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Modal Tableaux with Propagation Rules and Structural Rules

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Abstract. In this paper we generalize the existing tableau methods for modal logics.

First of all, while usual modal tableaux are based on trees, our basic structures are rooted directed acyclic graphs (RDAG). This allows natural tableau rules for some modal logics that are difficult to capture in the usual way (such as those having an accessibility relation that is dense or confluent). Second, tableau rules rewrite patterns, which are (schemas of) parts of a RDAG. A particular case of these rules are the single-step rules recently proposed by Massacci. This allows in particular tableau rule presentations for K5, KD5, K45, KD45, and S5 that respect the subformula property. Third, we divide modal tableau rules into propagation rules and structural rules. Structural rules construct new edges and nodes (without adding formulas to nodes), while propagation rules add formulas to nodes. This distinction allows to prove completeness in a modular way. -

Keywords: Tableaux, Modal logic, Structural Rules, Propagation Rules -

1. Introduction

Following Kripke, tableau rules should be designed in order to propagate formulas in a tree *so that* it *simulates* the properties of a Kripke model, which is not simply a tree, but has additional features. E.g., a tree with the S4 rule "if $\Box A$ is present in some node then transport it into all successors" should behave *as if* the tree where transitive.

In the standard approach the propagation of formulas is only top-down; moreover, using only trees as underlying structures is too restrictive: it is difficult to design tableaux methods for some logics like those based on a density axiom ($\Diamond p \to \Diamond\Diamond p$) or on a confluence axiom ($\Diamond\Box p \to \Box\Diamond p$); this seems to indicate that trees are not a good basis for such properties.

We present here a new basis that is characterized by two ideas:

1. the propagation of formulas need not to be top-down,
2. the underlying structure need not to be a tree.

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About point 1: recently, some authors have used the first idea ([De Giacomo, Massacci 1996], [Massacci 1994]). Their approach is intimely mixed with another feature: the so called single-step rules. Such rules allow to propagate formulas only from one node to one of its successors or predecessor in the tree. We claim that this desideratum is unnecessarily restrictive. Rather, we define here "pattern-driven rules" (even though they are very often single-step rules): rules apply if some elementary pattern in the mathematical structure has been matched.

The second point, to our knowledge, has never been identified before. We will show that while trees are a good basis for many usual modal logics, they fail to support confluent relations for example. We argue in this paper that rooted directed acyclic graphs (RDAG for short), which are DAGs having a distinguished node called the root, are better suited. They allow to naturally handle some properties that do not marry easily with tree structures (like confluence, density), while other properties (like transitivity, symmetry, ...) can still be handled by the propagation of formulas.

This leads us to identify two kinds of tableaux rules:

1. propagation rules
2. structural rules.

The former are formulated as "if in some node of such pattern there is such formula, then propagate such formula (the same or another one)", while the latter are "if there is such pattern then add some new node(s) and edge(s)".
They respectively correspond to two different families of axioms (relational properties):

- Propagation rules correspond to axioms T, 4, B and 5 (properties of reflexivity, transitivity, symmetry and euclideanity, respectively);
- Structural rules correspond to axioms D, De and C (respectively properties of seriality, density and confluence).

What do we gain by this new perspective? It holds in a few words: simplicity, naturality and modularity, both in the definition of a tableau calculus for a given system and in its correctness proof. First, for the classical connectives as well as for $\Diamond$, rules and correctness proof are common to all systems. There only remains the case of structural rules, and of propagation rules for $\Box$ that are treated in a really simple, natural and modular way.

Generally speaking, a tableau is a structure (usually a tree, in our case it will be an RDAG) whose nodes are labelled by sets of formulas. The completeness proof of a tableau method is in two main steps: the construction of a model from this structure, and the verification that this model satisfies the formulas of the nodes (the so-called Fundamental Lemma).

The first step is usually done by adding new arrows to the structure, according to the particular property of the accessibility relation of the logic under concern. For example, for the system S4 the accessibility relation is reflexive and transitive. Hence, given a tree (the underlying structure for S4), we must close it under reflexivity and transitivity in order to make an S4-model of it. In other terms, we must characterize when two nodes are related in the resulting closure. Then we can say that for a given node x another node y will be accessible from x if there is an $n \geq 0$ and $x_0, \ldots, x_i, x_{i+1}, \ldots, x_n$ such that x_0 is x, x_n is y and x_{i+1} is a child of x_i in the original tree. From this characterization of the closure of the initial tree under the additional properties of the logic under concern, we can "read off" the rules to be designed. Thus the rules will ensure the correct propagation of formulas, the proof being very easy[1]. This gives naturality and simplicity. In addition, the rules that we have obtained fit closely to the intuition.
Modularity is achieved since we obtain tableaux calculi whose completeness proofs are neatly separated into three components:

[1] The correctness proof mainly consists in results of relational calculus.

1. a first relation calculus part stated in the Relational Closure Lemma (lemma 4.1) where the properties of the closure of an RDAG under some relational properties are expressed in terms of the initial RDAG,

2. a second relation calculus part stated in the Structural Lemma (lemma 4.3) where we check that the closure of an RDAG under some relational properties preserves some of its initial features (e.g. the transitive closure of a confluent RDAG yields a confluent relation, but not necessarily an RDAG),

3. a "Box" part stated in the Box Lemma (lemma 7.1) where we check that whenever x and y are related in the closure, and $\Box A \in x$ then the set of associated rules ensures that A was transported into y.

The rest of the completeness proof is completely factorized. We also present a soundness result for the tableaux calculi we define.
Last, but not least, all our tableaux calculi verify the subformula property: only subformulas of the initial formula are propagated. Thus the usual argument of finiteness can be applied and provides a decidability result.

We assume that the reader is familiar with modal logic, Kripke semantics and tableau methods for modal logics as presented e.g. in [de Swart 1980], [Fitting 1983], [Fitting 1993].

2. Modal logics and relational properties

A modal logic can be specified syntactically or semantically. We recall what the links between these presentations are.

The modal logics we consider are all obtained by extending the basic modal logic K by one or several of the well-known axioms T, B, 4, 5, D, De (axiom of density: $\Diamond p \rightarrow \Diamond\Diamond p$) and C (axiom of confluence: $\Diamond\Box p \rightarrow \Box\Diamond p$). Thus KDC4 denotes the modal logic obtained by adding the axioms D, C and 4 to the basic system K.

With each of these axioms can be associated a relational property of the accessibility relation of the Kripke models:

Axiom	Property	Notation
T $= \Box p \rightarrow p$	reflexivity	*Ref*
4 $= \Box p \rightarrow \Box\Box p$	transitivity	*Tr*
B $= \Diamond\Box p \rightarrow p$	symmetry	*Sym*
5 $= \Diamond\Box p \rightarrow \Box p$	euclideanity	*Eucl*

Group 1: Properties handled by propagation rules

Axiom	Property	Notation
D $= \Box p \rightarrow \Diamond p$	seriality	*Ser*
De $= \Diamond p \rightarrow \Diamond\Diamond p$	density	*Dens*
C $= \Diamond\Box p \rightarrow \Box\Diamond p$	confluence	*Conf*

Group 2: Properties handled by structural rules

As a consequence of Sahlqvist's theorem [Sahlqvist 1975], a system based on K plus any combination of these axioms is characterized by the Kripke models whose accessibility relation satisfies the corresponding properties. Thus, KD4 is characterized by Kripke models where the accessibility relation is both serial and transitive; for KT5 reflexivity and euclideanity are required (and, as a consequence, transitivity, seriality and symmetry).

