Logics of contingency

Luis Fariñas del Cerro and Andreas Herzig
IRIT-CNRS, Université de Toulouse
farinas@irit.fr,herzig@irit.fr

Abstract

We introduce the logic of positive and negative contingency. Together with modal operators of necessity and impossibility they allow to dispense of negation. We study classes of Kripke models where the number of points is restricted, and show that the modalities reduce in the corresponding logics. We finally discuss how this provides an alternative language for answer set programming.

Introduction

Traditionally, modal logics are presented as extensions of classical propositional logic by modal operators of necessity L and possibility M, nowadays rather written $\Box$ and $\Diamond$. These operators are interpreted in Kripke models according to the following truth conditions:

$$M, w \Vdash \mathsf{L}\varphi \quad \text{iff} \quad M, v \Vdash \varphi \text{ for every } v \in R(w)$$
$$M, w \Vdash \mathsf{M}\varphi \quad \text{iff} \quad M, v \Vdash \varphi \text{ for some } v \in R(w)$$

where a Kripke model is a triple $M = \langle W, R, V \rangle$ such that W is a nonempty set of possible worlds (alias states, or points), $R : W \longrightarrow 2^W$ associates to every world $w \in W$ the set of worlds $R(w) \subseteq W$ that are accessible from w, and $V : W \longrightarrow 2^{\mathcal{P}}$ associates to every $w \in W$ the subset of the set of propositional variables $\mathcal{P}$ that is true at w.

Let $\mathcal{L}_{\mathsf{L},\mathsf{M}}$ denote the language built from L, M, and the Boolean operators $\neg$, $\vee$ and $\wedge$. Other languages to talk about Kripke models exist. Lewis and Langford —who were the first to systematically investigate axiom systems for modal logics and to which the names $\mathbf{S4}$ and $\mathbf{S5}$ are due— formulated axiomatic systems in terms of a primitive binary connector of strict implication $>$. Their formulas $\varphi > \psi$ are equivalent to $\mathsf{L}(\varphi \to \psi)$, see e.g. (Hughes and Cresswell 1968).

In this paper we study yet another set of primitives that is based on the notion of *contingency*. Contingency is the opposite of what might be called 'being settled', i.e. being either necessary or impossible. It is a natural concept that is important in commmonsense reasoning. Contingency of φ can be expressed in $\mathcal{L}_{\mathsf{L},\mathsf{M}}$ by the formula $\neg\mathsf{L}\varphi \wedge \neg\mathsf{L}\neg\varphi$. One may distinguish contingent truth of φ ($\neg\mathsf{L}\varphi \wedge \neg\mathsf{L}\neg\varphi \wedge \varphi$) and contingent falsehood of φ ($\neg\mathsf{L}\varphi \wedge \neg\mathsf{L}\neg\varphi \wedge \neg\varphi$). If the modal logic is at least $\mathbf{KT}$ then contingent truth of φ reduces to $\varphi \wedge \neg\mathsf{L}\varphi$, and contingent falsehood of φ reduces to $\neg\varphi \wedge \neg\mathsf{L}\neg\varphi$. We adopt the latter two as our official definitions of contingency: $\mathsf{C}^+\varphi$ denotes contingent truth of φ, and $\mathsf{C}^-\varphi$ denotes contingent falsity of φ. We take these two operators as primitive, together with necessity $\mathsf{L}^+\varphi$ and impossibility $\mathsf{L}^-\varphi$.

Actually the negation operator is superfluous in a language with our four primitives because the modal operators already contain negative information: $\neg\varphi$ is equivalent to $\mathsf{L}^-\varphi \vee \mathsf{C}^-\varphi$.

In this paper we focus on the fragment of formulas where modal depth is at most one. We show that this fragment is less expressive than the set of all formulas even when we restrict models to $\mathbf{S5}$ models, but does not loose expressivity in the case of models where there are at most two or three possible worlds. We are going to relate this to answer set programming.

The paper is organized as follows. We first give syntax and semantics and study some properties, in particular of the class of $\mathbf{S5}$ models. We then show that formulas whose modal depth is at most one only require three points models. Thereafter we show that in models with at most two points, every formula is equivalent to a formula of depth at most one. We finally establish the link with the intermediate logic of here and there as studied in answer set programming.

The logic of contingency

We introduce a logic with modal operators of positive and negative contingency and positive and negative necessity that are interpreted in reflexive Kripke models. We then study the validities in $\mathbf{S5}$ models.

Language and semantics

We define a language $\mathcal{L}_{pos}$ without negation: beyond the Boolean connectors $\wedge$ and $\vee$, our language has four unary modal operators: L^+ (necessity), L^- (impossibility), C^+ (contingent truth), and C^- (contingent falsehood), that we take all as primitive. Such primitives was already studied in (Fariñas del Cerro 1984) for the case of $\mathbf{S5}$ models in order to provide a resolution principle for the fragment of formulas where modal operators have only atomic formulas in their scope. L^+ and C^+ are called positive modal operators and L^- and C^- negative modal operators.

The *modal depth* of a formula φ is the maximum number of nested modal operators in φ.

