Deciding Non-Compressible Blocks in Sparse Direct Solvers using Incomplete Factorization

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Context: Sparse direct solvers

Current status for 3D problems of size n^3

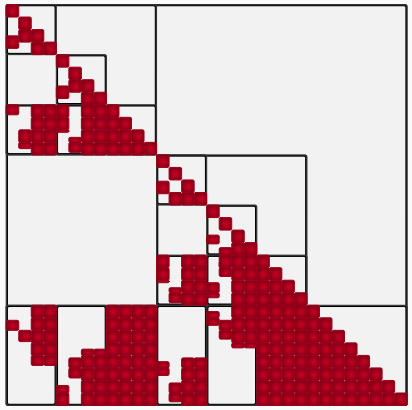
- $\Theta(n^2)$ time complexity
- $\Theta(n^{\frac{4}{3}})$ memory complexity
- BLAS Level 3 operations with computations on blocks

Exploit low-rank compression to reduce the cost

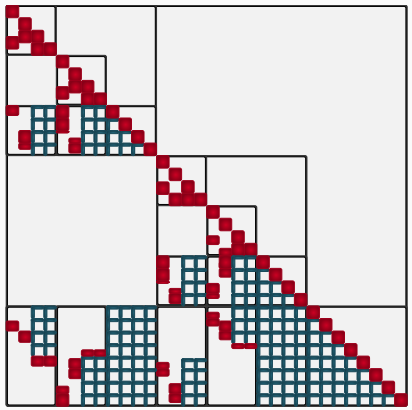
- Compress diagonal and/or off-diagonal blocks
- Use different compression formats: BLR, $\mathcal{H}$, HODLR, HSS, $\mathcal{H}^2$..
- Recently introduced in many solvers:
  MUMPS, PASTIX, STRUMPACK and many others

Block Low-Rank (BLR) compression scheme (in PaStiX)

Full rank:



Block low-rank:



Approach

- Keep the same structure as in full rank used to increase parallelism / block efficiency
- Diagonal blocks are kept dense
- Off-diagonal blocks are compressed wrt an admissibility criterion (block size)

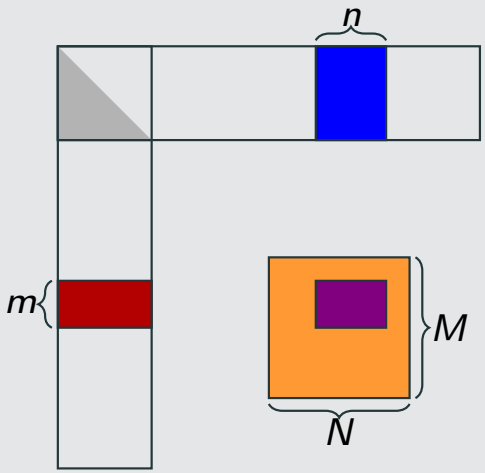
Operations

- TRSM are always performed on the **low-rank** form
- Two scenarios to apply the GEMM updates

Strategies to apply the updates

Cost of applying a $m \times n$ update to a $M \times N$ block.