From now on we will indistinctly denote a modal system by $\mathrm{KA}_1 \ldots \mathrm{A}_n$, where each A_i belongs to group 1 or 2, or by a set ρ of its accessibility relation properties; we will write $\rho = \rho_1 \cup \rho_2$ where ρ_1 is a maximal subset of properties of group 1 (maximal here means "including all those of group 1 which are a consequence of it": thus, symmetry and transitivity imply euclideanity: any set ρ_1 that contain *Sym* and *Tr* must also contains *Eucl*), and ρ_2 is a subset of properties of group 2. E.g. KCD4 will be denoted by $\{Ser, Tr, Conf\}$, KDeB4 by $\{Sym, Tr, Eucl, Dens\}$ (since euclideanity is a consequence of transitivity and symmetry).

Definition 1. Given a set ρ of relational properties among group 1 and 2, a ρ-model is a Kripke model whose accessibility relation satisfies ρ. A formula is ρ-satisfiable iff it is satisfiable in a ρ-model. It is ρ-valid iff it is valid in the class of all ρ-models, this will be denoted $\models_\rho A$. Thus A is a theorem of a system denoted by a set ρ of properties iff it is ρ-valid.

We note that Relational calculus has been used as a base for proof procedures in non-classical logis in a comprehensive way in [Orlowska 1997] and [Demri, Orlowska].

3. Preliminaries and notations

The tableau method we are going to present is based on RDAG (rooted directed acyclic graphs) having additional properties; let ρ be the set of these additional properties, we define:

Definition 2. A labelled ρ-RDAG is a triple $(\mathcal{N}, \Sigma, \text{FOR})$ where:

- $(\mathcal{N}, \Sigma)$ is a directed acyclic graph (DAG), i.e. a directed graph that contains no cycle, with a distinguished node called the *root* that can access every other node in the transitive closure of Σ,
- $(\mathcal{N}, \Sigma)$ satisfies all the properties of ρ,
- FOR is a function that associates additional information with each of the nodes: if x is a node, $\text{FOR}(x)$ is a set of formulas.

By abuse of notation and for the sake of notational economy, we will make no distinction between the nodes and their associated sets of formulas; thus we will write $A \in x$ instead of $A \in \text{FOR}(x)$. Also by abuse of notation, we will sometimes denote a ρ-RDAG $(\mathcal{N}, \Sigma)$ by the binary relation Σ. Thus we will make no distinction between labelled structures and structures.

This notion also extend to graphs:

Definition 3. An RGRAPH is a graph that has a root, and a ρ-RGRAPH is a RGRAPH that satisfies all properties of ρ.

As usual, $\Sigma(x)$ will denote the set of nodes accessible from x by Σ: $\Sigma(x) = \{y \in \mathcal{N} : (x, y) \in \Sigma\}$. Also, Σ^n will denote the pairs (x, y) such that there is a path of length n between x and y. The diagonal relation: $\{(x, x) : x \in \mathcal{N}\}$ will be denoted by I and also by Σ^0.

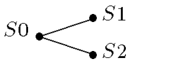
For the sake of clarity, we will use diagrammatic representation for RDAG. The figure below gives the intended meaning of those diagrammatic representations in which the edges are implicitely left-to-right directed[2]:

$\bullet^{S}$ denotes a node S

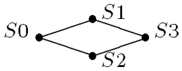
$S0 \bullet\!\!-\!\!-\!\!\bullet S1$ denotes $(S0, S1) \in \Sigma$

[2]Note that RDAG are of course antisymmetrical.

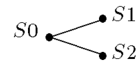
denotes $(S0, S1), (S0, S2), (S1, S2) \in \Sigma$

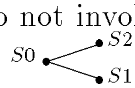


denotes $(S0, S1), (S0, S2) \in \Sigma$



denotes $(S0, S1), (S0, S2), (S1, S3), (S2, S3) \in \Sigma$

The last two diagrams do not involve any order between $S1$ and $S2$, e.g. 

can be represented as well by 

4. Closure of RGRAPH

We define the following closure operation on RGRAPH:

Definition 4. Let Σ be an RGRAPH over a set $\mathcal{N}$ and ρ a set of relational properties of group 1; the ρ-closure of Σ (denoted by Σ^{ρ}) is the least RGRAPH that contains Σ and which satisfies every property of ρ.

This ρ-closure always exists if the properties are among $\{Ref, Tr, Sym, Eucl\}$. A very important point is that for properties of group 1, the closure can be expressed in terms of the initial RGRAPH. E.g. the transitive closure of an RGRAPH Σ is defined by: $(x,y) \in \Sigma^{Tr}$ iff $\exists n \geq 1$ such that $(x,y) \in \Sigma^n$ (c.f. def 9). Note that we do not consider here properties of group 2: it makes no sense to talk about closure under a property of group 2. This is the reason why they are handled in a different way: no propagation rule can simulate them.

Lemma 4.1. (Relational Closure Lemma)
Let Σ be an RDAG over a set $\mathcal{N}$ of nodes:

- $(x,y) \in \Sigma^{Ref}$ iff $(x,y) \in \Sigma$ or $x = y$.
- $(x,y) \in \Sigma^{Sym}$ iff $(x,y) \in \Sigma$ or $(y,x) \in \Sigma$.
- $(x,y) \in \Sigma^{Tr}$ iff $\exists n \geq 1$ such that $(x,y) \in \Sigma^n$.
- $(x,y) \in \Sigma^{Eucl}$ iff $(x,y) \in \Sigma$ or $\exists u \in \mathcal{N}\ \exists n \geq 1\ \exists m \geq 1$ such that $(u,x) \in \Sigma^n$ and $(u,y) \in \Sigma^m$.
- $(x,y) \in \Sigma^{Ref,Sym}$ iff $(x,y) \in \Sigma$ or $x = y$ or $(y,x) \in \Sigma$.
- $(x,y) \in \Sigma^{Ref,Tr}$ iff $\exists n \geq 0$ such that $(x,y) \in \Sigma^n$.
- $(x,y) \in \Sigma^{Ref,Eucl}$ iff $\exists n \geq 0\ \exists x_0 = x, \ldots x_i, x_{i+1}, \ldots, x_n = y : (x_i, x_{i+1}) \in \Sigma$ or $(x_{i+1}, x_i) \in \Sigma$.
- $(x,y) \in \Sigma^{Sym,Tr}$ iff $\exists n \geq 1\ \exists x_0 = x, \ldots x_i, x_{i+1}, \ldots, x_n : (x_i, x_{i+1}) \in \Sigma$ or $(x_{i+1}, x_i) \in \Sigma$.
- $(x,y) \in \Sigma^{Tr,Eucl}$ iff $\exists u \in \mathcal{N}\ \exists n \geq 0\ \exists m \geq 1$ such that $(u,x) \in \Sigma^n$ and $(u,y) \in \Sigma^m$.

Proof:
Straightforward consequence of the lemmas 9.1 and 9.3 of the appendix.

Lemma 4.2. The remaining cases are reducible to those of the previous lemma:

- $\Sigma^{Sym,Eucl} = \Sigma^{Sym,Tr,Eucl} = \Sigma^{Sym,Tr}$

- $\Sigma^{Ref,Sym,Tr} = \Sigma^{Ref,Tr,Eucl} = \Sigma^{Ref,Sym,Eucl} = \Sigma^{Ref,Sym,Tr,Eucl} = \Sigma^{Ref,Eucl}$.

