

# **The Algebra of Multi-Agent Dynamic Belief Revision**

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## Multi-Agent Dynamic Belief Revision

- Belief Revision: a formal system about revising theories
- Dynamic Belief Revision: a formal system about revising theories **following actions**
- Multi-Agent Dynamic Belief Revision: a formal system about revising theories held by **different agents** following communication actions

Belief Revision is a particular form of epistemic update.

## Belief Revision and Epistemic Update

- Agent's view of the world is  $m$ .
- Communication action  $q$  happens.
- Agent's **updated** view of the world after the action is  $m \otimes q$ .
- If the updated view is contradictory  $m \otimes q = \perp$ , then we **revise** agent's view according to the action  $m * q$ .
- The revised view no longer contradicts the action

$$(m * q) \otimes q \neq \perp.$$

## Algebra of Dynamic Belief Revision

An extension of the algebra for Dynamic Epistemic Logic

- Precise mathematical structure
- A minimal language with a set of axioms
- Non-boolean: reflecting partiality of information
- Resource-sensitive
- Residuated modalities as in Moortgaat.

## Syntax

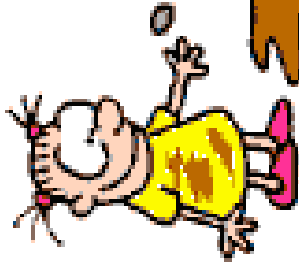
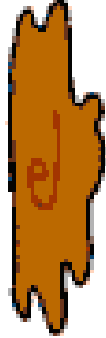
**Propositional Connectives:**  $\left\{ \begin{array}{ll} \top, \perp & \text{true and false propositions} \\ m \wedge m' & \text{conjunction} \\ m \vee m' & \text{disjunction} \\ f_A(m) & \text{agent } A\text{'s view of } m \\ \Box_A m & \text{agent } A \text{ knows that } m \\ p & \text{facts} \end{array} \right\}$

**Action Connectives:**  $\left\{ \begin{array}{ll} \epsilon, \perp & \text{skip and impossible actions} \\ q \vee q' & \text{non-deterministic choice} \\ q \bullet q' & \text{sequential composition} \\ f_A(q) & \text{agent } A\text{'s view of } q \\ \Box_A q & \text{agent } A \text{ knows that } q \text{ is happening} \end{array} \right\}$

**Mixed Connectives:**  $\left\{ \begin{array}{ll} m \otimes q & \text{update proposition } m \text{ by action } q \\ [q]m & \text{after action } q, \text{ proposition } m \text{ holds} \end{array} \right\}$

What do we have to specify?

Algebra		Kripke Semantics
Agents	$A, B$	agents
Facts	$p$	facts
Propositions	$m, m'$	sets of states
Partial order on props	$m \leq m'$	entailment
Views of agents about props	$f_A(m)$	accessibility relation
Actions	$q$	sets of states
Views of agents about actions	$f_A(q)$	accessibility relation
Kernel of each action	$ker(q)$	precondition



**Example**

## Encoding the muddy children puzzle

**Agents:** four children, three of them dirty  $\{1, 2, 3, 4\}$

**Facts:**  $D_i$  : child  $i$  is dirty,  $\bar{D}_i$  : child  $i$  is clean

**Propositions:** all possible situations  $s_{2,3} \quad s_{1,2,3}$

**Partial order:**  $s_{1,2,3} \leq D_1$

**Views:** in each situation each child thinks either he is dirty or clean

$$\begin{aligned} f_i(s_\beta) &= s_{\beta \cup i} \vee s_{\beta \setminus i} \\ f_1(s_{1,2,3}) &= s_{1,2,3} \vee s_{2,3} \end{aligned}$$

**Father's announcement:**  $q_0$

$$f_i(q_0) = q_0, \quad \ker(q_0) = D_0$$

**Children's replies:**  $q$

$$f_i(q) = q, \quad \ker(q) = \wedge_i \square_i D_i$$



## Axioms of the Algebra

Propositions  $\left\{ \begin{array}{l} f_A(\perp) = \perp \\ f_A(m \vee m') = f_A(m) \vee f_A(m') \end{array} \right.$  contradiction  
disjunction-preserving

