Subdivision as a Sequence of Sampled C^p Surfaces

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Summary. This article deals with practical conditions for tuning a subdivision scheme in order to control its artifacts in the vicinity of a mark point. To do so, we look for good behaviour of the limit vertices rather than a good mathematical property of the limit surface. The good behaviour of the limit vertices is characterized with the definition of C^2-convergence of a scheme. We propose necessary explicit conditions for C^2-convergence of a scheme in the vicinity of any mark point being a vertex of valency greater or equal to three.

1 Introduction

A bivariate subdivision scheme defines a sequence of polygonal meshes each of whose vertices is a linear combination of vertices belonging to the previous mesh in the sequence. Such a scheme is interesting if the sequence converges to a surface which is as regular as possible. Tuning is the choosing of coefficients for the linear combinations used in constructing new points. This article deals with conditions which may be used for tuning a scheme in order to get such a sequence.

Some schemes (Loop [6], Catmull-Clark [3], Doo-Sabin [4]...) are defined so that each polygonal mesh is the control polyhedron of a Box-Spline surface which is the limit surface of the sequence. In this case the convergence and regularity problems are solved by definition, except around extraordinary vertices. An extraordinary vertex is a vertex of the mesh whose valency is not equal to six if the mesh faces are triangles, or not equal to four if the mesh faces are quadrilaterals. For extraordinary vertices in a Box-Spline based scheme and for all vertices in other schemes, the convergence of the scheme and the regularity of its limit surface in the vicinity of a vertex need to be analysed.

This analysis may lead to a tuning of the scheme. In most cases, the coefficients of the linear combinations depend only on the local topology of the mesh, and not on its geometry. Moreover, we assume the scheme to be stationary : the coefficients remain the same through the sequence of polygonal meshes.

The first analysis of the behaviour of the limit surface around an extraordinary vertex was by Doo and Sabin [4]. They give necessary conditions for a scheme being convergent towards a C^2-continuous limit surface. These conditions are derived from estimate of its first and second derivatives around the extraordinary vertex. Subsequently, the majority of works interpret this question as follows: the mesh around but excluding the extraordinary vertex is the control polyhedron of continuous patches. At each subdivision step, a new ring of such patches fill in a part of the n-sided hole created by the virtual removal of the extraordinary vertex. The authors analyse how this iterative insertion of new rings converges and fills in completely this hole. Ball and Storry [1] give sufficient conditions for the surface being tangent plane continuous. Reif [10] remarks that in terms of differential geometry, a surface is C^p-continuous if there exists a C^p function which parameterizes it. If the limit surface is tangent plane continuous, the surface can be parametrized over a characteristic map of this tangent plane around the extraordinary vertex. Reif gives necessary and sufficient conditions for any stationary scheme to be convergent towards a C^1-continuous surface. Independently, Prautzsch [8] and Zorin [15] proposed in the late 90's necessary and sufficient conditions for a scheme to be convergent towards a C^p-continuous surface. They both use more or less the parametrization over the characteristic map proposed by Reif.

Most of the prior work on tuning subdivision schemes alter local coefficient in order to fulfil the previous necessary and sufficient conditions [12]. But a mathematical C^p-continuity of the limit surface is perhaps not the best target to aim for. We may look for good behaviour of the limit vertices rather than a good mathematical property of the limit surface. Good behaviour of the limit vertices may mean less artifacts on the limit surface [13]. For instance, Prautzsch and Umlauf tuned the Loop and Butterfly schemes in order to make them C^1 and C^2-continuous around an extraordinary vertex by creating a flat spot [9]; but a flat spot may be considered as an artifact. Furthermore, the necessary and sufficient conditions for C^2-continuity of the limit surface are not explicit if the scheme is not Box-Spline based.

In this paper, we characterize the good behaviour of the limit vertices with the definition of C^2-convergence of a scheme. This definition is based on the interpretation proposed implicitly by Doo and Sabin [4]. Each control mesh is viewed not as the control polyhedron of a Box-Spline surface but as the sampling of a continuous surface. Thus the sequence of meshes are samplings of a sequence of continuous surfaces which converges uniformly towards the limit surface. Naturally, C^2-convergence of a scheme is related to the C^2-continuity of the limit surface: it is a sufficient condition for it.

