) \notin R_1\}$

Graphically,

$$\begin{tikzcd} u \arrow[r, bend left] & v \end{tikzcd} \qquad \begin{tikzcd} u \arrow[r, dotted, bend right] & v \end{tikzcd} \qquad \begin{tikzcd} u \arrow[r, bend left] \arrow[r, dotted, bend right] & v \end{tikzcd}$$

Will give:

$$\begin{tikzcd} f_1(u) \arrow[r] \arrow[dr] & f_1(v) \\ f_2(u) \arrow[r] \arrow[ur] & f_2(v) \end{tikzcd} \qquad \begin{tikzcd} f_1(u) \arrow[r, dotted] \arrow[dr, dotted] & f_1(v) \\ f_2(u) \arrow[r, dotted] \arrow[ur, dotted] & f_2(v) \end{tikzcd} \qquad \begin{tikzcd} f_1(u) \arrow[r] \arrow[dr, dotted] & f_1(v) \\ f_2(u) \arrow[r] \arrow[ur, dotted] & f_2(v) \end{tikzcd}$$

Where dotted lines denote R2 and full lines R1.

Finally, let m'' be defined by: $f_i(w) \in m''(F)$ iff $w \in m(F)$, for F a propositional variable.

Then let $\varphi = f_1^{-1} \cup f_2^{-1}$. φ is a pseudo-epimorphism from (V, R_1'', R_2'', m'') onto (W', R_1', R_2', m) (Hughes and Cresswell [14]). This ensures that F is satisfiable in (V, R_1'', R_2'', m''). Moreover, it can easily be shown that $R_1'' \cup R_2''$ is an equivalence relation over V, and even that $R_1'' \cup R_2'' = V^2$, and that $R_1'' \cap R_2'' = \varnothing$. Hence the lemma is proved. ■

LEMMA 19 *Let (W, R_1, R_2) be a frame for $KT5[1 \cup 2]$ wherein $\tau(F)$ is satisfiable. Then there is an inaccessible worlds frame wherein F is satisfiable.*

Proof. The proof is similar to that in (Humberstone [15]). ■

Suppose $M = (W, R_1, R_2, m)$, and $M, w_0 \models \tau(F)$ for some $w_0 \in W$. By the above Lemma 18, we can suppose that $R_1 \cap R_2 = \varnothing$, and that $R_1 \cup R_2 = W^2$, i. e. R_2 is the complement of R_1.

Let $M_{iw} = (W, R_1, m)$. Clearly M_{iw} is an inaccessible worlds model. We prove by induction that for every $w \in W$ and every formula F, we have $M, w \models \tau(F)$ iff $M_{iw}, w \models^{iw} F$. The only non-trivial case is: $F = \Box G$, i. e. $\tau(F) = [1]\tau(G) \wedge [2]\neg\tau(G)$.

- From the left to right, suppose $M, w \models [1]\tau(G) \wedge [2]\neg\tau(G)$. Let v be any world from W. If $v \in R_1(w)$, as $M, w \models [1]\tau(G)$, we have that $M, v \models \tau(G)$, and by induction hypothesis, $M_{iw}, v \models^{iw} G$. If $v \notin R_1(w)$ then $v \in R_2(w)$ (because $R_1 \cup R_2 = W^2$). As $M, w \models [2]\neg\tau(G)$, we have that $M, v \models \neg\tau(G)$, and by induction hypothesis, $M_{iw}, v \models^{iw} \neg G$. Putting both together we get that $M_{iw}, w \models^{iw} \Box G$.

- From the right to the left, suppose $M_{iw}, w \models^{iw} \Box G$. Then v is in $R_1(w)$ iff $M, v \models^{iw} G$. By induction hypothesis, $v \in R_1(w)$ iff $M, v \models \tau(G)$. Hence $M, w \models [1]\tau(G)$. As $R_1 \cap R_2 = \varnothing$, we have that $M, w \models [2]\neg\tau(G)$. Putting both together we get that $M, w \models [1]\tau(G) \wedge [2]\neg\tau(G)$.