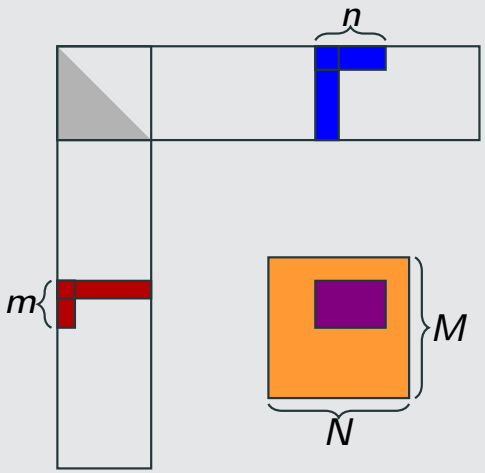
Full rank



- A single GEMM kernel
- Cost depends on mn

Non-fully structured

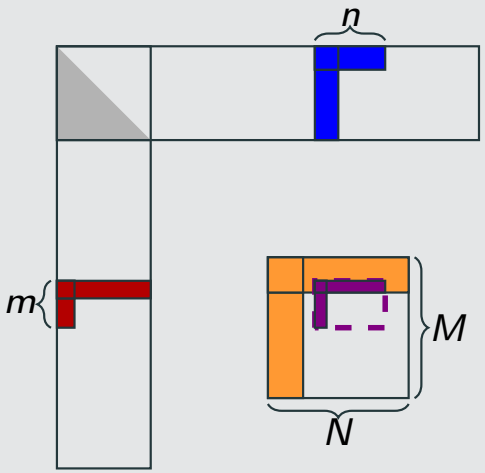
Just-In-Time



- Small GEMM kernels
- Cost depends on mn

Fully structured

Minimal Memory



- Complex operation
- Cost depends on MN

The two compression scenarios

Algorithm 1 Minimal Memory - Just in Time Scenarios

```
1: for each off-diagonal block A_ij in A do
2:     Compress(A_ij)                          /* Compress if admissible */
3: for k=1 to N_cblk do
4:     Factorize
5:     for each off-diagonal block A_ij in cblk do
6:         Compress(A_ij)                      /* Compress if admissible */
7:     Solve
8:     Update
```

Just in Time

- ALAP Compression
  - *After all Write*
  - *Before all Read tasks*
- Non fully structured updates

✗ Do not reduce the memory peak

✓ Reduce the flop count

✓ Reduce the time to solution

Minimal Memory

- ASAP Compression
  - *Before any other operation*
- Fully structured updates

✓ Reduce memory footprint

✗ May increase the flop count

✗ May increase the time to solution

Objective

Identify the high-rank blocks to reduce the fully structured updates overhead

Main ideas

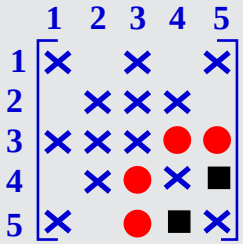
- Allow low memory overhead to save flops
- Do not attempt to compress blocks that end up full-rank
- Identify blocks with numerous updates of smaller sizes
- Identify blocks with few updates of similar sizes

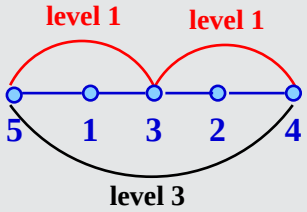
Proposed idea

Exploit the block Incomplete LU (ILU) factorization levels

Block ILU Factorization

Recalls on ILU





The levels of the fill-in values (blocks) in L somehow represent the *distance* to the original matrix A.

Easy and cheap heuristic to apply in a block-wise manner.