And because the definition of C^2-convergence of a scheme is *theoretical* and formal, we propose in this paper explicit but only necessary conditions for C^2-convergence.

In the following section, we present the theoretical tools we use in Sect. 3 to establish the necessary conditions for a scheme to C^2-converge.

2 Theoretical Tools

We first describe our notation and then we propose the definition for the C^p-convergence of a scheme. From this definition, we derive a description of the limit points. Finally the eigenanalysis of the Fourier transformed subdivision matrix gives a description of the limit frequencies.

2.1 Notation

We study the convergence of a subdivision scheme towards a regular surface in the vicinity of a vertex being a mark point. A mark point is a point of a mesh whose vicinity keeps the same topology throughout subdivision. For instance, mesh vertices are mark points in the case of Loop or Catmull-Clark subdivision schemes, and face centres but not mesh vertices are mark points in the case of Doo-Sabin refinement. As a consequence, our analysis does not apply to Doo-Sabin nor to other schemes where the vertices are not mark points. The generalization of this analysis to any markpoint being a vertex or a face centre can be found in a technical report [5].

Let A be the markpoint, and n its valency (number of outgoing edges from A). We assume that the vicinity of A is made up of ordinary vertices. This hypothesis is relevant because after a subdivision step, the vertices of the mesh map to vertices with the same valency, and new vertices are created which are all ordinary. Thus, after several subdivision steps, every extraordinary vertex is surrounded by a sea of ordinary vertices. As a consequence, the vicinity of A may be divided into n topologically equivalent sectors. In the jth sector, let B_j, C_j, $D_j\ldots$ be an infinite number of vertices sorted from the topologically nearest vertex from A to the farthest. If there exist two vertices in one sector on the same ring which are in complementary positions then they are labelled with the same letter, but with a prime put on the vertex which is further anticlockwise from the positive x-axis. An example is E and E' in Fig. 1. However, if the points are not in complementary positions, then they are given distinct letters.

Let $A^{(k)}$ be the markpoint and $\left\{B_j^{(k)}, C_j^{(k)}, D_j^{(k)}\ldots\right\}_{j\in 1\ldots n}$ its vicinity after k subdivision steps. All these vertices are put into an infinite vector $\mathbf{P^{(k)}} := \left[A^{(k)}B_1^{(k)}\cdots B_n^{(k)}C_1^{(k)}\cdots C_n^{(k)}D_1^{(k)}\cdots D_n^{(k)}\cdots\right]^T$.

Finally, a surface is C^p-continuous if there exists a C^p-diffeomorphic parametrization of it from a subset of $\mathbb{R}^2$. We define a parametrization domain

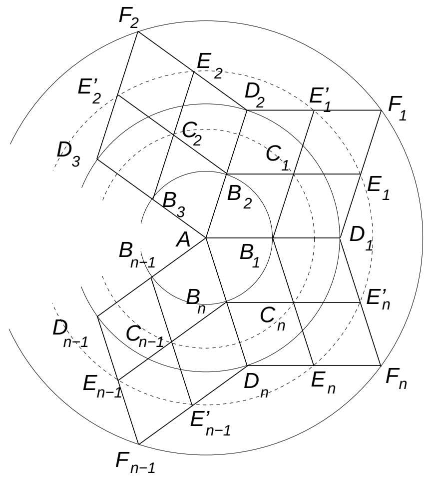

Fig. 1. Labelling of the vicinity of a markpoint

by projecting onto $\mathbb{R}^2$ without folding the polygonal mesh around the markpoint. $A^{(k)}$ is projected onto $(0,0)$, and $\forall X \in \{B, C, D, \ldots\}$, $\forall j \in \{1, \ldots, n\}$, $X_j^{(k)}$ is projected onto $(x_j^{(k)}, y_j^{(k)})$. For simplicity, we ask $(x_j^{(k)}, y_j^{(k)})$ to lie on the same circle for given k and X, and to lie on the same radial axis for given j and X:

$$(x_j^{(k)}, y_j^{(k)}) := (\varrho_X^{(k)} \cos(\theta_{(X,j)}), \varrho_X^{(k)} \sin(\theta_{(X,j)}) ,$$

where

$$\theta_{(X,j)} := \frac{2\pi}{n}(j + \alpha_X) ,$$

Furthermore, because the vertices $X_j^{(k)}$ converge to the limit mark point if the scheme converges [10], we ask that $\lim_{k\to\infty}(\varrho_X^{(k)}) = 0$. The choice of the phases α_X and the radii $\varrho_X^{(k)}$ remains free. These degrees of freedom will be used in the characterization of C^1-convergence in Sect. 3.

2.2 C^p-Convergence and Behaviour of the Limit Points

We propose the following definition for the C^p-convergence of a scheme. The scheme C^p-converges in the vicinity of A if

- for every X in the infinite vicinity $\{B, C, D, \ldots\}$ of A, there exist phases α_X and, for all j in $\{1, \ldots, n\}$, for every k, radii $\varrho_X^{(k)}$ and a C^p-continuous function $\mathcal{F}^{(k)}(x, y)$ such that

$$\begin{aligned} A^{(k)} &= \mathcal{F}^{(k)}(0,0) , \\ X_j^{(k)} &= \mathcal{F}^{(k)}(\varrho_X^{(k)} \cos(\theta_{(X,j)}), \varrho_X^{(k)} \sin(\theta_{(X,j)})) . \end{aligned}$$

- Furthermore, the sequence of pth differentials $\left(d^p\mathcal{F}^{(k)}\right)_k$ converges uniformly onto $d^p\mathcal{F}$ which is the pth differential of a C^p-continuous parameterization $\mathcal{F}(x,y)$ of the limit surface in the vicinity of the limit mark point.
- Finally, for all $q \in 0 \ldots p-1$, the sequence $\left(d^q\mathcal{F}^{(k)}(0,0)\right)_k$ converges onto $d^q\mathcal{F}(0,0)$.

In this definition, an infinite vicinity $\{B, C, D, \ldots\}$ is taken into account. In any practical application, we will consider only a finite number of vertices. More precisely, we choose the set of vertices which will influence the limit position of the mark point and its neighbourhood. This practical restriction is not inconsistent with finding only necessary conditions for C^p-convergence. From the definition, we see that if the scheme C^p-converges in the vicinity of A, then the sequence of meshes converges towards a C^p-continuous surface around the limit markpoint. But the converse is not true: a scheme, which converges towards a C^p-continuous surface is not *necessarily* C^p-convergent. Note also that the definition domain of $\mathcal{F}^{(k)}$ shrinks as k grows since $\lim_{k\to\infty}(\varrho_X^{(k)}) = 0$ from Sect. 2.1.

In Sect. 3 we will be considering the necessary conditions for C^2-convergence. Therefore, consider a scheme which C^2-converges in the vicinity of the markpoint A. The parameterization $\mathcal{F}(x,y)$ is C^2-continuous. From its Taylor expansion around $(0,0)$, we may describe the behaviour of the limit points in the vicinity of A. In the following lines, we detail this behaviour with derivatives of the limit function and according to the regularity of the scheme convergence. If the scheme C^0-converges then $\forall X \in \{B, C, D, \ldots\}$, $\forall j \in \{1, \ldots, n\}$,

$$\lim_{k\to\infty}(A^{(k)}) = \mathcal{F}(0,0) \ , \qquad \text{and} \qquad \lim_{k\to\infty}(X_j^{(k)}) = \mathcal{F}(0,0) \ . \tag{1}$$