Proof:
Straightforward.

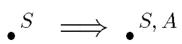
The above lemma will be a powerful tool for proving completeness: it will allow to define a model for a formula from an open tableau. But this is not the whole story. As we previously said, some properties are handled structurally; roughly speaking seriality, density and confluence are treated by the underlying "kind" of RDAG of the tableaux. When in the completeness proof we must close the RDAG under one or several properties of group 1 (note that after this closure operation, the initial RDAG is no longer an RDAG but an RGRAPH), we must also check that its structural properties are preserved after this closure (i.e. that it is still of the same "kind"). E.g. we must prove that the transitive closure of a confluent RDAG is still confluent. This is the aim of the lemma below:

Lemma 4.3. (Structural Lemma) Let ρ_2 be a subset of group 2, ρ_1 a subset of group 1 and let Σ be a ρ_2-RGRAPH over a set $\mathcal{N}$ of nodes. Then Σ^{ρ_1} is also a ρ_2-RGRAPH and hence is a $(\rho_1 \cup \rho_2)$-RGRAPH.

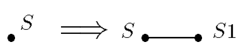
Proof:
See appendix II.

5. Rewriting RDAG

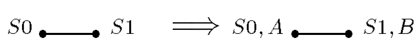
Usually, tableaux calculi consist in rewriting a structure by using some appropriate set of rewriting rules (or simply rules). But before presenting our rules, we want to propose some visual conventions. The rules we will use will all be of one of the following forms (the intended meaning is given below the rule); as usual, S, A denotes $S \cup \{A\}$:



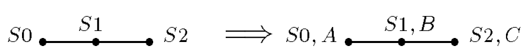
rewrite the node S into the node $S \cup \{A\}$, i.e. add the formula A to the node S,



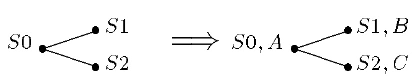
add the new node $S1$ to the successors of the node S,



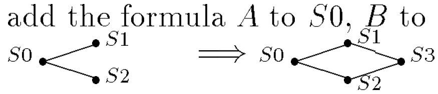
add the formula A to the node $S0$ and B to $S1$,



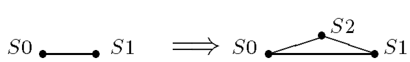
add the formula A to $S0$, B to $S1$ and C to $S2$,



add the formula A to $S0$, B to $S1$ and C to $S2$.

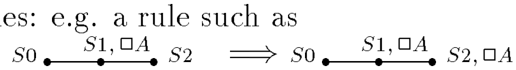


add the new node $S3$ as a common successor of the node $S1$ and $S2$,



add the new node $S2$ between $S0$ and $S1$.

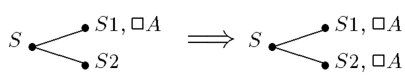
This presentation allows to implicitly take into account constraints on the applicability of rules: e.g. a rule such as

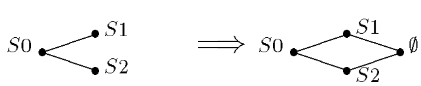


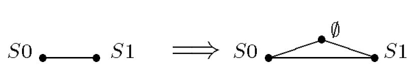
reads "add $\Box A$ to any successor of $S1$ if $S1$ has a predecessor and contains $\Box A$".

6. Rules

Here are the rules we need:

- Classical and $\Diamond$ rules:
  - Rule $\bot$: $\bullet^{A,\neg A,S} \Longrightarrow \bullet^{A,\neg A,\bot,S}$
  - Rule $\neg$: $\bullet^{\neg\neg A,S} \Longrightarrow \bullet^{\neg\neg A,A,S}$
  - Rule $\wedge$: $\bullet^{A\wedge B,S} \Longrightarrow \bullet^{A\wedge B,A,B,S}$
  - Rule $\vee$: $\bullet^{\neg(A\wedge B),S} \Longrightarrow \bullet^{\neg(A\wedge B),C,S}$
    where C is one among $\neg A$ and $\neg B$
  - Rule $\Diamond$: $\bullet^{\Diamond A,S} \Longrightarrow \Diamond A, S \bullet\!\!-\!\!\!-\!\!\bullet\ A$
- Propagation rules:
  - Rule K: $\Box A, S \bullet\!\!-\!\!\!-\!\!\bullet\ S1 \Longrightarrow \Box A, S \bullet\!\!-\!\!\!-\!\!\bullet\ A, S1$
  - Rule T: $\bullet^{\Box A,S} \Longrightarrow \bullet^{\Box A,A,S}$
  - Rule 4: $S, \Box A \bullet\!\!-\!\!\!-\!\!\bullet\ S1 \Longrightarrow S, \Box A \bullet\!\!-\!\!\!-\!\!\bullet\ S1, \Box A$
  - Rule B: $S \bullet\!\!-\!\!\!-\!\!\bullet\ S1, \Box A \Longrightarrow S, A \bullet\!\!-\!\!\!-\!\!\bullet\ S1, \Box A$
  - Rule $5_{\rightarrow}$: 

  - Rule $5_{\uparrow}$: $S \bullet\!\!-\!\!\!-\!\!\bullet\ S1, \Box A \Longrightarrow S, \Box A \bullet\!\!-\!\!\!-\!\!\bullet\ S1, \Box A$
  - Rule $5_{\downarrow}$: $S \bullet\!\!-\!\!\!-\!\!\overset{S1,\Box A}{\bullet}\!\!-\!\!\!-\!\!\bullet\ S2 \Longrightarrow S \bullet\!\!-\!\!\!-\!\!\overset{S1,\Box A}{\bullet}\!\!-\!\!\!-\!\!\bullet\ S2, \Box A$
- Structural rules:
  - Rule D: $\bullet^{S} \Longrightarrow S \bullet\!\!-\!\!\!-\!\!\bullet\ \emptyset$
  - Rule $C0$: 

  - Rule $C1$: $S \bullet\!\!-\!\!\!-\!\!\bullet\ S1 \Longrightarrow S \bullet\!\!-\!\!\!-\!\!\overset{S1}{\bullet}\!\!-\!\!\!-\!\!\bullet\ \emptyset$
  - Rule De: 

In order to define a tableau calculus for a system denoted by $\rho_1 \cup \rho_2$, we must associate a set of rules with it. All the tableaux calculi we are going to define contain: the classical rules and the rule $\Diamond$ plus the rule K (as these rules are common to all tableaux calculi, we will henceforth omit them) plus none or some structural and propagation rules.

A tableau calculus for a system denoted by $(\rho_1 \cup \rho_2)$ is obtained by taking (in addition to classical, $\Diamond$ and K rules) the rules corresponding to properties of $(\rho_1 \cup \rho_2)$; this correspondance is given in the figure below.

	Properties	Rules	
Group 1	*Ref*	T	Propagation
	Sym	B	Rules
	Tr	4	
	Eucl	$5_\uparrow$ $5_\downarrow$ $5_\rightarrow$	
Group 2	*Ser*	D	Structural
	Dens	De	Rules
	Conf	C0 C1	

Definition 5. A $(\rho_1 \cup \rho_2)$-tableau for a formula A is the limit of a sequence $\Upsilon_0, \ldots, \Upsilon_i, \Upsilon_{i+1}, \ldots$ where:

- Υ_0 is an RDAG consisting of only one node whose associated set of formulas is $\{A\}$,
- Υ_{i+1} is obtained from Υ_i by applying either a classical rule, or the $\Diamond$ rule, or the rule K, or a rule of $(\rho_1 \cup \rho_2)$
- and in which every applicable rule has been applied.