Low level blocks

- Have larger ranks
- Marginally improve memory footprint
- Receive many updates

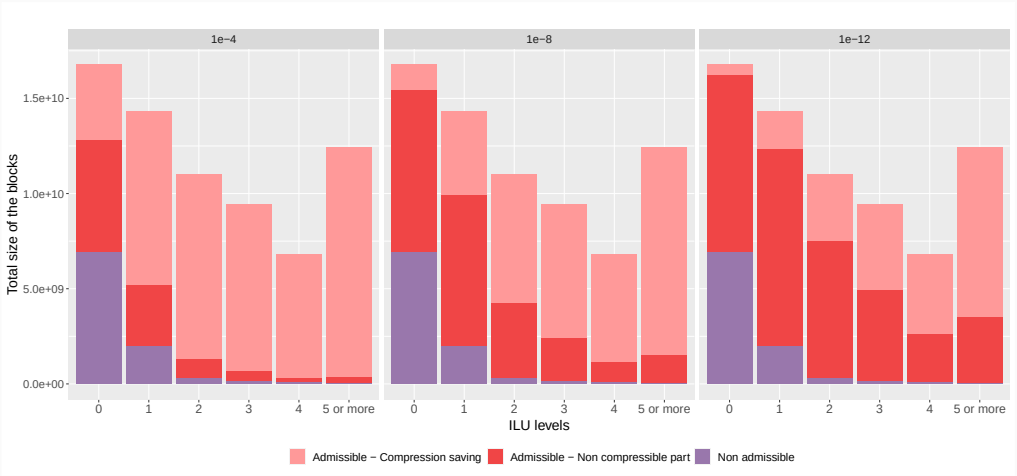
$\rightarrow$ Can be compressed ALAP

High level blocks

- Have smaller ranks
- Greatly improve memory footprint
- Receive few updates

$\rightarrow$ Can be compressed ASAP

Compression ratio wrt ILU block levels and tolerance



- Statistics on a set of 31 matrices from SuiteSparse Matrix Collection
- Dark red represents the memory used by the admissible low-rank blocks after compression
- Light red shows the amount of memory that can be saved by compressing the blocks
- Purple represents the memory used by the non-admissible blocks

- The higher the level, the higher the memory saving
- For a low tolerance criterion, good memory savings can be obtained considering only small levels of fill

Block ILU level definition

Algorithm 2 Symbolic ILU(*maxlevel*) Factorization

```
1:  /* Initialize the levels */
2:  for all L_ij in L do
3:      if A_ij ≠ 0 then
4:          lvl(L_ij) = 0
5:      else
6:          lvl(L_ij) = ∞
7:  /* Simulate the block factorization */
8:  for k = 1 to n do
9:      for i = k + 1 to n do
10:         for j = k + 1 to n do
11:             lvl(L_ij) = min(lvl(L_ij), lvl(L_ik) + lvl(L_kj) + 1)
```

BLR algorithm with the new ILU(*maxlevel*) heuristic

Algorithm 3 Minimal Memory - Just in Time - ILU(*maxlevel*) Scenarios

```
1:  for each off-diagonal block A_ij in A do
2:      if level(A_ij) > maxlevel then
3:          Compress(A_ij)                    /* Compress if admissible */
4:  for k=1 to N_cblk do
5:      Factorize
6:      for each off-diagonal block A_ij in cblk do
7:          if level(A_ij) <= maxlevel then
8:              Compress(A_ij)                /* Compress if admissible */
9:      Solve
10:     Update
```

maxlevel defines the criterion to switch from ASAP to ALAP compression according to the input matrix and tolerance