Given a reflexive Kripke model $M = \langle W, R, V\rangle$, the truth conditions of our modal operators are as follows:

$$\begin{array}{lcl} M,w \Vdash \mathsf{L}^+\varphi & \text{iff} & M,v \Vdash \varphi \text{ for every } v \in R(w) \\ M,w \Vdash \mathsf{L}^-\varphi & \text{iff} & M,v \nVdash \varphi \text{ for every } v \in R(w) \\ M,w \Vdash \mathsf{C}^+\varphi & \text{iff} & M,w \Vdash \varphi \text{ and } M,w \nVdash \mathsf{L}^+\varphi \\ M,w \Vdash \mathsf{C}^-\varphi & \text{iff} & M,w \nVdash \varphi \text{ and } M,w \nVdash \mathsf{L}^-\varphi \end{array}$$

The last two conditions could also be written:

$$\begin{array}{lcl} M,w \Vdash \mathsf{C}^+\varphi & \text{iff} & M,w \Vdash \varphi \text{ and } \exists v \in R(w), M,v \nVdash \varphi \\ M,w \Vdash \mathsf{C}^-\varphi & \text{iff} & M,w \nVdash \varphi \text{ and } \exists v \in R(w), M,v \Vdash \varphi \end{array}$$

Validity and satisfiability are defined as usual.

Although negation is *not* among the primitives of $\mathcal{L}_{pos}$, we can still reason with negative information by defining $\neg\varphi$ to be an abbreviation of $\mathsf{L}^-\varphi \vee \mathsf{C}^-\varphi$. It can be readily checked that for our definition of negation it holds that $M,w \Vdash \neg\varphi$ iff $M,w \nVdash \varphi$, for every reflexive model M and every world w in M.

Falsehood ($\bot$), material implication ($\to$) and equivalence ($\leftrightarrow$) are then defined by the usual abbreviations.[1] So in terms of the standard modal operator L, the new operators $\mathsf{L}^+\varphi$ and $\mathsf{L}^-\varphi$ are nothing but $\mathsf{L}\varphi$ and $\mathsf{L}\neg\varphi$; and $\mathsf{C}^+\varphi$ and $\mathsf{C}^-\varphi$ are nothing but $\varphi \wedge \neg\mathsf{L}\varphi$ and $\neg\varphi \wedge \neg\mathsf{L}\neg\varphi$. The other way round, in reflexive models $\mathsf{M}\varphi$ is nothing but $\mathsf{L}^+\varphi \vee \mathsf{C}^+\varphi \vee \mathsf{C}^-\varphi$.

Some properties

The modal operators C^+ and C^- are neither normal boxes nor normal diamonds in Chellas's sense (Chellas 1980). Actually neither $\mathsf{C}^+\top$ nor $(\mathsf{C}^+\varphi \vee \mathsf{C}^+\psi) \to \mathsf{C}^+(\varphi \vee \psi)$ nor $\mathsf{C}^+(\varphi \vee \psi) \to (\mathsf{C}^+\varphi \vee \mathsf{C}^+\psi)$ are valid.

Our four modal operators are mutually exclusive and exhaustive.

Proposition 1 *The formulas*

$$\mathsf{L}^+\varphi \vee \mathsf{L}^-\varphi \vee \mathsf{C}^+\varphi \vee \mathsf{C}^-\varphi$$

$$\begin{array}{ll} \neg(\mathsf{L}^+\varphi \wedge \mathsf{L}^-\varphi) & \neg(\mathsf{L}^-\varphi \wedge \mathsf{C}^+\varphi) \\ \neg(\mathsf{L}^+\varphi \wedge \mathsf{C}^+\varphi) & \neg(\mathsf{L}^-\varphi \wedge \mathsf{C}^-\varphi), \\ \neg(\mathsf{L}^+\varphi \wedge \mathsf{C}^-\varphi) & \neg(\mathsf{C}^+\varphi \wedge \mathsf{C}^-\varphi) \end{array}$$

are valid in the set of all Kripke models.

Observe also that $(\neg\mathsf{L}^+\varphi \wedge \neg\mathsf{L}^+\neg\varphi) \leftrightarrow (\mathsf{C}^-\varphi \vee \mathsf{C}^+\varphi)$ is valid: mere contingency corresponds therefore to the $\mathcal{L}_{pos}$ formula $\mathsf{C}^-\varphi \vee \mathsf{C}^+\varphi$.

Here are some other properties w.r.t. conjunction and disjunction.

Proposition 2 *The equivalences*

$$\begin{array}{lcl} \mathsf{L}^+(\varphi \wedge \psi) & \leftrightarrow & \mathsf{L}^+\varphi \wedge \mathsf{L}^+\psi \\ \mathsf{L}^-(\varphi \vee \psi) & \leftrightarrow & \mathsf{L}^-\psi \wedge \mathsf{L}^-\psi \\ \mathsf{C}^+(\varphi \wedge \psi) & \leftrightarrow & \varphi \wedge \psi \wedge (\mathsf{C}^+\varphi \vee \mathsf{C}^+\psi) \\ & \leftrightarrow & (\varphi \wedge \mathsf{C}^+\psi) \vee (\mathsf{C}^+\varphi \wedge \psi) \\ \mathsf{C}^-(\varphi \vee \psi) & \leftrightarrow & \neg\varphi \wedge \neg\psi \wedge (\mathsf{C}^-\varphi \vee \mathsf{C}^-\psi) \\ & \leftrightarrow & (\neg\varphi \wedge \mathsf{C}^-\psi) \vee (\mathsf{C}^-\varphi \wedge \neg\psi) \\ & \leftrightarrow & (\mathsf{C}^-\varphi \wedge \mathsf{C}^-\psi) \vee \\ & & (\mathsf{C}^-\varphi \wedge \mathsf{L}^-\psi) \vee (\mathsf{L}^-\varphi \wedge \mathsf{C}^-\psi) \end{array}$$

are valid.