If the scheme C^1-converges then $\forall X \in \{B, C, D, \ldots\}$, $\forall j \in \{1, \ldots, n\}$,

$$\lim_{k\to\infty}\left(\frac{X_j^{(k)} - \mathcal{F}^{(k)}(0,0)}{\varrho_X^{(k)}}\right) = \cos(\theta_{(X,j)})\frac{\partial\mathcal{F}}{\partial x}(0,0) + \sin(\theta_{(X,j)})\frac{\partial\mathcal{F}}{\partial y}(0,0) \ . \tag{2}$$

If the scheme C^2-converges then $\forall X \in \{B, C, D, \ldots\}$, $\forall j \in \{1, \ldots, n\}$,

$$\begin{aligned}\lim_{k\to\infty}\left(\frac{\Delta_{X,j}^{(k)}}{{\varrho_X^{(k)}}^2}\right) &= \left(\frac{\partial^2\mathcal{F}}{\partial x^2}(0,0) + \frac{\partial^2\mathcal{F}}{\partial y^2}(0,0)\right)\frac{1}{4} + \frac{\partial^2\mathcal{F}}{\partial x\partial y}(0,0)\frac{\sin(2\theta_{(X,j)})}{2} \\ &\quad + \left(\frac{\partial^2\mathcal{F}}{\partial x^2}(0,0) - \frac{\partial^2\mathcal{F}}{\partial y^2}(0,0)\right)\frac{\cos(2\theta_{(X,j)})}{4} \ .\end{aligned} \tag{3}$$

with

$$\Delta_{X,j}^{(k)} := X_j^{(k)} - \mathcal{F}^{(k)}(0,0) - \varrho_X^{(k)}\left(\cos(\theta_{(X,j)})\frac{\partial\mathcal{F}^{(k)}}{\partial x}(0,0) + \sin(\theta_{(X,j)})\frac{\partial\mathcal{F}^{(k)}}{\partial y}(0,0)\right) \ .$$

2.3 Eigenanalysis of the Transformed Subdivision Matrix

Consideration of the relationship between the spatial and frequency domains allows us to produce necessary conditions for C^2-convergence. In this section we introduce the necessary notation for the subdivision matrix transformed into the frequency domain. We may write the discrete rotational frequencies $\tilde{X}^{(k)}(\omega)$ of each set of vertices $\left\{X_j^{(k)}\right\}_{j\in 1\ldots n}$ by applying a Discrete Fourier Transform. It is well-known [1] that there exists a matrix $\tilde{\mathsf{M}}(\omega)$ such that for all ω in $\left\{-\lfloor\frac{n-1}{2}\rfloor,\ldots,\lfloor\frac{n}{2}\rfloor\right\}$,

$$\tilde{\mathbf{P}}^{(\mathbf{k+1})}(\omega)=\tilde{\mathsf{M}}(\omega)\tilde{\mathbf{P}}^{(\mathbf{k})}(\omega)$$

where, if $\omega\neq 0$, $\tilde{\mathbf{P}}^{(\mathbf{k})}(\omega):=\left[\tilde{B}^{(k)}(\omega)\tilde{C}^{(k)}(\omega)\tilde{D}^{(k)}(\omega)\cdots\right]^T$ and otherwise $\tilde{\mathbf{P}}^{(\mathbf{k})}(0):=\left[\tilde{A}^{(k)}(0)\tilde{B}^{(k)}(0)\tilde{C}^{(k)}(0)\tilde{D}^{(k)}(0)\cdots\right]^T$.

For every discrete rotational frequency ω, the matrix $\tilde{\mathsf{M}}(\omega)$ is supposed to be non defective (otherwise we should use the canonical Jordan form).