Experiments

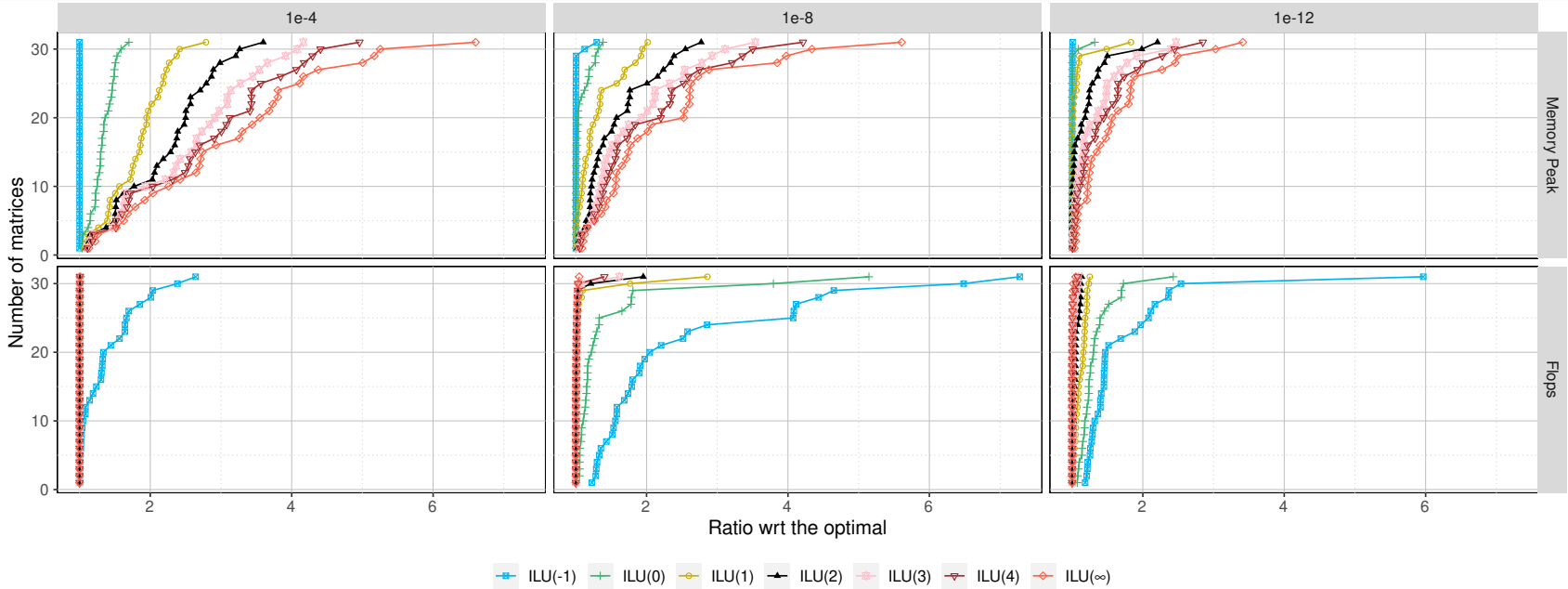
Context

- On a set of 31 matrices from the SuiteSparse Matrix Collection
- On a two INTEL Xeon E5-2680v3 12-cores running at 2.50 GHz and 128 GB of memory

Presentation of results

- Performance profile curve: each curve represents the number of matrices within a percentage overhead of the best solution for each metric and matrix
- We compare the existing approaches, *Minimal Memory* (or $ILU(-1)$) and *Just-In-Time* (or $ILU(\infty)$), with the new heuristic based on levels of fill

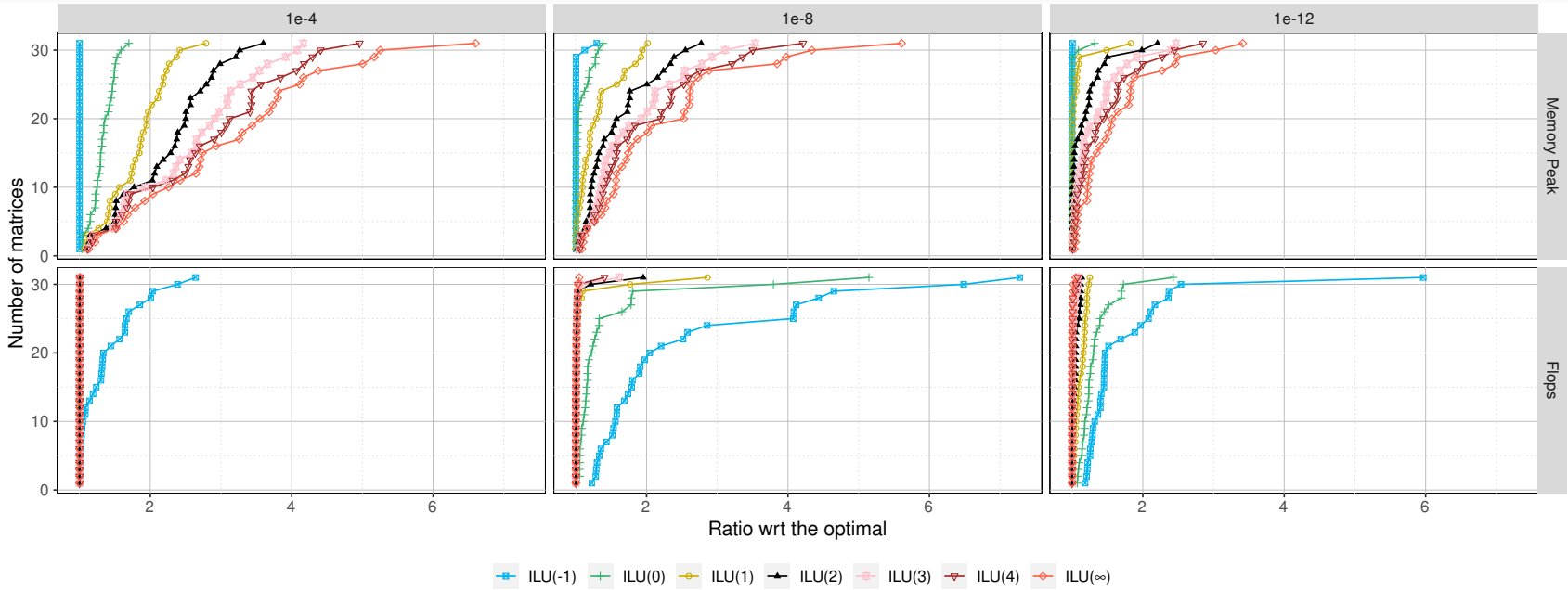
Impact of the ILU(k) Heuristic on the memory and flops