For the other combinations of contingency operators and Boolean connectors we only have implications:

$$\begin{array}{lcl} \mathsf{C}^+(\varphi \vee \psi) & \to & \mathsf{C}^+\psi \vee \mathsf{C}^+\psi \\ \mathsf{C}^-(\varphi \wedge \psi) & \to & \mathsf{C}^-\varphi \vee \mathsf{C}^-\psi \end{array}$$

The converse implications are not valid. More generally, there seems to be no formulas without Boolean operators in the scope of modalities that are equivalent to $\mathsf{C}^+(p \wedge q)$ and $\mathsf{C}^-(p \vee q)$.

Every $\mathcal{L}_{pos}$ formula can be rewritten to a formula such that every propositional atom is in the scope of at least one modal operator. This is possible because the equivalence $\varphi \leftrightarrow \mathsf{L}^+\varphi \vee \mathsf{C}^+\varphi$ is valid.

Reducing modalities in S4

We now check which modalities our language $\mathcal{L}_{pos}$ has in **S4**. Modalities are understood in the sense of (Hughes and Cresswell 1968) as non-reducible sequences of modal operators. Remember that **S4** models have accessibility relations that are reflexive and transitive.

In **S4**, modalities starting with positive modal operators reduce to length at most one:

Proposition 3 *The equivalences*

$$\begin{array}{lclclcl} \mathsf{L}^+\mathsf{L}^+\varphi & \leftrightarrow & \mathsf{L}^+\varphi & \quad & \mathsf{C}^+\mathsf{L}^+\varphi & \leftrightarrow & \bot \\ \mathsf{L}^+\mathsf{L}^-\varphi & \leftrightarrow & \mathsf{L}^-\varphi & & \mathsf{C}^+\mathsf{L}^-\varphi & \leftrightarrow & \bot \\ \mathsf{L}^+\mathsf{C}^+\varphi & \leftrightarrow & \bot & & \mathsf{C}^+\mathsf{C}^+\varphi & \leftrightarrow & \mathsf{C}^+\varphi \\ \mathsf{L}^+\mathsf{C}^-\varphi & \leftrightarrow & \bot & & \mathsf{C}^+\mathsf{C}^-\varphi & \leftrightarrow & \mathsf{C}^-\varphi \end{array}$$

are **S4** *valid.*

Reducing modalities in S5

Remember that **S5** models have accessibility relations that are equivalence relations: they are reflexive, transitive and Euclidean. Put together, these constraints say that $w \in R(w)$ and that if $v \in R(w)$ then $R(v) = R(w)$.

Just as in the standard modal language $\mathcal{L}_{\mathsf{L},\mathsf{M}}$, in **S5** modalities of $\mathcal{L}_{pos}$ can be reduced to length at most one:

Proposition 4 *The equivalences*

$$\begin{array}{lclclcl} \mathsf{L}^-\mathsf{L}^+\varphi & \leftrightarrow & \mathsf{L}^-\varphi \vee \mathsf{C}^+\varphi \vee \mathsf{C}^-\varphi & \quad & \mathsf{C}^-\mathsf{L}^+\varphi & \leftrightarrow & \bot \\ \mathsf{L}^-\mathsf{L}^-\varphi & \leftrightarrow & \mathsf{L}^+\varphi \vee \mathsf{C}^+\varphi \vee \mathsf{C}^-\varphi & & \mathsf{C}^-\mathsf{L}^-\varphi & \leftrightarrow & \bot \\ \mathsf{L}^-\mathsf{C}^+\varphi & \leftrightarrow & \mathsf{L}^+\varphi \vee \mathsf{L}^-\varphi & & \mathsf{C}^-\mathsf{C}^+\varphi & \leftrightarrow & \mathsf{C}^-\varphi \\ \mathsf{L}^-\mathsf{C}^-\varphi & \leftrightarrow & \mathsf{L}^+\varphi \vee \mathsf{L}^-\varphi & & \mathsf{C}^-\mathsf{C}^-\varphi & \leftrightarrow & \mathsf{C}^+\varphi \end{array}$$

are valid in reflexive and transitive models.

[1] Alternatively we may also define $\bot$ as $\mathsf{L}^+p \wedge \mathsf{L}^-p$, for some propositional variable $p \in \mathcal{P}$.