$$\tilde{\mathsf{M}}(\omega)=\tilde{\mathsf{V}}(\omega)^{-1}\tilde{\Lambda}(\omega)\tilde{\mathsf{V}}(\omega)$$

where the columns $\tilde{\mathbf{v}}_{\mathbf{l}}(\omega)$ of $\tilde{\mathsf{V}}(\omega)^{-1}$ are the right eigenvectors of $\tilde{\mathsf{M}}(\omega)$, the rows $\tilde{\mathbf{u}}_{\mathbf{l}}^{\mathbf{T}}(\omega)$ of $\tilde{\mathsf{V}}(\omega)$ are the left eigenvectors of $\tilde{\mathsf{M}}(\omega)$, and $\tilde{\Lambda}(\omega)$ is diagonal whose diagonal components $\tilde{\lambda}_l(\omega)$ are the eigenvalues of $\tilde{\mathsf{M}}(\omega)$, with $l\geq 1$. Let $L_l^-(\omega)$, $L_l(\omega)$, and $L_l^+(\omega)$ be sets of indices such that if $q\in L_l^-(\omega)$, then $\left|\tilde{\lambda}_q(\omega)\right|<\left|\tilde{\lambda}_l(\omega)\right|$, if $q\in L_l(\omega)$, then $\left|\tilde{\lambda}_q(\omega)\right|=\left|\tilde{\lambda}_l(\omega)\right|$, and if $q\in L_l^+(\omega)$, then $\left|\tilde{\lambda}_q(\omega)\right|>\left|\tilde{\lambda}_l(\omega)\right|$, where $\left|\tilde{\lambda}\right|$ is the modulus of the complex $\tilde{\lambda}$. Then with $\mathcal{P}(q,\omega)=\tilde{u}_q(\omega)^T\tilde{P}^{(0)}(\omega)$, we get for every l,

$$\begin{aligned}\tilde{\mathbf{P}}^{(\mathbf{k})}(\omega)&-\sum_{q\in L_l^+(\omega)}\tilde{\lambda}_q(\omega)^k\mathcal{P}(q,\omega)\tilde{\mathbf{v}}_{\mathbf{q}}(\omega)\\&=\tilde{\lambda}_l(\omega)^k\left(\sum_{q\in L_l(\omega)}\mathcal{P}(q,\omega)\tilde{\mathbf{v}}_{\mathbf{q}}(\omega)+\sum_{q\in L_l^-(\omega)}\left(\frac{\tilde{\lambda}_q(\omega)}{\tilde{\lambda}_l(\omega)}\right)^k\mathcal{P}(q,\omega)\tilde{\mathbf{v}}_{\mathbf{q}}(\omega)\right).\end{aligned}\tag{4}$$

Thus, as k grows to infinity, $\tilde{\lambda}_l(\omega)^k\sum_{q\in L_l(\omega)}\mathcal{P}(q,\omega)\tilde{\mathbf{v}}_{\mathbf{q}}(\omega)$ is a good estimate of the frequency $\tilde{\mathbf{P}}^{(\mathbf{k})}(\omega)-\sum_{q\in L_l^+(\omega)}\tilde{\lambda}_q(\omega)^k\mathcal{P}(q,\omega)\tilde{\mathbf{v}}_{\mathbf{q}}(\omega)$ in the same way that $\sum_{a+b=l}\frac{x^ay^b}{l!}\frac{\partial^l\mathcal{F}}{\partial x^a\partial y^b}(0,0)$ is a good estimate of the function $\mathcal{F}(x,y)-\sum_{a+b<l}\frac{x^ay^b}{l!}\frac{\partial^l\mathcal{F}}{\partial x^a\partial y^b}(0,0)$ as (x,y) converges to $(0,0)$.

3 Necessary Conditions for C^2-Convergence and Derivatives of the Limit Surface

Equations (1), (2) and (3) describe the behaviour of the limit points. Applying the Discrete Fourier Transform on these equations gives a description of the limit frequencies. The consistency between this description with the one given by equation (4) implies necessary conditions for the C^2-convergence of the scheme. It gives also the partial derivatives of the limit surface in the mark point. As a notation, if $\tilde{X}^{(k)}(\omega)$ is the mth component of $\tilde{\mathbf{P}}^{(\mathbf{k})}(\omega)$, then $(\tilde{v}_l(\omega))_X$ is the mth component of $\tilde{\mathbf{v}}_{\mathbf{l}}(\omega)$. We assume also without any restriction that for every fixed ω, $\tilde{\lambda}_2(\omega)$ is the eigenvalue of $\tilde{\mathsf{M}}(\omega)$ with the greatest modulus after $\tilde{\lambda}_1(\omega)$ and the possibly others eigenvalues with same modulus as $\tilde{\lambda}_1(\omega)$: for all ω, $L_1(\omega) = L_2^+(\omega)$.