Memory consumption

- *Minimal Memory* reaches the smallest memory consumption, while *Just-In-Time* reaches the highest
- The lower the level, the higher the memory consumption
- *ILU*(1) reaches a good compromise in average

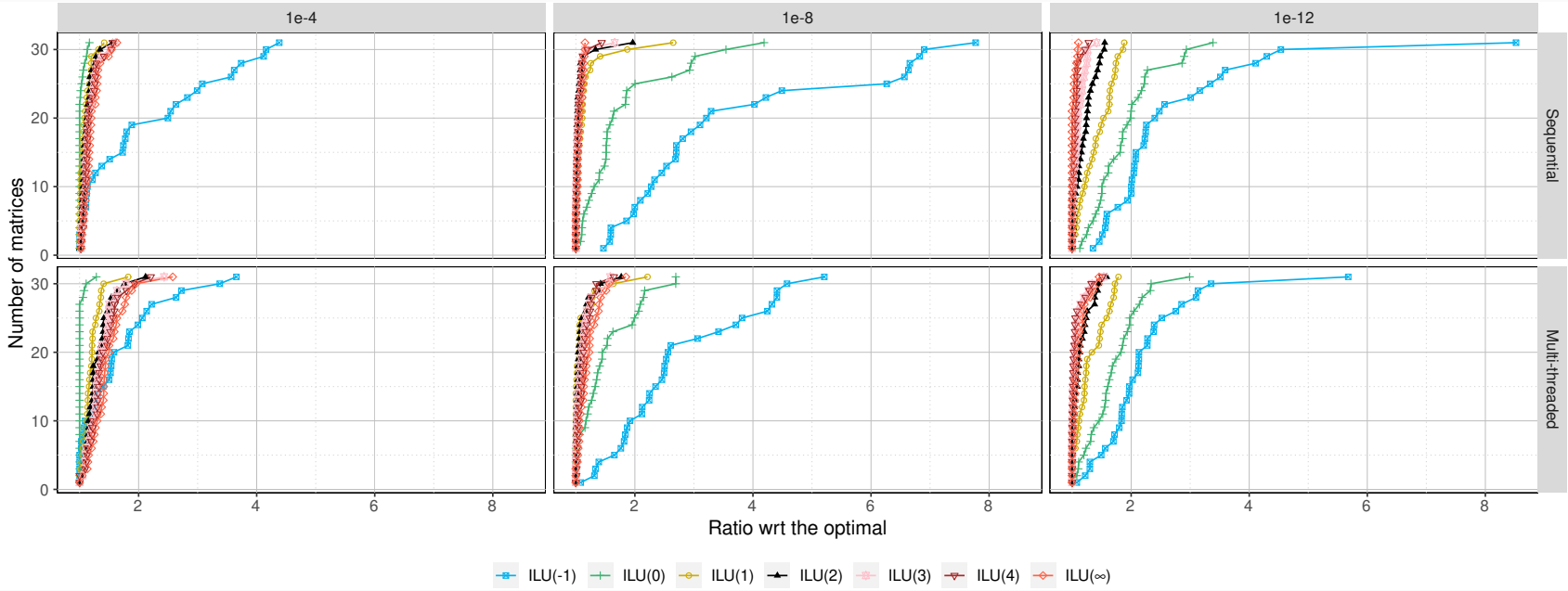
Impact of the ILU(k) Heuristic on the memory and flops