REMARK 20 *It is clear that this lemma cannot handle extensions of the basic inaccessible world logic e.g. with axiom D. The reason is that we cannot ensure that our construction of the inaccessible world model preserves the accessibility relation properties.*

Now the theorem follows immediately from Lemma 17 and 19.

4.4 Classical Modal Logics

The Translation

The translation τ from classical modal logics into multi-modal logics combines the above translations of monotonic modal logics and Humberstone's inaccessible worlds logic. It goes as follows:

- $\tau(F) = F$ if F is a propositional variable
- $\tau(\Box F) = \langle 1\rangle([2]\tau(F) \wedge [3]\neg\tau(F))$

and homomorphic for the cases of the classical connectives.
Then we have

THEOREM 21 $\models_e F$ *iff* $\vdash_{K[1,2,3]} \tau(F)$.

Hence a formula F is valid in classical modal logic E iff $\tau(F)$ is valid in normal multi-modal logic $K[1,2,3]$.

REMARK 22 *Note that at least for the basic classical modal logic E we are able to strengthen our translation to* $\tau(\Box F) = \langle 1\rangle([2]\tau(F) \wedge [1]\neg\tau(F))$.

In the next theorem we extend our result to ET. We translate into $K[1,2,3] + B_{1,2}$, where $B_{1,2}$ is the axiom $\langle 1\rangle[2]G \rightarrow G$. $B_{1,2}$ axiomatizes a condition on frames that we call $b_{1,2} : R_1 \subseteq R_2^{-1}$.

THEOREM 23 $\models_{et} F$ *iff* $\vdash_{K[1,2,3]+B_{1,2}} \tau(F)$.

Note that axiom $B_{1,2}$ is in fact the monotonic modal logic translation of the T-axiom.
The generalization towards extensions of E by other axioms such as $D, 4, B, T$ seems to be much more difficult. This is due to the fact that the monotonic translations of these axioms do not completely axiomatize the translated models (see Lemma 25 in the proofs). On the other hand, the classical translation of e.g. axiom T yielding $\langle 1\rangle([2]G \wedge [3]\neg G) \rightarrow G$ would also be a candidate for the target logic axioms[7], but it is difficult to devise a semantics for such complex multimodal axioms. (Note nevertheless that we would only need a soundness result for the multimodal axiom).

Examples

EXAMPLE 24 The formula $\Box\top$ is translated into
$\langle 1\rangle([2]\top \wedge [3]\neg\top)$,
which in $K[1,2,3]$ is equivalent to $\langle 1\rangle[3]\bot$. This is clearly not a theorem of $K[1,2,3]$. Hence $\Box\top$ is not valid in classical modal logic.

[7] and not (i): $\langle 1\rangle([2]\tau(G) \wedge [3]\neg\tau(G)) \rightarrow \tau(G)$ as a particular instance of T is $\Box G \rightarrow G$, for G propositional variable, whose translation is $\langle 1\rangle([2]G \wedge [3]\neg G \rightarrow G$ which give (by substitution) $\langle 1\rangle([2]G \wedge [3]\neg G) \rightarrow G$ for any formula G.

Proofs

LEMMA 25 *Let $M_{min} = (W, N, m)$ be a minimal model.*
Let $M = (V, R_1, R_2, R_3, m)$ be a model such that

- $V = W \cup \bigcup_{w \in W} N(w)$,
- $R_1(w) = N(w)$, *for every* $w \in W$,
- $R_2(U) = U$, *for every* $w \in W$ *and* $U \in N(w)$,
- $R_3(U) = W - U$, *for every* $w \in W$ *and* $U \in N(w)$,

Then for every formula F and world $w \in W$, $M_{min}, w \models^{min} F$ iff $M, w \models \tau(F)$.