3.1 C^0-Convergence

If the scheme C^0-converges, then

$$\begin{cases} \tilde{\lambda}_1(0) = 1 \quad , \\ \left|\tilde{\lambda}_1(\omega)\right| < 1 \;\; \text{if } \omega \neq 0, \end{cases}$$

and if $L_1(0) = \{1\}$,

$$(\tilde{v}_1(0))_X = \nu_0$$

with ν_0 being a constant, and

$$\mathcal{F}(0,0) = \frac{\mathcal{P}(1,0)}{n}\,(\tilde{v}_1(0))_X \,.$$

Not only we get necessary conditions on eigenvalues and eigenvectors of $\tilde{\mathsf{M}}(\omega)$, but we get also the value of $\mathcal{F}(0,0)$, that is the limit mark point.

3.2 C^1-Convergence

If the scheme C^1-converges and the markpoint is a vertex, then

$$\left|\tilde{\lambda}_2(0)\right| < \left|\tilde{\lambda}_1(\pm 1)\right| \;\text{ and }\; \left|\tilde{\lambda}_1(\omega)\right| < \left|\tilde{\lambda}_1(\pm 1)\right| \;,$$

with $\omega \notin \{-1, 0, 1\}$.

Furthermore, when k is *large*, if $L_1(1) = L_1(-1) = \{1\}$, the moduli of the eigencomponents $|(\tilde{v}_1(1))_X|$ and $|(\tilde{v}_1(-1))_X|$ are sorted like the rings radii ϱ_X.

If the scheme is rotational invariant, the modulus of the eigenvalue $\left|\tilde{\lambda}_1(1)\right| = \left|\tilde{\lambda}_1(-1)\right|$ gives the speed of the parameters' shrinkage during the subdivision process. The freedom we had in the choice of the parameters $\varrho_X^{(k)}$

and the phase α_k is restricted. But there remains enough freedom to write things quite simply. For simplicity, we can define the radii $\varrho_X^{(k)}$ as follows:

$$\varrho_X^{(k)} = \left|\tilde{\lambda}_1(1)\right|^k \left|(\tilde{v}_1(1))_X\right| = \left|\tilde{\lambda}_1(-1)\right|^k \left|(\tilde{v}_1(-1))_X\right| \ .$$

Besides, if we define α_X as

$$\alpha_X = \frac{n}{2\pi}\varphi_{(\tilde{v}_1(-1))_X}$$

with $\varphi_{(\tilde{v}_1(-1))_X}$ being the phase of $(\tilde{v}_1(-1))_X$, and if

$$\frac{\partial\mathcal{F}}{\partial x}(0,0) \pm \mathrm{i}\frac{\partial\mathcal{F}}{\partial y}(0,0) \neq 0 \ ,$$

then

$$\begin{cases} \frac{\partial\mathcal{F}}{\partial x}(0,0) = \frac{2}{n}\,\Re\left(\mathcal{P}(1,1)\right) = \frac{2}{n}\,\Re\left(\mathcal{P}(1,-1)\right) \ , \\ \frac{\partial\mathcal{F}}{\partial y}(0,0) = \frac{2}{n}\,\Im\left(\mathcal{P}(1,1)\right) = -\frac{2}{n}\,\Im\left(\mathcal{P}(1,-1)\right) \ , \end{cases}$$

with $\Re\left(\mathcal{P}(1,1)\right)$ and $\Im\left(\mathcal{P}(1,1)\right)$ being respectively the real and the imaginary part of $\mathcal{P}(1,1)$.