The fragment of formulas of depth at most n

Let $\mathcal{L}^n_{pos}$ be the set of formulas of $\mathcal{L}_{pos}$ of modal depth at most n. These sets are defined by the following BNFs.

$$\begin{array}{lcl} \varphi^0 & ::= & p \mid \varphi^0 \wedge \varphi^0 \mid \varphi^0 \vee \varphi^0 \\ \varphi^{n+1} & ::= & \varphi^n \mid \mathsf{L}^+\varphi^n \mid \mathsf{L}^-\varphi^n \mid \mathsf{C}^+\varphi^n \mid \mathsf{C}^-\varphi^n \mid \\ & & \varphi^{n+1} \wedge \varphi^{n+1} \mid \varphi^{n+1} \vee \varphi^{n+1} \end{array}$$

where p ranges over the set of propositional variables $\mathcal{P}$. The elements of $\mathcal{L}^0_{pos}$ are positive Boolean formulas: formulas that are both negation-free and modality-free; the elements of $\mathcal{L}^1_{pos}$ are modal formulas of depth at most one.

Every formula of $\mathcal{L}_{pos}$ is in $\mathcal{L}^n_{pos}$ for some integer n. In the rest of the paper we are going to show some results for these languages:

- In **S5** models, every $\mathcal{L}_{pos}$ formula is equivalent to some $\mathcal{L}^2_{pos}$ formula (theorem 1).
- Every satisfiable $\mathcal{L}^1_{pos}$ formula has a model with at most three points and satisfying a persistence condition: beyond the actual world there are worlds $w_\forall$ and $w_\exists$ such that $w_\exists$ inherits $w_\forall$'s truths, in the sense that a propositional variable that is true in $w_\forall$ is also true in $w_\exists$ (theorem 2).
- In the class of models having at most two points, every $\mathcal{L}_{pos}$ formula is equivalent to some $\mathcal{L}^1_{pos}$ formula (theorem 3).

Reducing S5 formulas to modal depth two

It is well-known that in **S5**, every $\mathcal{L}_{\mathsf{L,M}}$ formula is equivalent to a formula of modal depth at most one (Hughes and Cresswell 1968). For our language $\mathcal{L}_{pos}$ we can only prove a weaker property.

Theorem 1 *In* **S5** *models, every $\mathcal{L}_{pos}$ formula is equivalent to a $\mathcal{L}^2_{pos}$ formula.*

PROOF. The following steps are one way to obtain the result: (1) rewrite φ to a $\mathcal{L}_{\mathsf{L,M}}$ formula φ_1; (2) rewrite φ_1 to a $\mathcal{L}_{\mathsf{L,M}}$ formula φ_2 of modal depth at most one; (3) rewrite φ_2 by replacing L by L^+, resulting in φ_3; (4) rewrite φ_3 by replacing every subformula $\mathsf{M}\psi$ by $\mathsf{L}^+\psi \vee \mathsf{C}^+\psi \vee \mathsf{C}^-\psi$, resulting in a formula φ_4 that is still of depth at most one; (5) rewrite φ_4 by replacing every subformula $\neg\psi$ by $\mathsf{L}^-\psi \vee \mathsf{C}^-\psi$, resulting in a formula φ_5 of depth at most two. The formula φ_5 is a $\mathcal{L}_{pos}$ formula. ■

One may wonder whether this can be pushed further, i.e. whether one may reduce to a formula of depth at most one. That this is not the case will follow from our results in the next section (see proposition 5).

Models with at most three points

We now show that the models of the language $\mathcal{L}^1_{pos}$ can be restricted to a special class of Kripke models: models with three points where beyond the actual world there is a 'positive world' $w_\forall$ and a 'negative world' $w_\exists$, such that every propositional variable that is true in the former is also true in the latter.

Definition 1 *Let (M, w) be a pointed Kripke model with R reflexive. Its* 3-points projection *is the pointed Kripke model (M^3, w) where $M^3 = \langle W^3, R^3, V^3\rangle$, $W^3 = \{w, w_\forall, w_\exists\}$, $R^3 = W^3 \times W^3$, and V^3 is such that*

- *$V^3(w) = V(w)$;*
- *$V^3(w_\forall) = \bigcap_{v \in R(w)} V(v)$;*
- *$V^3(w_\exists) = \bigcup_{v \in R(w)} V(v)$.*

In such a 3-points model we have: $M^3, w_\forall \Vdash p$ iff $M, v \Vdash p$ for every $v \in R(w)$, and $M^3, w_\exists \Vdash p$ iff $M, v \Vdash p$ for some $v \in R(w)$, for every propositional variable $p \in \mathcal{P}$. This generalizes to positive Boolean formulas of $\mathcal{L}^0_{pos}$.

Lemma 1 *Let (M^3, w) be the 3-points projection of a pointed Kripke model (M, w). Let $\varphi^0 \in \mathcal{L}^0_{pos}$ be a positive Boolean formula, i.e. a formula that is both negation-free and modality-free.*

1. *$M^3, w \Vdash \varphi^0$ iff $M, w \Vdash \varphi^0$.*
2. *$M^3, w_\forall \Vdash \varphi^0$ iff $M, v \Vdash \varphi^0$ for every $v \in R(w)$.*
3. *$M^3, w_\exists \Vdash \varphi^0$ iff $M, v \Vdash \varphi^0$ for some $v \in R(w)$.*

PROOF. The proof is by induction on the form of φ^0. ■

Moreover, 3-points projections preserve the properties of the accessibility relation.