Proof. First, M is a model of $K[1, 2, 3]$. Then the proof is straightforward by induction on F. The only non-trivial case is that of $F = \Box G$. We have: $M_{min}, w \models^{min} \Box G$

iff there is $U \in N(w)$ such that for every $v \in W$, ($v \in U$ iff M_{min}, $v \models^{min} G$)

iff (by induction hypothesis) there is $U \in N(w)$ s. th. for every $v \in W$, ($v \in U$ iff $M, v \models \tau(G)$)

iff there is a $U \in N(w)$ s. th. for every $v \in W$

- if $v \in R_2(U)$ then $M, v \models \tau(G)$ (by the definition of R_2), and
- if $v \in R_3(U)$ then $M, v \models \neg\tau(G)$) (by the definition of R_3)

iff there is $U \in R_1(w)$ such that $M, U \models [2]\tau(G) \wedge [3]\neg\tau(G)$

iff $M, w \models \tau(F)$.

■

LEMMA 26 *(Chellas [2], p. 261) If $\models_{et} F$ then $\vdash_{ET} F$.*

The next lemma is proven syntactically, because we can state it thus in a general way for extensions of E by any axiom $\mathcal{A}$.

LEMMA 27 *Let $\mathcal{A}$ be any axiom schema. If $\vdash_{E+\mathcal{A}} F$ then $\vdash_{K[1,2,3]+\tau(\mathcal{A})} \tau(F)$.*

Proof. We use induction on the proof of F: Whenever the axiom $\mathcal{A}$ is used, we can replace it by $\tau(\mathcal{A})$. The other axioms are classical. Whenever the non-normal inference rule RE:

$$\frac{F \leftrightarrow G}{\Box F \leftrightarrow \Box G}$$

is used, we can replace it by

$$\frac{\tau(F) \leftrightarrow \tau(G)}{\langle 1\rangle([2]\tau(F) \wedge [3]\neg\tau(F)) \leftrightarrow \langle 1\rangle([2]\tau(G) \wedge [3]\neg\tau(G))}$$

which is a derived inference rule of $K[1,2,3]$ (by substitution of equivalences). ■

LEMMA 28 *(Catach [1])* $\vdash_{K[1,2,3]+B_{1,2}} F$ *implies* $\models_{k[1,2,3]+b_{1,2}} F$.

THEOREM 29 $\models_e F$ *iff* $\vdash_{K[1,2,3]} \tau(F)$.

Proof. The proof follows from Lemmas 25 and 27, using the completeness of E (Lemma 26). ■

THEOREM 30 $\models_{et} F$ *iff* $\vdash_{K[1,2,3]+B_{1,2}} \tau(F)$.

Proof. From the left to the right, the proof follows from Lemmas 27 and 26.

From the right to the left, the proof is semantical: Given a minimal model, we must warrant that the normal model that we construct in Lemma 25 is a model for
$\tau(T) = (\langle 1\rangle([2]\tau(F) \wedge [3]\neg\tau(F))) \rightarrow F$.
Now the translated model of Lemma 25 already satisfies the required property. [8] Then we take advantage of the soundness of the normal modal logic $K[1,2,3] + B_{1,2}$.

■

5 CONCLUSION

We have given a translation from classical modal logics into normal modal logics. For particular classes of non-normal modal logics we have given specialised translations. Precisely, we have proved the exactness of the translation for the following logics:

- the basic classical modal logic E,
- the basic monotonic modal logic EM,
- the basic regular modal logic EMC,
- Humberstone's inaccessible worlds logic.

We have also given exactness proofs for the extension of E with axiom T and for extensions of EM with axioms

$D: \Box F \rightarrow \Diamond F$

[8]Note that this is not always the case (a simple example being the axiom $\Diamond\top$). This makes it difficult to prove the exactness of other extension of E e.g. by axioms 5 or B.

$T: \Box F \rightarrow F$

$4: \Box F \rightarrow \Box\Box F$

Moreover we have proved exacteness of the translation for any extension of the basic regular modal logic EMC.

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