As a conclusion, the necessary conditions for C^1-convergence are a domination of the main eigenvalues $\tilde{\mathsf{M}}(1)$ and $\tilde{\mathsf{M}}(-1)$, after the main eigenvalue from $\tilde{\mathsf{M}}(0)$, and a configuration of the elements of the associated eigenvectors which defines the vertices coordinates in an injective parametric space. Furthermore, these conditions give us the values of $\frac{\partial\mathcal{F}}{\partial x}(0,0)$ and $\frac{\partial\mathcal{F}}{\partial y}(0,0)$.

3.3 C^2-Convergence

If the scheme C^2-converges and the mark point is a vertex, then

$$\tilde{\lambda}_2(0) = \left|\tilde{\lambda}_1(\pm 2)\right| = \left|\tilde{\lambda}_1(\pm 1)\right|^2 \ , \text{ and } \left|\tilde{\lambda}_1(\omega)\right| < \left|\tilde{\lambda}_1(\pm 1)\right|^2 \ ,$$

with $\omega \notin \{-2,-1,0,1,2\}$. Furthermore,

$$\left|\tilde{\lambda}_2(\pm 1)\right| < \left|\tilde{\lambda}_1(\pm 1)\right|^2$$

iff

$$\lim_{k\to\infty}\left(\frac{\left|\frac{\partial\mathcal{F}^{(k)}}{\partial x}(0,0) \mp \mathrm{i}\frac{\partial\mathcal{F}^{(k)}}{\partial y}(0,0)\right| - \left|\frac{\partial\mathcal{F}}{\partial x}(0,0) \mp \mathrm{i}\frac{\partial\mathcal{F}}{\partial y}(0,0)\right|}{\left|\tilde{\lambda}_1(\pm 1)\right|^k}\right) = 0 \ .$$

Besides, if $L_2(0) = \{2\}$, then

$$\frac{(\tilde{v}_2(0))_X - (\tilde{v}_2(0))_A}{|(\tilde{v}_1(1))_X|^2} \quad \text{and} \quad \frac{(\tilde{v}_2(0))_X - (\tilde{v}_2(0))_A}{|(\tilde{v}_1(-1))_X|^2}$$

depend neither on X nor on k, and if $L_1(2) = L_1(-2) = \{1\}$, then the ratios

$$\frac{|(\tilde{v}_1(2))_X|}{|(\tilde{v}_1(1))_X|^2} , \quad \frac{|(\tilde{v}_1(2))_X|}{|(\tilde{v}_1(-1))_X|^2} , \quad \frac{|(\tilde{v}_1(-2))_X|}{|(\tilde{v}_1(1))_X|^2} , \quad \text{and} \quad \frac{|(\tilde{v}_1(-2))_X|}{|(\tilde{v}_1(-1))_X|^2}$$

and the differences

$$\varphi_{(\tilde{v}_1(2))_X} - \varphi_{(\tilde{v}_1(1))_X} , \quad \text{and} \quad \varphi_{(\tilde{v}_1(-2))_X} - \varphi_{(\tilde{v}_1(-1))_X}$$

do not depend on X.

Furthermore, if we define $\varrho_X^{(k)}$ and α_X as proposed in Sect. 3.2 for a rotational invariant scheme, then we can scale the eigenvectors such that

$$(\tilde{v}_1(2))_X = (\tilde{v}_1(1))_X^2 , \qquad (\tilde{v}_1(-2))_X = (\tilde{v}_1(-1))_X^2 ,$$

and, if

$$\frac{\partial^2 \mathcal{F}}{\partial x^2}(0,0) - \frac{\partial^2 \mathcal{F}}{\partial y^2}(0,0) \mp 2\mathrm{i}\frac{\partial^2 \mathcal{F}}{\partial x \partial y}(0,0) \neq 0 ,$$

then

$$\begin{aligned} \frac{\partial^2 \mathcal{F}}{\partial x^2}(0,0) + \frac{\partial^2 \mathcal{F}}{\partial y^2}(0,0) &= 4\frac{\mathcal{P}(2,0)}{n} , \\ \frac{\partial^2 \mathcal{F}}{\partial x^2}(0,0) - \frac{\partial^2 \mathcal{F}}{\partial y^2}(0,0) &= \frac{8}{n}\Re\left(\mathcal{P}(1,2)\right) = \frac{8}{n}\Re\left(\mathcal{P}(1,-2)\right) , \\ \frac{\partial^2 \mathcal{F}}{\partial x \partial y}(0,0) &= -\frac{4}{n}\Im\left(\mathcal{P}(1,2)\right) = \frac{4}{n}\Im\left(\mathcal{P}(1,-2)\right) . \end{aligned}$$