Lemma 2 *Let M be a Kripke model. If R is transitive (resp. Euclidean, symmetric) then R^3 is transitive (resp. Euclidean, symmetric).*

We now can prove:

Theorem 2 *Let φ^1 be a $\mathcal{L}^1_{pos}$ formula. Let (M, w) be a pointed Kripke model, and let (M^3, w) be its 3-points projection. Then $M, w \Vdash \varphi^1$ iff $M^3, w \Vdash \varphi^1$.*

PROOF. We proceed by induction on the form of φ^1. There are several base cases.

1. ψ^1 is a Boolean formula ψ^0. Then by lemma 1, $M, w \Vdash \varphi^0$ iff $M^3, w \Vdash \varphi^0$.
2. ψ^1 is of the form $\mathsf{L}^+\psi^0$. Then

$$\begin{array}{lll} M, w \Vdash \mathsf{L}^+\psi^0 & \text{iff} & M, v \Vdash \psi^0 \text{ for every } v \in R(w) \\ & \text{iff} & M^3, w_\forall \Vdash \psi^0 \quad \text{(by lemma 1)} \\ & \text{iff} & M^3, w_\forall \Vdash \psi^0 \text{ and } M^3, w_\exists \Vdash \psi^0 \\ & \text{iff} & M^3, w \Vdash \psi^0 \text{ and} \\ & & M^3, w_\forall \Vdash \psi^0 \text{ and } M^3, w_\exists \Vdash \psi^0 \\ & \text{iff} & M^3, w \Vdash \mathsf{L}^+\psi^0 \end{array}$$

3. ψ^1 is of the form $\mathsf{L}^-\psi^0$. Then

$$\begin{array}{lll} M, w \Vdash \mathsf{L}^-\psi^0 & \text{iff} & M, v \nVdash \psi^0 \text{ for every } v \in W \\ & \text{iff} & M^3, w \nVdash \psi^0 \text{ and } M^3, w_\forall \nVdash \psi^0 \text{ and} \\ & & M^3, w_\exists \nVdash \psi^0 \quad \text{(by lemma 1)} \\ & \text{iff} & M^3, w \Vdash \mathsf{L}^-\psi^0 \end{array}$$

4. ψ^1 is of the form $\mathsf{C}^+\psi^0$. Then

$$
\begin{array}{lll}
M,w \Vdash \mathsf{C}^+\psi^0 & \text{iff} & M,w \Vdash \psi^0 \text{ and} \\
& & M,v \not\Vdash \psi^0 \text{ for some } v \in W \\
& \text{iff} & M^3,w \Vdash \psi^0 \text{ and } M^3,w_\forall \not\Vdash \psi^0 \\
& & \qquad\qquad \text{(by lemma 1)} \\
& \text{iff} & M^3,w \Vdash \mathsf{C}^+\psi^0
\end{array}
$$

5. ψ^1 is of the form $\mathsf{C}^-\psi^0$. Then

$$
\begin{array}{lll}
M,w \Vdash \mathsf{C}^-\psi^0 & \text{iff} & M,w \not\Vdash \psi^0 \text{ and} \\
& & M,v \Vdash \psi^0 \text{ for some } v \in W \\
& \text{iff} & M^3,w \not\Vdash \psi^0 \text{ and } M^3,w_\exists \Vdash \psi^0 \\
& & \qquad\qquad \text{(by lemma 1)} \\
& \text{iff} & M^3,w \Vdash \mathsf{C}^-\psi^0
\end{array}
$$

As to the induction step, the cases of conjunctions and disjunctions are straightforward. ■

It follows that formulas of our language $\mathcal{L}_{pos}$ cannot be reduced to modal depth one.

Proposition 5 *There is no negation-free $\mathcal{L}_{pos}$ formula of depth less than 2 that is equivalent to the $\mathcal{L}_{\mathsf{L},\mathsf{M}}$ formula $\mathsf{M}(p \wedge \neg q)$. The same is the case for the $\mathcal{L}_{pos}$ formulas $\mathsf{C}^+(p \wedge \mathsf{C}^+q)$ and $\mathsf{L}^+(p \vee \mathsf{C}^-q)$.*

PROOF. If such a formula existed then **S5** models could be restricted to models with at most 3 points. ■

One cannot do better than theorem 2: there are satisfiable $\mathcal{L}^1_{pos}$ formulas which require three points. Consider for example

$$\mathsf{C}^+p \wedge \mathsf{C}^-q \wedge \mathsf{C}^-(p \wedge q).$$

It is satisfiable, and requires an actual world where p is true and q is false, another possible world where both p and q are true, and some third world where p is false.

In the rest of the section we are going to restrict models to an even smaller class: two points models with persistence. We will show that in that class, for every $\mathcal{L}_{pos}$ formula there is an equivalent $\mathcal{L}^1_{pos}$ formula.

Models with at most two points

In the rest of the paper we consider reflexive models with at most two points. For that class we are going to establish a strong normal form.

First, observe that reflexive models with at most two points are also transitive. Therefore the equivalences of proposition 3 apply: every modality starting with a positive operator can be reduced. The next proposition allows to also reduce modalities starting with a negative operator.