Definition 6. A tableau is closed if some node in it contains $\perp$; it is open otherwise. A formula is $\rho_1 \cup \rho_2$-closed iff all its $(\rho_1 \cup \rho_2)$-tableaux are closed [3].

7. Completeness

In this section we prove the completeness of our tableaux calculi[4]. We show how, from a given open $(\rho_1 \cup \rho_2)$-tableau for A we can construct a $(\rho_1 \cup \rho_2)$-model for A.

Let Υ be an open $(\rho_1 \cup \rho_2)$-tableau for A. Υ is a ρ_2-RDAG where $\Upsilon = (\mathcal{N}, \Sigma, \text{FOR})$ with root r, since structural rules corresponding to ρ_2 ensure that Υ satisfies ρ_2.

Now let $\mu = (W, R, \tau)$ be the Kripke model defined as follows:

Definition 7.

- $W = \mathcal{N}$
- R is the ρ_1-closure of Σ, i.e. $R = \Sigma^{\rho_1}$
- for all $w \in W$, $w \in \tau(p)$ iff $p \in w$ (in fact iff $p \in For(w)$).

By construction, μ satisfies properties of ρ_1 and, by the Structural Lemma (lemma 4.3), it also satisfies the properties of ρ_2; hence it is a $(\rho_1 \cup \rho_2)$-model. What remains is to prove that it satisfies the formula A. We first establish the following important lemma:

[3] Due to the rule $\vee$, a formula may have several distinct tableaux.
[4] We make the usual assumption of *fairness*.

Lemma 7.1. (Box Lemma) Let $\Upsilon = (\mathcal{N}, \Sigma, \text{FOR})$ be a $(\rho_1 \cup \rho_2)$-tableau with root r. Let x, y be such that $(x, y) \in \Sigma^{\rho_1}$ and $\Box A \in x$; then $A \in y$.

Proof:
There are nine cases, according to ρ_1; we only prove the lemma for some of the most complex cases (all involving euclideanity):

- $\rho_1 = \{Eucl\}$: if $(x, y) \in \Sigma^{\rho_1}$ then by the Relational Closure lemma, we have either $(x, y) \in \Sigma$ and then $A \in y$ (by rule K), or $\exists u\ \exists n \geq 1\ \exists m \geq 1$ such that $(u, x) \in \Sigma^n$ and $(u, y) \in \Sigma^m$ from. Hence
  - $\exists x_0 = x, \ldots, x_i, x_{i+1}, \ldots, x_n = u$: $(x_{i+1}, x_i) \in \Sigma$; then $\Box A \in x_i$ for $0 \leq i \leq n$ (by rule $5_{\uparrow}$ n times), in particular: $\Box A \in x_{n\perp 1}$ and $\Box A \in x_n = u$.
  - $\exists y_0 = u, \ldots, y_i, y_{i+1}, \ldots, y_m = y$: $(y_i, y_{i+1}) \in \Sigma$; hence $\Box A \in y_1$ (by rule $5_{\rightarrow}$ since $\Box A \in x_{n\perp 1}$) from which we get $\Box A \in y_i$ for $1 \leq i \leq m$ (by rule $5_{\downarrow}$ $m \perp 1$ times) and since $\Box A \in x_n = u = y_0$ it comes: $\Box A \in y_i$ for $0 \leq i \leq m$. Hence $A \in y_i$ for $1 \leq i \leq m$ (by rule K), in particular $A \in y$.
- $\rho_1 = \{Tr, Eucl\}$: if $(x, y) \in \Sigma^{\rho_1}$ then by the Relational Closure lemma, we have $\exists u \in \mathcal{N}\ \exists n \geq 0\ \exists m \geq 1$ such that $(u, x) \in \Sigma^n$ and $(u, y) \in \Sigma^m$. This implies that:
  - $\exists n \geq 0\ \exists x_0 = x, \ldots, x_i, x_{i+1}, \ldots, x_n = u$: $(x_{i+1}, x_i) \in \Sigma$; then $\Box A \in x_0$ implies $\Box A \in u$ (by rule $5_{\uparrow}$, n times)
  - $\exists m \geq 0\ \exists y_0 = u, \ldots, y_i, y_{i+1}, \ldots, y_{m+1} = y$: $(x_i, x_{i+1}) \in \Sigma$; hence $\Box A \in u$ implies $\Box A \in y_m$ (by rule 4, m times) and $A \in y$ (by rule K).
- $\rho_1 = \{Sym, Tr, Eucl\}$: if $(x, y) \in \Sigma^{\rho_1}$ then by the Relational Closure lemma, we have $\exists n \geq 1\ \exists x_0 = x, \ldots, x_i, x_{i+1}, \ldots, x_n = y$: $(x_i, x_{i+1}) \in \Sigma$ or $(x_{i+1}, x_i) \in \Sigma$; but $\Box A \in x_0$ and $\Box A \in x_i \Rightarrow \Box A \in x_{i+1}$ (by rule 4 or $5_{\uparrow}$, according to whether $(x_i, x_{i+1}) \in \Sigma$ or $(x_{i+1}, x_i) \in \Sigma$). Thus $\Box A \in x_i$ for $0 \leq i \leq n$ and hence $A \in x_i$ for $0 \leq i \leq n+1$ (by rule K or B). Thus $A \in y$.

The following fundamental lemma brings us to the desired conclusion:

Lemma 7.2. (Fundamental Lemma) Let Υ be an open $(\rho_1 \cup \rho_2)$-tableau for A, let μ be the $(\rho_1 \cup \rho_2)$-model defined as in definition 7 w.r.t. Υ and let $B \in Subformulas(A)$ then: (i) if $B \in x$ then $\mu, x \models B$.

Proof:
(By induction on the structure of B: W.l.o.g we can suppose that B is written with only $\neg$, $\wedge$, $\perp$ and $\Box$).

Induction initialization: let B be an atom; then (i) holds by definition of τ.

Induction step[5]:

- B cannot be $\perp$, otherwise x would be closed.
- Let B be $\neg\neg C$.
  $\neg\neg C \in x$
  $\Rightarrow C \in x$ (by rule $\neg$)
  $\Rightarrow \mu, x \models C$ (by IH)
  $\Rightarrow \mu, x \models \neg\neg C$.

[5]In this proof, when we say "by rule R" we mean "by rule R and by the fairness assumption that rule R has been applied".

- Let B be $(C \wedge D)$.
  $(C \wedge D) \in x$
  $\Rightarrow C \in x$ and $D \in x$ (by rule $\wedge$)
  $\Rightarrow \mu, x \models C$ and $\mu, x \models D$ (by IH)
  $\Rightarrow \mu, x \models (C \wedge D)$.

- Let B be $\neg(C \wedge D)$.
  $\neg(C \wedge D) \in x$
  $\Rightarrow \neg C \in x$ or $\neg D \in x$ (by rule $\vee$)
  $\Rightarrow \mu, x \models \neg C$ or $\mu, x \models \neg D$ (by IH)
  $\Rightarrow \mu, x \models \neg(C \wedge D)$.

- Let B be $\neg\Box C$
  $\neg\Box C \in x$
  $\Rightarrow$ there exists y such that $(x, y) \in \Sigma$ and $\neg C \in y$ (by rule $\Diamond$)
  $\Rightarrow$ there exists y such that $(x, y) \in R$, and $\mu, y \models \neg C$ (by IH and definition of R)
  $\Rightarrow \mu, x \models \neg\Box C$.