Actions  $\left\{ \begin{array}{l} q \bullet \epsilon = \epsilon \bullet q = q \\ f_A(\epsilon) = \epsilon \\ f_A(q \vee q') = f_A(q) \vee f_A(q') \\ f_A(q \bullet q') = f_A(q) \bullet f_A(q') \end{array} \right.$  skip action  
skip is public  
disjunction-preserving  
composition-preserving

Mixed  $\left\{ \begin{array}{l} p \otimes q \leq p \\ m \otimes (q \vee q') = (m \otimes q) \vee (m \otimes q') \\ (m \vee m') \otimes q = (m \otimes q) \vee (m' \otimes q) \\ m \otimes (q \bullet q') = (m \otimes q) \otimes q' \\ f_A(m \otimes q) \leq f_A(m) \otimes f_A(q) \end{array} \right.$  facts are update-stable  
disjunction-preserving  
disjunction-preserving  
associativity of update  
update inequality

$f_A(m) \leq m' \Leftrightarrow m \leq \square_A m'$  epistemic adjunction  
 $m \otimes q \leq m' \Leftrightarrow m \leq [q]m'$  dynamic adjunction

## Solving the muddy children puzzle

### Proposition

$$s_{1,2,3} \leq [q_0 \bullet q \bullet q] \square_1 D_1$$

### (I) Induction

$$s_{2,3} \otimes q_0 \otimes q \in \ker(q) \Rightarrow s_{2,3} \otimes q_0 \otimes q \otimes q = \perp \leq D_1$$

### (II) Stability of Fact

$$s_{1,2,3} \leq D_1 \Rightarrow s_{1,2,3} \otimes q_0 \otimes q \otimes q \leq D_1$$

$$\text{(I) and (II):} \quad (s_{1,2,3} \vee s_{2,3}) \otimes q_0 \otimes q \otimes q \leq D_1$$

### Apply assumptions and axioms (taste of heaven!)

$$\begin{aligned} f_1(s_{1,2,3}) \otimes f_1(q_0) \otimes f_1(q) \otimes f_1(q) &\leq D_1 \\ f_1(s_{1,2,3} \otimes q_0 \otimes q \otimes q) &\leq D_1 \\ s_{1,2,3} \otimes q_0 \otimes q \otimes q &\leq \square_1 D_1 \\ s_{1,2,3} &\leq [q_0 \bullet q \bullet q] \square_1 D_1 \end{aligned}$$

## Cheating Muddy Children

After father's announcement: **action**  $q_0$ , in the first round all 4 reply 'No!': **action**  $q$

Then, children 2 and 3 cheat by secretly communicating to each other that they are dirty: **action**  $\pi$ . As a result, 1 and 4 reply 'No!', and 2 and 3 reply 'Yes!' in the second round: **mixed action**  $q'$ . This makes children 1 and 4 confused!

Child 1 wrongly concludes that he is clean

$$s_{1,2,3,4} \leq [q_0 \bullet q \bullet \pi \bullet q'] \square_1 \overline{D}_1.$$

Child 4 goes crazy

$$s_{1,2,3,4} \leq [q_0 \bullet q \bullet \pi \bullet q'] \square_4 \perp.$$

## Help Child 4: Belief Revision

**Goal:** We do not want agents to get contradictory views from actions  $f_A(m \otimes q) = \perp$  when there is no contradiction in the real world  $m \otimes q \neq \perp$ .

$$f_A(m \otimes q) = \perp \Rightarrow f_A(m) \otimes f_A(q) = \perp$$

**Method:**

Use agent's Revised Belief instead of his updated belief

replace  $f_A(m) \otimes f_A(q) = \perp$  with  $f_A(m) * f_A(q) \neq \perp$

## Axioms of Belief Revision

1.  $\exists m' \in M, m * q = m' \otimes q$

Consistency of revised view with action.