Proposition 6 *The equivalences*

$$
\begin{array}{lll}
\mathsf{L}^-\mathsf{L}^+\varphi & \leftrightarrow & \mathsf{L}^-\varphi \vee \mathsf{C}^+\varphi \\
\mathsf{L}^-\mathsf{L}^-\varphi & \leftrightarrow & \mathsf{L}^+\varphi \vee \mathsf{C}^-\varphi \\
\mathsf{L}^-\mathsf{C}^+\varphi & \leftrightarrow & \mathsf{L}^+\varphi \vee \mathsf{L}^-\varphi \vee \mathsf{C}^-\varphi \\
\mathsf{L}^-\mathsf{C}^-\varphi & \leftrightarrow & \mathsf{L}^+\varphi \vee \mathsf{L}^-\varphi \vee \mathsf{C}^+\varphi \\
\mathsf{C}^-\mathsf{L}^+\varphi & \leftrightarrow & \mathsf{C}^-\varphi \\
\mathsf{C}^-\mathsf{L}^-\varphi & \leftrightarrow & \mathsf{C}^+\varphi \\
\mathsf{C}^-\mathsf{C}^+\varphi & \leftrightarrow & \bot \\
\mathsf{C}^-\mathsf{C}^-\varphi & \leftrightarrow & \bot
\end{array}
$$

are valid in Kripke models having at most two-points.

PROOF. One may prove each of the equivalences semantically, exploring the 4 different possible truth values of φ in two-points models. ■

Together, propositions 3 and 6 allow to reduce every modality to length at most one.

Observe that while the formulas $\mathsf{C}^-\mathsf{L}^+\varphi$ and $\mathsf{C}^-\mathsf{L}^-\varphi$ are satisfiable in reflexive models with at most two points, they are both unsatisfiable in **S5** according to proposition 4.

Beyond the reduction of modalities, reflexive models with at most two points also allow for the distribution of modal operators over conjunctions and disjunctions. Standard modal equivalences already allows us to distribute L^+ over conjunctions and L^- over disjunctions. Proposition 2 allows us to distribute C^+ over conjunctions and C^- over disjunctions. The next proposition deals with the resulting cases.

Proposition 7 *The equivalences*

$$
\begin{array}{lll}
\mathsf{L}^+(\varphi \vee \psi) & \leftrightarrow & \mathsf{L}^+\varphi \vee \mathsf{L}^+\psi \vee (\mathsf{C}^+\varphi \wedge \mathsf{C}^-\psi) \vee (\mathsf{C}^-\varphi \wedge \mathsf{C}^+\psi) \\
\mathsf{L}^-(\varphi \wedge \psi) & \leftrightarrow & \mathsf{L}^-\varphi \vee \mathsf{L}^-\psi \vee (\mathsf{C}^+\varphi \wedge \mathsf{C}^-\psi) \vee (\mathsf{C}^-\varphi \wedge \mathsf{C}^+\psi) \\
\mathsf{C}^+(\varphi \vee \psi) & \leftrightarrow & (\mathsf{C}^+\varphi \wedge \mathsf{C}^+\psi) \vee (\mathsf{C}^+\varphi \wedge \mathsf{L}^-\psi) \vee (\mathsf{L}^-\varphi \wedge \mathsf{C}^+\psi) \\
\mathsf{C}^-(\varphi \wedge \psi) & \leftrightarrow & (\mathsf{C}^-\varphi \wedge \mathsf{C}^-\psi) \vee (\mathsf{C}^-\varphi \wedge \mathsf{L}^+\psi) \vee (\mathsf{L}^+\varphi \wedge \mathsf{C}^-\psi)
\end{array}
$$

are valid in the class of Kripke models having at most two points.

A strong normal form

Distributing the modal operators over conjunctions and disjunctions according to proposition 7 results in a formula made up of modal atoms —modalities followed by a propositional variable— that are combined by conjunctions and disjunctions. These modal atoms can then be reduced by propositions 3 and 6. This results in a very simple normal form.

Theorem 3 *In the class of Kripke models with reflexive and persistent accessibility relation having at most two points, every $\mathcal{L}_{pos}$ formula is equivalent to a $\mathcal{L}^1_{pos}$ formula that is built according to the following BNF:*

$$\varphi \quad ::= \quad \mathsf{L}^+p \mid \mathsf{L}^-p \mid \mathsf{C}^+p \mid \mathsf{C}^-p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi$$

where p ranges over the set of propositional variables $\mathcal{P}$.

PROOF. Let φ be a $\mathcal{L}_{pos}$ formula. First, distribute the modal operators in φ over conjunctions and disjunctions (according to propositions 2 and 7), resulting in a formula built from *modal atoms* by means of conjunction and disjunction. Second, reduce the modalities using the equivalences of propositions 3 and 6, resulting in a formula built from modal atoms of depth at most one by means of conjunction and disjunction. Third, replace every atom p that is not in the scope of any modal operator by $\mathsf{L}^+p \vee \mathsf{C}^+p$, resulting in fully modal combination of modal atoms of the form L^+p, L^-p, C^+p and C^-p. ■

A proof procedure for formulas in normal form

We can translate formulas in strong normal form into the language of classical propositional logic by means of the following transformation t:

$$\begin{array}{rcl} t(\mathsf{L}^+p) & = & p^{11} \\ t(\mathsf{L}^-p) & = & p^{00} \\ t(\mathsf{C}^+p) & = & p^{10} \\ t(\mathsf{C}^-p) & = & p^{01} \end{array}$$

and homomorphic for conjunction and disjunction.