- Let B be $\Box C$ and suppose $(x, y) \in R$; then by the Box Lemma (7.1), $C \in y$. Then by IH, it comes $\mu, y \models C$. Hence, $\mu, x \models \Box C$.

As a direct consequence of the previous lemma, we have:

Corollary 7.1. If A has a fair open $(\rho_1 \cup \rho_2)$-tableau then A is $(\rho_1 \cup \rho_2)$-satisfiable. Hence our tableaux calculi are complete under the fairness assumption.

8. Soundness

In this section, we prove the soundness of our tableaux calculi: if a formula A is $(\rho_1 \cup \rho_2)$-closed then A is $(\rho_1 \cup \rho_2)$-unsatisfiable. The technique we use for proving the soundness of our tableaux is simple. We prove that all rules preserve the "satisfiability" of the pattern involved in its application. In our sense, a pattern is $(\rho_1 \cup \rho_2)$-satisfiable iff there exists a $(\rho_1 \cup \rho_2)$-model that contains it and satisfies its formulas. We formally develop this below.

Definition 8. Let $\Upsilon = (\mathcal{N}, \Sigma, \text{FOR})$ be a labelled $(\rho_1 \cup \rho_2)$-RGRAPH and $\mu = (W, R, \tau)$ be a $(\rho_1 \cup \rho_2)$-model; let h be a function such that $h(\mathcal{N}) \subseteq W$ and $\forall n_1, n_2 \in \mathcal{N} \colon (n_1, n_2) \in \Sigma \Rightarrow (h(n_1), h(n_2)) \in R$.

- h is called an *embedding* from Υ to μ (or h matches Υ to μ);

- μ satisfies Υ via h iff $\forall n \in \mathcal{N} \colon A \in \text{FOR}(n) \Rightarrow \mu, h(n) \models A$;

- μ satisfies Υ iff there exists an embedding h from Υ to μ such that μ satisfies Υ via h.

Lemma 8.1. Let $\Upsilon \Longrightarrow \Upsilon'$ be a rule of some set ρ (resp. $\Upsilon \Longrightarrow \Upsilon'$ or Υ'' for rule $\vee$); then if some ρ-model μ satisfies Υ then it satisfies Υ' (resp. then it satisfies Υ' or Υ'').

Proof:
If we suppose that μ satisfies Υ via some embedding h we just have to exhibit an embedding h' such that μ satisfies Υ' via h' (resp. such that μ satisfies Υ' or Υ'' via h'). This is done by analysing every rule. We only do it for the $\Diamond$ rule, for one structural rule and for one propagation rule. For classical rules, it is immediate: just take $h' = h$.

- Rule $\Diamond$: $\Upsilon = (\mathcal{N} = \{n_0\}, \Sigma = \emptyset, \text{FOR} = \{(n_0, \Diamond A)\})$ rewrites into $\Upsilon' = (\mathcal{N} \cup \{n_1\}, \Sigma \cup \{(n_0, n_1)\}, \text{FOR} \cup \{(n_1, A)\})$.
  If μ satisfies Υ via h then $\mu, h(n_0) \models \Diamond A$, hence $\exists y \in R(h(n_0)){:}\, \mu, y \models A$; let y_1 be such a y, and define $h'(n_1) = y_1$ and $h'(n_0) = h(n_0)$. μ satisfies Υ' via h', since $(h'(n_0), h'(n_1)) \in R$ and $\mu, h'(n_1) \models A$.

- Rule De: $\Upsilon = (\mathcal{N} = \{n_0, n_1\}, \Sigma = \{(n_0, n_1)\}, \text{FOR} = \{(n_0, S_0), (n_1, S_1)\})$ rewrites into $\Upsilon' = (\mathcal{N} \cup \{n_2\}, \Sigma \cup \{(n_0, n_2), (n_2, n_1)\}, \text{FOR} \cup \{(n_2, \emptyset)\})$.
  If μ satisfies Υ via h then $(h(n_0), h(n_1)) \in R$, and since R is dense $\exists z{:}\, (h(n_0), z) \in R$ and $(z, h(n_1)) \in R$. Let z_2 be such a z and define $h'(n_2) = z_2$ and $h'(n) = h(n)$ for $n \neq n_2$. μ satisfies Υ' via h', since $(h'(n_0), h'(n_2)) \in R$ and $(h'(n_2), h'(n_1)) \in R$, and $For(n_2) = \emptyset$.

  For propagation rules, we just have to prove that we are done by taking $h' = h$.

- Rule $5_{\rightarrow}$: $\Upsilon = (\mathcal{N} = \{n_0, n_1, n_2\}, \Sigma = \{(n_0, n_1), (n_0, n_2)\}, \text{FOR} = \{(n_0, S_0), (n_1, S_1 \cup \{\Box A\}), (n_2, S_2)\})$ rewrites into $\Upsilon' = (\mathcal{N}, \Sigma, \text{FOR} \cup \{(n_2, \Box A)\})$.
  If μ satisfies Υ via h then $\mu, h(n_1) \models \Box A$;
  Also, since R is euclidean, we have:
  $\mu, h(n_0) \models \Box(\Box A \rightarrow \Box\Box A)$ (valid formula of euclidean models)
  $\Rightarrow \mu, h(n_1) \models \Box A \rightarrow \Box\Box A$ (since $(h(n_0), h(n_1) \in R)$
  $\Rightarrow \mu, h(n_1) \models \Box\Box A$ (since $\mu, h(n_1) \models \Box A$)
  $\Rightarrow \mu, h(n_0) \models \Diamond\Box\Box A$ (since $h(n_0), h(n_1) \in R$)
  $\Rightarrow \mu, h(n_0) \models \Box\Box A$ ($\Diamond\Box\Box A \rightarrow \Box\Box A$ is valid in euclidean models)
  $\Rightarrow \mu, h(n_2) \models \Box A$ (since $(h(n_0), h(n_2)) \in R$).

Corollary 8.1. If A is $(\rho_1 \cup \rho_2)$-satisfiable then it has an open $(\rho_1 \cup \rho_2)$-tableau. Hence our tableaux calculi are sound.

Proof:
If A is $(\rho_1 \cup \rho_2)$-satisfiable by some world x of some $(\rho_1 \cup \rho_2)$-model μ, then its starting labelled RGRAPH: $(\{n_0\}, \emptyset, \{(n_0, A)\})$ is satisfied by μ (via the embedding $h{:}\, n_0 \mapsto x$). Hence, at least one of its $(\rho_1 \cup \rho_2)$-tableaux must be open since no closed tableau is satisfiable by μ.

9. Concluding remarks

Decidability and Termination All rules we use only propagate subformulas of the initial formula. Thus only finitely many distinct nodes can be generated, therefore there is a finite model if the formula is satisfiable. Hence all the logics we have considered are decidable. For termination of the tableau calculi, the usual argument as for S4 (cf. [Fitting 1983]) applies with the help of a loop-test (which consists in blocking the development of nodes already in the RGRAPH), the problem being to efficiently implement it.