2. If  $m \otimes q \neq \perp$  then  $m * q = m \otimes q$

3. If  $m * q = \perp$  then  $q = \perp$

4.  $f_A(m \otimes q) \leq f_A(m) * f_A(q)$

Replace the updated belief with revised belief.

## Axioms of Belief Revision

**Lemma:**

The revised belief is the weakest belief consistent with the action

$$m * q = \bigvee \{m'' \mid m'' \otimes q \neq \perp\}.$$

Note: actions can also be revised  $q * q'$ , details in the proceedings.

## Revise Belief of Child 4

In the cheating muddying children, child 4 is fine until the last round of answers:

$$f_4(s_{1,2,3} \otimes q_0 \otimes q \otimes \pi) \neq \perp$$

But after the last round, child 4 gets a contradiction

$$\begin{aligned} f_4(s_{1,2,3} \otimes q_0 \otimes q \otimes \pi) \otimes f_4(q') &= \perp \\ f_4(s_{1,2,3} \otimes q_0 \otimes q \otimes \pi) \otimes q' &= \perp \end{aligned}$$

In order to avoid this, we revise his beliefs

$$f_4(s_{1,2,3} \otimes q_0 \otimes q \otimes \pi) * q' \neq \perp$$

So child 4 does not get a contradiction any more

$$s_{1,2,3} \otimes q_0 \otimes q \otimes \pi \otimes q' \not\vdash \square_4 \perp.$$

## Calculate the Revised Belief of Child 4

$$f_4(s_{1,2,3} \otimes q_0 \otimes q \otimes \pi) * q' = m \otimes q'$$

Make child 4 suspect the cheating:

$$\text{change } f_4(\pi) = \epsilon \text{ to } f_4(\pi) = \pi \vee \epsilon$$

Contradiction gets eliminated

$$f_4(s_{1,2,3} \otimes q_0 \otimes q \otimes \pi) \otimes q' = f_4(s_{1,2,3}) \otimes q_0 \otimes q \otimes (\pi \vee \epsilon) \otimes q' \neq \perp$$

The revised belief of child 4 is

$$m = f_4(s_{1,2,3}) \otimes q_0 \otimes q \otimes (\pi \vee \epsilon).$$

instead of the original

$$f_4(s_{1,2,3}) \otimes q_0 \otimes q \otimes \epsilon.$$



## The Mathematical Structure behind all this

The mathematical structure that governs all this is a

### Belief Revision Epistemic System

- A system, i.e. a pair module-quantale  $(M, Q)$  that encodes dynamics,
- with an update action  $- \otimes - : M \times Q \rightarrow M$  satisfying some axioms,
- and a belief revision operator  $- * - : M \times Q \rightarrow M$  also satisfying some axioms.

- The system is endowed with pairs of homomorphisms

$$f_A = (f_A^M : M \rightarrow M, f_A^Q : Q \rightarrow Q)$$

that encode epistemics.

- It satisfies the revision inequality:

$$f_A(m \otimes q) \leq f_A(m) * f_A(q).$$

## Conclusion

- Belief Revision is a particular form of epistemic update
- Epistemic update is usually done using Kripke structures
- I have presented an algebraic way to do it
  - Strong mathematical structure as its basis
  - Explicit structure on epistemic actions
  - Uniform way of dealing with modalities: residuation
  - **Multi-agent belief revision**

## Future Work

- Connections with AGM axioms
- Sequent calculus

## References

- 1- A. Baltag and L.S. Moss, 'Logics for epistemic programs', *Synthese* **139**, 2004.
- 2- A. Baltag, M. Sadrzadeh, 'The algebra of multi-agent dynamic belief revision'.
- 3- A. Baltag, B. Coecke and M. Sadrzadeh, 'Algebra and sequent calculus for epistemic actions', in *ENTCS*, Vol. 126, March 2005, pp. 27-52.