Proposition 8 *Let φ be a formula in strong normal form. φ is valid in here-and-there models if and only if $\Gamma_\varphi \models t(\varphi)$, where Γ_φ is the union of the sets of formulas*

$$\begin{array}{rl} \Gamma_p = \{ & \neg(p^{00} \wedge p^{01}), \neg(p^{00} \wedge p^{10}), \neg(p^{00} \wedge p^{11}), \\ & \neg(p^{01} \wedge p^{10}), \neg(p^{01} \wedge p^{11}), \neg(p^{10} \wedge p^{11}), \\ & p^{00} \vee p^{01} \vee p^{10} \vee p^{11} \quad \} \end{array}$$

such that p occurs in φ.

The length of Γ_φ is bound by the length of φ, and t is a linear transformation. We may therefore check in coNP time whether a formula φ in strong normal form is valid in reflexive models with at most two points by checking whether $\Gamma_\varphi \models t(\varphi)$ in classical propositional logic.

It moreover follows that the validities of the class of reflexive models with at most two points can be axiomatized as follows:

- some axiomatization of classical propositional logic;
- the formulas of proposition 1 (which match the hypotheses of proposition 8);
- the modality reduction axioms of propositions 3 and 6;
- the distribution axioms of propositions 2 and 7;
- rules of equivalence

$$\frac{\varphi \leftrightarrow \psi}{\mathsf{L}^+\varphi \leftrightarrow \mathsf{L}^+\psi} \qquad \frac{\varphi \leftrightarrow \psi}{\mathsf{C}^+\varphi \leftrightarrow \mathsf{C}^+\psi}$$

$$\frac{\varphi \leftrightarrow \psi}{\mathsf{L}^-\varphi \leftrightarrow \mathsf{L}^-\psi} \qquad \frac{\varphi \leftrightarrow \psi}{\mathsf{C}^-\varphi \leftrightarrow \mathsf{C}^-\psi}$$

(The formulas of the propositions mentioned above have to be taken as axiom schemas.)

Answer set programming in terms of contingency

In the rest of the paper we consider models with at most two points where the accessibility relation is reflexive and *persistent*, aka *hereditary*. Just as in intuitionistic logic, we say that R is persistent if $\langle u, v\rangle \in R$ implies $V(u) \subseteq V(v)$. Such models were studied since Gödel in order to give semantics to an implication $\Rightarrow$ with strength between intuitionistic and material implication (Heyting 1930). More recently these models were baptized *here-and-there models* and their logic was investigated under the denomination *equilibrium logic*[2] by Pearce, Cabalar, Lifschitz, Ferraris and others as a semantical framework for answer set programming (Lifschitz, Pearce, and Valverde 2001; Pearce, de Guzmán, and Valverde 2000; Cabalar and Ferraris 2007; Cabalar, Pearce, and Valverde 2007; Ferraris, Lee, and Lifschitz 2007; Lifschitz 2010).

Here-and-there models

Precisely, here-and-there models are models $M = \langle W, R, V\rangle$ with $W = \{H, T\}$, $V(H) \subseteq V(T)$, and $R = \{\langle H, H\rangle, \langle H, T\rangle, \langle T, T\rangle\}$. The other reflexive and persistent two-points models are bisimilar to here-and-there models: (1) two points models with a symmetric relation are bisimilar to one point models due to persistence; (2) models where the two points are not related are bisimilar to one point models; (3) one point models are bisimilar to here-and-there models.

In persistent models the formula C^+p is unsatisfiable for every propositional variable $p \in \mathcal{P}$. In the normal form of theorem 3 we may therefore replace every subformula C^+p by $\mathsf{L}^+p \wedge \mathsf{L}^-p$, resulting in a formula built from modal atoms of depth one where C^+ does not occur.

Theorem 4 *Let φ be a $\mathcal{L}_{pos}$ formula. φ is valid in the class of here-and-there models if and only if*

$$\varphi \vee \bigvee_{p\,:\,p \text{ occurs in } \varphi} \mathsf{C}^+p$$

is valid in the class of reflexive models with at most two points.

From implication to contingency

Let us call $\mathcal{L}^{\Rightarrow}_{pos}$ the language resulting from the addition of a binary modal connector $\Rightarrow$ to our language $\mathcal{L}_{pos}$. The language of equilibrium logic is the fragment $\mathcal{L}^{\Rightarrow}$ of $\mathcal{L}^{\Rightarrow}_{pos}$ without our four unary modal connectors.

The truth condition for the intermediate implication can be written as:[3]

$$M, w \Vdash \varphi \Rightarrow \psi \quad \text{iff} \quad \forall v \in R(w), M, v \nVdash \varphi \text{ or } M, v \Vdash \psi$$

The traditional language of logic programming is Horn clauses. Given that C^+p is unsatisfiable for every $p \in \mathcal{P}$, theorem 3 shows that the above language made up of Boolean combinations of modal atoms of the form L^+p, L^-p, C^-p is an alternative language.