Extensions to other properties Our work extends easily to other properties of group 1 (almost-reflexivity: $\forall x(\exists u{:}\, (u, x) \in R \Rightarrow (x, x) \in R)$, almost-transitivity: $\forall x, y, z, u\, ((x, y) \in R \wedge (y, z) \in R \wedge (z, u) \in R) \Rightarrow (y, u) \in R, \ldots)$. First complete the Relational Closure lemma (4.1) and then check that the closure under this new property of a ρ_2-RGRAPH is still a ρ_2-RGRAPH (Structural lemma). Then design one or several rules for this property e.g. for almost-reflexivity, the natural rule such as:
$S0 \bullet\!\!-\!\!\bullet S1, \Box A \implies S0 \bullet\!\!-\!\!\bullet S1, \Box A, A$
(it is obviously sound). Then prove that this/these rule(s) allow(s) to correctly propagate

formulas (Box lemma) with the help of the Relational Closure lemma.
For new properties of group 2 (like 3-density: $(x, y) \in R \Rightarrow \exists u, v\colon (x, u) \in R \wedge (u, v) \in R \wedge (v, y) \in R$), one must first define the underlying structure (here 3-dense RGRAPH) and extend the Structural lemma (if possible). Then designing a corresponding sound structural rule is straightforward, and completeness is for free.

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Appendix I: Some properties about binary relations

From now on, we will make use of *relations* (binary relations) and *rooted relations* instead of graphs and rooted graphs. The set of all relations over a given set will be denoted by $\mathcal{R}$ while that of rooted relations will be denoted by $\mathcal{RR}$.

Definition 9. Let R be a relation over a set $\mathcal{N}$: $R(x)$ will denote the set of nodes accessible from x by R: $R(x) = \{y \in \mathcal{N} : (x,y) \in R\}$, $\overline{R}$ will denote its inverse, R^+ will denote its transitive closure and R^* its transitive and reflexive closure. Also, R^n will denote the pairs (x,y) such that there is a path of length n between x and y. The diagonal relation: $\{(x,x) : x \in \mathcal{N}\}$ will be denoted by I and also by R^0. The composition of two relations R and S (which is defined as $\{(x,y) : \exists z (x,z) \in R$ and $(z,y) \in S\}$ will be denoted by $(R \circ S)$. The total relation $\mathcal{N}^2$ is denoted by $\mathcal{U}$. The empty relation is denoted by $\mathcal{O}$.

Property 1. (About $\mathcal{R}$) Let R, S and $T \in \mathcal{R}$, let ρ be a subset of group 1:

1. $R^+ = \bigcup_{i \geq 1} R^i$
2. $\overline{\overline{R}} = R$
3. $\overline{R \cup S} = \overline{R} \cup \overline{S}$
4. $\overline{R \circ S} = \overline{S} \circ \overline{R}$
5. $\overline{R^n} = \overline{R}^n$ (for $n \geq 0$)
6. $\overline{R^+} = \overline{R}^+$
7. $\overline{R^*} = \overline{R}^*$
8. $(R \cup I)^+ = R^*$
9. $(R \cup S) \circ T = (R \circ T) \cup (S \circ T)$
10. $T \circ (R \cup S) = (T \circ R) \cup (T \circ S)$
11. $I^+ = I^* = \overline{I} = I$
12. If $R \neq \mathcal{O}$ then $R \circ \overline{R} \neq \mathcal{O}$
13. If $R \neq \mathcal{O}$ then $\mathcal{U} \circ R \circ \mathcal{U} = \mathcal{U}$
14. $(R^n)^+ \subseteq (R^+)^n$ (for $n \geq 0$)
15. $R \subseteq R^\rho$ (growth)
16. $R \subseteq S \Rightarrow R^\rho \subseteq S^\rho$ (monotonicity)
17. If $P \in \rho$ then $(R^\rho)^P = R^\rho$ (idempotence); and of course: $(R^\rho)^\rho = R^\rho$
18. R is reflexive iff $I \subseteq R$
19. R is symmetrical iff $\overline{R} \subseteq R$

20. R is transitive iff $R^2 \subseteq R$

21. R is euclidean iff $(\overline{R} \circ R) \subseteq R$, or iff $(\overline{R} \circ R) \subseteq \overline{R}$

22. R is dense iff $R \subseteq R^2$

23. R is serial iff $I \subseteq (R \circ \overline{R})$

24. R is confluent iff $(\overline{R} \circ R) \subseteq (R \circ \overline{R})$

25. R is rooted iff $(\overline{R}^{*} \circ R^{*}) = \mathcal{U}$

26. R is connected iff $(\overline{R} \cup R)^{*} = \mathcal{U}$, and rooted implies connected.

Proof:
All are well-known or obvious properties except maybe 14 for which it suffices to prove that $(R^2)^+ \subseteq (R^+)^2$:
$(R^+)^2 = (\bigcup_{i \geq 1} R^i)^2 = (\bigcup_{i \geq 1} R^i) \circ (\bigcup_{i \geq 1} R^i) = \bigcup_{i \geq 1} \bigcup_{j \geq 1} (R^i \circ R^j)$
$= \bigcup_{i \geq 1} \bigcup_{j \geq 1} (R^{i+j}) = \bigcup_{i \geq 2} (R^i)$,
and, $(R^2)^+ = \bigcup_{i \geq 1} R^{2i} \subseteq \bigcup_{i \geq 2} (R^i)$.

Property 2. (About $\mathcal{RR}$) Let $R \in \mathcal{RR}$:

1. Let ρ be a subset of group 1 then R^{ρ} is also in $\mathcal{RR}$.

2. $(\overline{R}^{+} \circ R^{+} \circ \overline{R}^{+} \circ R^{+}) = (\overline{R}^{+} \circ R^{+})$

3. If $(\overline{R} \circ R) \subseteq (R \circ \overline{R})$ then $(\overline{R}^{+} \circ R^{+}) \subseteq (R^{+} \circ \overline{R}^{+})$

4. $(\overline{R}^{*} \circ R^{+})^{+} = (\overline{R}^{*} \circ R^{+})$

Proof:

1. Trivial since the root r of R is still a root in R^{ρ}.

2. If $R = \mathcal{O}$ then 2 holds trivially, else we have:
   $(\overline{R}^{+} \circ R^{+} \circ \overline{R}^{+} \circ R^{+}) = (\overline{R} \circ \overline{R}^{*} \circ R^{*} \circ R \circ \overline{R} \circ \overline{R}^{*} \circ R^{*} \circ R) = (\overline{R} \circ \mathcal{U} \circ R \circ \overline{R} \circ \mathcal{U} \circ R) = (\overline{R} \circ \mathcal{U} \circ R)$; (since $R \neq \mathcal{O} \Rightarrow R \circ \overline{R} \neq \mathcal{O}$) $= (\overline{R}^{+} \circ R^{+})$

3. We show that $(\overline{R} \circ R) \subseteq (R \circ \overline{R}) \Rightarrow \forall k, l \geq 1 \colon (\overline{R}^{k} \circ R^{l}) \subseteq (R^{l} \circ \overline{R}^{k})$ by induction on $k + l$.
   Induction base:
   if $k + l = 2$, the property hold by hypothesis.
   Induction step:
   if $k > 1$ then $(\overline{R}^{k} \circ R^{l}) = (\overline{R} \circ \overline{R}^{k-1} \circ R^{l}) \subseteq (\overline{R} \circ R^{l} \circ \overline{R}^{k-1})$(by IH) $\subseteq (R^{l} \circ \overline{R} \circ \overline{R}^{k-1})$(by IH) $\subseteq (R^{l} \circ \overline{R}^{k})$.
   else if $k = 1$ and $l > 1$ then
   $(\overline{R}^{k} \circ R^{l}) = (\overline{R}^{k} \circ R \circ R^{l-1}) \subseteq (R \circ \overline{R}^{k} \circ R^{l-1})$(by IH) $\subseteq (R \circ R^{l-1} \circ \overline{R}^{k})$(by IH) $\subseteq (R^{l} \circ \overline{R}^{k})$.