Flops

- *Just-In-Time* has the smallest flops count, while *Minimal Memory* has the highest
- The lower the level, the lower the flops count
- In many cases, the new heuristic is almost mingled with *Just-In-Time*

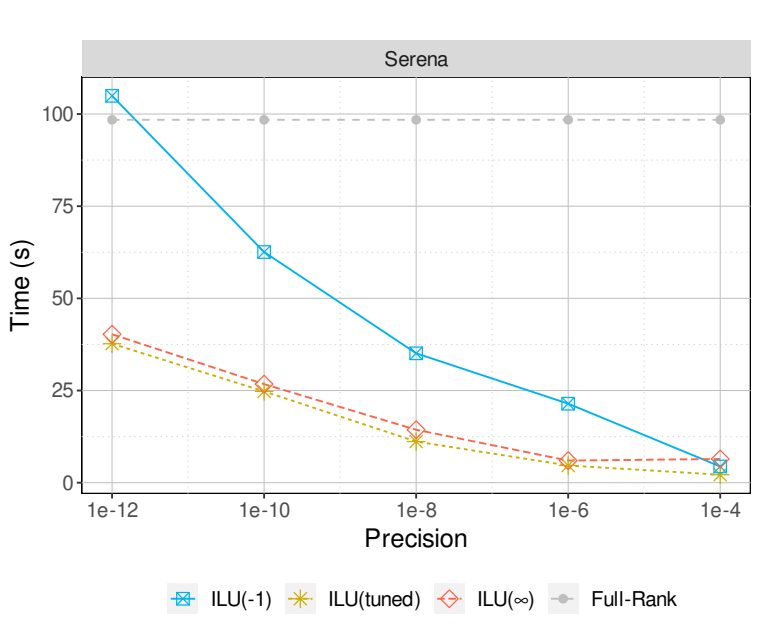
Impact of the ILU(k) Heuristic on the factorization time



Sequential vs multi-threaded

- The trend on the factorization time follows the trend on flops
- The ILU(k) heuristic behaves better than the *Just-In-Time* strategy in parallel: its execution time if often lower, especially for low tolerance

Tolerance scalability on the Serena Matrix



Tolerance	$ILU(tuned)$
$1e^{-4}$	$ILU(0)$
$1e^{-6}$	$ILU(2)$
$1e^{-8}$	$ILU(2)$
$1e^{-10}$	$ILU(4)$
$1e^{-12}$	$ILU(4)$

Table 1: Corresponding levels for $ILU(tuned)$ at each precision.

Observations

- By tuning correctly the level of fill for each tolerance, we can always outperform both the *Minimal Memory* and the *Just-In-Time* strategies
- Recall that when $ILU(k)$ is faster than *Just-In-Time*, it also reduces significantly its memory consumption

Conclusion

The ILU(k) heuristic

- Allows to reach good trade-offs between the *Minimal Memory* and the *Just-In-Time* strategies
- The new heuristic in sequential provides huge speedup and reduces flops with only a slight memory increase
- The new heuristic in parallel is even faster than *Just-In-Time*
- It is possible to tune the level of fill to always consume as low memory as *Minimal Memory* or to reduce execution time compared to *Just-In-Time*

Future work

- Infer automatically the level depending on the user's requirements
- Order unknowns to gather contributions with a similar level of fill