Theorem 5 *The $\mathcal{L}^{\Rightarrow}_{pos}$ formula*

$$\varphi \Rightarrow \psi \quad \leftrightarrow \quad \mathsf{L}^-\varphi \vee \mathsf{L}^+\psi \vee (\mathsf{C}^-\varphi \wedge \mathsf{C}^-\psi) \vee (\mathsf{C}^+\varphi \wedge \mathsf{C}^+\psi)$$

[2] http://www.equilibriumlogic.net

[3] We preferred a more standard presentation; in the equilibrium logic literature it is:

$$\begin{array}{lll} M \Vdash p & \text{iff} & V(T) \Vdash p, \text{ for } p \in \mathcal{P} \\ M \Vdash \varphi \Rightarrow \psi & \text{iff} & (M \nVdash \varphi \text{ or } M \Vdash \psi) \text{ and} \\ & & (V(T) \nVdash \varphi \text{ or } V(T) \Vdash \psi) \end{array}$$

where $V(H)$ and $V(T)$ are understood as models of classical propositional logic. $V(T) \Vdash \varphi$ therefore stands for truth of φ in the model $V(T)$. (Hence formulas are always evaluated at H.) A formula φ is valid in the class of here-and-there models if and only if $M \Vdash \varphi$ for every here-and-there model M.

is valid in here-and-there models.

PROOF. By the truth condition of $\Rightarrow$, $\varphi \Rightarrow \psi$ is equivalent to $\mathsf{L}^+(\neg\varphi \vee \psi)$. By proposition 7 the latter is equivalent to

$$\mathsf{L}^+\neg\varphi \vee \mathsf{L}^+\psi \vee (\mathsf{C}^-\neg\varphi \wedge \mathsf{C}^+\psi) \vee (\mathsf{C}^+\neg\varphi \wedge \mathsf{C}^-\psi),$$

which is equivalent to the right-hand side. ■

Together, theorems 5 and 4 say that instead of reasoning with an intermediate implication $\Rightarrow$ one might as well use our fairly simple modal logic of contingency having reflexive two-points models. It has to be noted that these models are not necessarily persistent. Once one has moved to that logic, one can put formulas in strong normal form (theorem 3), apply proposition 8, and directly work in classical propositional logic.

Conclusion

We have presented a modal logic of positive and negative contingency and have studied its properties in different classes of models. We have in particular investigated models whose number of points is restricted. We have established a link with equilibrium logics as studied in answer set programming. Our negation-free language in terms of contingency provides an alternative to the usual implication-based language.

Our logic can also be seen as a way of combining intuitionistic and classical implication. Other such approaches can be found in (Vakarelov 1977; Fariñas del Cerro and Raggio 1983; Došen 1985; Fariñas del Cerro and Herzig 1996).

As to the perspectives, our logic can be extended straightforwardly to first-order logic if we restrict formulas to prenex formulas. The link to answer sets for programs with variables (Ferraris, Lee, and Lifschitz 2007) could then be studied.

References

Cabalar, P., and Ferraris, P. 2007. Propositional theories are strongly equivalent to logic programs. *Theory and Practice of Logic Programming (TPLP)* 7(6):745–759.

Cabalar, P.; Pearce, D.; and Valverde, A. 2007. Minimal logic programs. In Dahl, V., and Niemelä, I., eds., *Proc. ICLP*, volume 4670 of *LNCS*, 104–118. Springer.

Chellas, B. 1980. *Modal logic: An introduction*. Cambridge University Press.

Došen, K. 1985. Models for stronger normal intuitionistic modal logics. *Studia Logica* 44:39–70.

Fariñas del Cerro, L., and Herzig, A. 1996. Combining classical and intuitionistic logic, or: intuitionistic implication as a conditional. In Baader, F., and Schulz, K. U., eds., *Frontiers in Combining Systems*, number 3 in Applied Logic Series, 93–102. Kluwer Academic Publishers.

Fariñas del Cerro, L., and Raggio, A. 1983. Some results in intuitionistic modal logic. *Logique et Analyse* 26(102):219–224.

Fariñas del Cerro, L. 1984. Un principe de résolution en logique modale. *R.A.I.R.O. Informatique théorique* 18(2):161–170.

Ferraris, P.; Lee, J.; and Lifschitz, V. 2007. A new perspective on stable models. In Veloso, M. M., ed., *IJCAI*, 372–379.

Heyting, A. 1930. Die Formalen Regeln der intuitionistischen Logik. *Sitzungsber. preuss. Akad. Wiss.* 42-71:158–169.

Hughes, G. E., and Cresswell, M. J. 1968. *An introduction to modal logic*. Methuen&Co. Ltd, London.

Lifschitz, V.; Pearce, D.; and Valverde, A. 2001. Strongly equivalent logic programs. *ACM Transactions on Computational Logic* 2(4):526–541.

Lifschitz, V. 2010. Thirteen definitions of a stable model. In Blass, A.; Dershowitz, N.; and Reisig, W., eds., *Fields of Logic and Computation*, volume 6300 of *Lecture Notes in Computer Science*, 488–503. Springer.

Pearce, D.; de Guzmán, I. P.; and Valverde, A. 2000. A tableau calculus for equilibrium entailment. In Dyckhoff, R., ed., *TABLEAUX*, volume 1847 of *Lecture Notes in Computer Science*, 352–367. Springer.

Vakarelov, D. 1977. Notes on constructive logic with strong negation. *Studia Logica* 36:110–125.