4. It suffices to show that $(\overline{R}^{*} \circ R^{+})^2 = (\overline{R}^{*} \circ R^{+})$:
   If $R = \mathcal{O}$ then it holds trivially, else we have: $(\overline{R}^{*} \circ R^{+})^2 = (\overline{R}^{*} \circ R^{*} \circ R)^2 = (\mathcal{U} \circ R)^2 = (\mathcal{U} \circ R \circ \mathcal{U} \circ R) = (\mathcal{U} \circ R) = (\overline{R}^{*} \circ R^{+})$.

Appendix II: Properties of closure operations

Lemma 4.1 (Relational Closure Lemma)
This lemma stated at page 5 is a straightforward consequence of the lemmas 9.1 and 9.3 below.

Lemma 9.1. (Closure under one property) Let $R \in \mathcal{RR}$:

1. $R^{Ref} = R \cup I$

2. $R^{Sym} = R \cup \overline{R}$

3. $R^{Tr} = R^+$

4. $R^{Eucl} = R \cup (\overline{R}^+ \circ R^+)$

Proof:
Only 4. is not obvious and well-known (it uses the fact that R has a root). We prove it by showing:

i) $R \cup (\overline{R}^+ \circ R^+) \subseteq R^{Eucl}$

ii) $R \cup (\overline{R}^+ \circ R^+)$ is euclidean

and we will get the conclusion since R^{Eucl} is the least superset of R being euclidean and, as such, it contains any other euclidean superset of R.

i) First we prove by induction on $i + j$ that $\forall i, j\colon (\overline{R}^i \circ R^j) \subseteq (\overline{R^{Eucl}} \circ R^{Eucl})$.
Induction base:
$i + j = 2$, i.e. $i = j = 1$: $(\overline{R} \circ R) \subseteq (\overline{R^{Eucl}} \circ R^{Eucl})$ (since $R \subseteq R^{Eucl}$ and hence $\overline{R} \subseteq \overline{R^{Eucl}}$).
Induction step:
if $j > 1$
then $(\overline{R}^i \circ R^j) = (\overline{R}^i \circ R^{j\perp 1} \circ R) \subseteq (\overline{R^{Eucl}} \circ R^{Eucl} \circ R)$ (by IH) $\subseteq (\overline{R^{Eucl}} \circ R) \subseteq (\overline{R^{Eucl}} \circ R^{Eucl})$ (by growth).
else
if $j = 1$ and $i > 1$
then $(\overline{R}^i \circ R^j) = (\overline{R} \circ \overline{R}^{i\perp 1} \circ R^j) \subseteq (\overline{R} \circ \overline{R^{Eucl}} \circ R^{Eucl})$ (by IH) $\subseteq (\overline{R} \circ R^{Eucl}) \subseteq (\overline{R^{Eucl}} \circ R^{Eucl})$.

Now, since $(\overline{R}^+ \circ R^+) = (\bigcup_{i \geq 1} \overline{R}^i) \circ (\bigcup_{j \geq 1} R^j) = \bigcup_{i,j \geq 1} (\overline{R}^i \circ R^j)$
$\subseteq \bigcup_{i,j \geq 1} (\overline{R^{Eucl}} \circ R^{Eucl}) = (\overline{R^{Eucl}} \circ R^{Eucl}) \subseteq R^{Eucl}$:
we obtain $R \cup (\overline{R}^+ \circ R^+) \subseteq R \cup R^{Eucl} \subseteq R^{Eucl}$.

ii) We show that indeed $R \cup (\overline{R}^+ \circ R^+)$ is euclidean by using lemma 1:
$\overline{(R \cup (\overline{R}^+ \circ R^+))} \circ (R \cup (\overline{R}^+ \circ R^+)) = (\overline{R} \cup \overline{(\overline{R}^+ \circ R^+)}) \circ (R \cup (\overline{R}^+ \circ R^+))$
$= (\overline{R} \cup (\overline{R^+} \circ \overline{\overline{R}^+})) \circ (R \cup (\overline{R}^+ \circ R^+)) = (\overline{R} \cup (\overline{R}^+ \circ R^+)) \circ (R \cup (\overline{R}^+ \circ R^+))$
$= (\overline{R} \circ R) \cup (\overline{R} \circ \overline{R}^+ \circ R^+) \cup (\overline{R}^+ \circ R^+ \circ R) \cup (\overline{R}^+ \circ R^+ \circ \overline{R}^+ \circ R^+)$
$\subseteq (\overline{R}^+ \circ R^+) \cup (\overline{R}^+ \circ R^+ \circ \overline{R}^+ \circ R^+)$
(since $(\overline{R} \circ R)$, $(\overline{R} \circ \overline{R}^+ \circ R^+)$ and $(\overline{R}^+ \circ R^+ \circ R) \subseteq (\overline{R}^+ \circ R^+)$)
$\subseteq (\overline{R}^+ \circ R^+)$ (Abour $\mathcal{RR}$: 2) $\subseteq R \cup (\overline{R}^+ \circ R^+)$.

Thanks to the previous lemma, we know how to compute the closure of an $\mathcal{RR}$ under one property of group 1, but how to do it for several properties ? The following lemma will provide us with a tool for this computation. It states that if some fix-point is reached by performing alternatively the closures under each of the properties of some subset ρ of group 1, then this fix-point *is* the closure under ρ. Before, we recall that if $\rho = \{P_1, \ldots, P_n\}$ is a set of properties, a relation S is said to be the ρ-closure of some relation R (i.e. $S = R^\rho$) if and only if S is the least relation containing R and closed under each P_i.

Lemma 9.2. Let $\rho = \{P_1, \ldots, P_n\}$ be a subset of group 1, and $R \in \mathcal{RR}$. Let $R_0 = R$ and $R_{i+1} = (\ldots(R_i^{P_1})\ldots)^{P_n}$; then if there exists m such that $R_{m+1} = R_m$ then $R_m = R^\rho$.

Proof:
We have by growth: $R_m \subseteq R_m^{P_1} \subseteq (R_m^{P_1})^{P_2} \subseteq \ldots \subseteq (\ldots((R_m^{P_1})^{P_2})\ldots)^{P_n} = R_{m+1}$. Now, since $R_m = R_{m+1}$ it comes: $R_m = R_m^{P_i}$, for $1 \leq i \leq n$ (otherwise growth would be falsified) and thus R_m is closed under each P_i $(1 \leq i \leq n)$. Hence R_m is closed under ρ. To conclude, take note that R^ρ is the least superset of R closed under ρ and as such is contained in R_m which, in its turn, is contained in R^ρ since $R_0 \subseteq R^\rho$ (by growth) and $R_i \subseteq R^\rho \Rightarrow R_{i+1} \subseteq (\ldots(((R^\rho)^{P_1})^{P_2})\ldots)^{P_n} = R^\rho$ (by idempotence).

Lemma 9.3. (Closure under several properties) Let R be any $\mathcal{RR}$:

1. $R^{Ref,Sym} = R \cup \overline{R} \cup I$
2. $R^{Ref,Tr} = (R \cup I)^+$
3. $R^{Ref,Sym,Tr} = (R \cup \overline{R} \cup I)^+$
4. $R^{Sym,Tr} = (R \cup \overline{R})^+$
5. $R^{Tr,Eucl} = (\overline{R}^* \circ R^+)$

Due to lemma 4.2, the other cases reduce to one of the previous.

Proof:
We indicate a closure by some property ρ by $\stackrel{\rho}{\Longrightarrow}$:

1. Case of $R^{Ref,Sym}$: $R \stackrel{Ref}{\Longrightarrow} R \cup I \stackrel{Sym}{\Longrightarrow} R \cup I \cup \overline{R \cup I} = R \cup \overline{R} \cup I \stackrel{Ref}{\Longrightarrow} R \cup \overline{R} \cup I$. A fix-point has been obtained.
2. Case of $R^{Ref,Tr}$: $R \stackrel{Ref}{\Longrightarrow} R \cup I \stackrel{Tr}{\Longrightarrow} = (R \cup I)^+ \stackrel{Ref}{\Longrightarrow} (R \cup I)^+ \cup I = (R \cup I)^+ = R^*$.
3. Case of $R^{Ref,Sym,Tr}$: $R \stackrel{Ref}{\Longrightarrow}$ cf. case 1 $\stackrel{Sym}{\Longrightarrow} R \cup \overline{R} \cup I \stackrel{Tr}{\Longrightarrow} (R \cup \overline{R} \cup I)^+$
$\stackrel{Ref}{\Longrightarrow} (R \cup \overline{R} \cup I)^+ \cup I = (R \cup \overline{R} \cup I)^+ = (R \cup \overline{R})^* = \mathcal{U} \stackrel{Sym}{\Longrightarrow} \mathcal{U} \cup \overline{\mathcal{U}} = \mathcal{U} = (\overline{R}^* \circ R^*)$.
4. Case of $R^{Sym,Tr}$: $R \stackrel{Sym}{\Longrightarrow} (R \cup \overline{R}) \stackrel{Tr}{\Longrightarrow} (R \cup \overline{R})^+ \stackrel{Sym}{\Longrightarrow} (R \cup \overline{R})^+ \cup \overline{(R \cup \overline{R})^+}$
$= (R \cup \overline{R})^+ \cup \overline{(R \cup \overline{R})}^+ = (R \cup \overline{R})^+ \cup (\overline{R} \cup R)^+ = (R \cup \overline{R})^+$.
5. Case of $R^{Tr,Eucl}$: $R \stackrel{Tr}{\Longrightarrow} R^+ \stackrel{Eucl}{\Longrightarrow} = R^+ \cup ((\overline{R^+})^+ \circ (R^+)^+) = R^+ \cup (\overline{R}^+ \circ R^+) = (\overline{R}^* \circ R^+) \stackrel{Tr}{\Longrightarrow}$
$(\overline{R}^* \circ R^+)^+ = (\overline{R}^* \circ R^+)$ (About $\mathcal{RR}$: 4).

We need to prove now the stability of group 2 with respect to closure under several properties of group 1. We first prove the following lemma concerning this stability with respect to closure under one property of group 1, and then (lemma 4.3) shows that the same holds for several properties.

Lemma 9.4. Let ρ_2 be a subset of group 2, ρ_1 a property of group 1, let $R \in \mathcal{RR}$ satisfying ρ_2 then R^{ρ_1} is in $\mathcal{RR}$ and satisfies ρ_2; hence it satisfies $\rho_1 \cup \rho_2$.

Proof:
The proof is case-based:

Case $\rho_2 = \mathit{Ser}:$ Immediate since $R \subseteq R^{\rho_1}$ (by monotonicity).

Case $\rho_2 = \mathit{Dens}:$ we must show that $R^{\rho_1} \subseteq (R^{\rho_1})^2$:

- If $\rho_1 = \mathit{Ref}:$ Trivial since reflexivity implies density;

- If $\rho_1 = \mathit{Sym}:$
  $(R^{\rho_1})^2 = R^2 \cup (R \circ \overline{R}) \cup (\overline{R} \circ R) \cup \overline{R}^2 \supseteq R^2 \cup \overline{R}^2 \supseteq R \cup \overline{R}$,
  hence $(R^{\rho_1})^2 \supseteq R \cup \overline{R} = R^{\rho_1}$;

- If $\rho_1 = \mathit{Tr}:$
  $(R^{\rho_1})^2 = (R^+)^2 \supseteq (R^2)^+$ (About $\mathcal{R}$: 14) $\supseteq R^+ = R^{\rho_1}$;

- If $\rho_1 = \mathit{Eucl}:$ Trivial since euclideanity implies density;

Case $\rho_2 = \mathit{Conf}:$ we must show that $(\overline{R^{\rho_1}} \circ R^{\rho_1}) \subseteq (R^{\rho_1} \circ \overline{R^{\rho_1}})$:

- If $\rho_1 = \mathit{Ref}:$
  $(\overline{R^{\rho_1}} \circ R^{\rho_1}) = (\overline{(R \cup I)} \circ (R \cup I)) = (\overline{R} \cup I) \circ (R \cup I) = (\overline{R} \circ R) \cup R \cup \overline{R} \cup I$
  $\subseteq (R \circ \overline{R}) \cup R \cup \overline{R} \cup I$ (since R is confluent)
  On the other hand, $(R^{\rho_1} \circ \overline{R^{\rho_1}}) = (R \circ \overline{R}) \cup R \cup \overline{R} \cup I$
  hence $(\overline{R^{\rho_1}} \circ R^{\rho_1}) \subseteq (R^{\rho_1} \circ \overline{R^{\rho_1}})$;

- If $\rho_1 = \mathit{Sym}:$ Trivial since symmetry implies confluence;

- If $\rho_1 = \mathit{Tr}:$
  $(\overline{R^{\rho_1}} \circ R^{\rho_1}) = (\overline{R}^+ \circ R^+) \subseteq (R^+ \circ \overline{R}^+)$ (About $\mathcal{RR}$: 3) $= (R^{\rho_1} \circ \overline{R^{\rho_1}})$

- If $\rho_1 = \mathit{Eucl}:$ Trivial since euclideanity implies confluence.

Lemma 4.3 (Structural Lemma)
Let ρ_2 be a subset of group 2, ρ_1 a subset of group 1, let $R \in \mathcal{RR}$ satisfying ρ_2 then R^{ρ_1} is in $\mathcal{RR}$ and satisfies ρ_2; hence it satisfies $\rho_1 \cup \rho_2$.

Proof:
If ρ_1 is empty it is trivial. Now suppose (IH1): the lemma is true for some ρ_1; let P be a property of group 1; we must prove (C): the lemma holds for $\rho_1 \cup \{P\}$. But $R^{\rho_1 \cup \{P\}}$ is the fixpoint of the sequence $(((...((R^{\rho_1})^P)^{\rho_1})^P...)^{\rho_1})^P$ that will be denoted by $((R^{\rho_1})^P)_{n \text{ times}}$ where n is the number of closure operations to be done before to reach the fixpoint. If $n = 0$ we trivially have (C). Now suppose (IH2): (C) holds for N, we must prove that it holds for $N+1$. We have: $((R^{\rho_1})^P)_{N+1 \text{ times}} = ((((R^{\rho_1})^P)_{N \text{ times}})^{\rho_1})^P$. By (IH2), $((R^{\rho_1})^P)_{N \text{ times}}$ satisfies ρ_2, then by (IH1) $(((R^{\rho_1})^P)_{N \text{ times}})^{\rho_1}$ also satisfies ρ_2 and by lemma 9.4 $((((R^{\rho_1})^P)_{N \text{ times}})^{\rho_1})^P = ((R^{\rho_1})^P)_{N+1 \text{ times}}$ satisfies ρ_2 too.