As a conclusion, the necessary condition for C^2-convergence is that among all the eigenvalues of all $\tilde{\mathsf{M}}(\omega)$, the global subsubdominant eigenvalues are the subdominant eigenvalue of $\tilde{\mathsf{M}}(0)$ and the dominant eigenvalue of $\tilde{\mathsf{M}}(2)$ and $\tilde{\mathsf{M}}(-2)$. These global subsubdominant eigenvalues are equal to the square of $\left|\tilde{\lambda}_1(\pm 1)\right|$, the global subdominant eigenvalues. Furthermore, the elements of the associated eigenvectors are in a quadratic configuration. These conditions have been already proposed by Sabin [11] as a condition related to the C^2-continuity of the limit surface. They let us get also the values of the partial derivatives $\frac{\partial^2 \mathcal{F}}{\partial x^2}(0,0)$, $\frac{\partial^2 \mathcal{F}}{\partial y^2}(0,0)$ and $\frac{\partial^2 \mathcal{F}}{\partial x \partial y}(0,0)$.

4 Discussion

Many authors interpret a subdivision scheme as a linear map between patches which fill in progressively a n-sided hole around an extraordinary point. Prautzsch [8] and Zorin [15] proposed necessary and sufficient conditions for C^p-regularity of the limit surface, on the eigenvalues and eigenbasis functions of this linear map. In contrast, we interpret a subdivision scheme as a linear map between samplings of two successive surfaces from a sequence of C^p surfaces. If this sequence converges with sufficient regularity (C^p-converges) these samplings may be used to approximate the derivatives of the limit surface. We propose necessary conditions for the C^2-convergence of a scheme, which is itself a sufficient condition for the C^2-continuity of the limit surface, on the eigenvalues and eigenvectors of the transformed subdivision matrix. As already stated, a scheme which converges toward a C^2-continuous limit surface does not necessarily C^2-converge. But it is interesting to understand the difference between our necessary conditions for a C^p-convergence, and the condition for the C^p-regularity of the limit surface proposed by Reif, Prautzsch and Zorin.

C^0-regularity We find the same conditions.

C^1-regularity Because we ask the subdominant eigenvalues to come from $\tilde{\mathsf{M}}(1)$ and $\tilde{\mathsf{M}}(-1)$, we assure the orthoradial injectivity of Reif's characteristic map as described in [7]; and because we ask the components of the associated eigenvectors to be sorted like the parameters $\varrho_X^{(k)}$, we assure good conditions for the radial injectivity of this map.

C^2-regularity Reif's characteristic map [10] is given by the subdominant eigenbasis functions. If the scheme is Box-Spline based, the eigenbasis functions are Box-Splines with our eigenvectors as control points (more precisely, our eigenvectors provide their radial coordinate). One of the conditions proposed by Prautzsch [8] and Zorin [15] for C^2-regularity, is that the eigenbasis functions z associated to the subsubdominant eigenvalue should belong to span $\left\{x^i y^j ; i+j=2\right\}$ where x and y are the eigenbasis functions associated to the subdominant eigenvalue. Our condition is the same, but with the eigenvectors instead of the eigenbasis functions. And the eigenvectors provide the altitude over the characteristic map of the control points of z. Around an ordinary vertex, we have checked that the quadratic configuration of the eigenvectors is fulfilled for Loop and Catmull-Clark schemes. Stam does so for the quadratic configuration of eigenbasis functions [14]. The possibility to get quadratic configuration of both eigenvectors and eigenbasis functions around an extraordinary vertex remains to be investigated.

5 Conclusion

In this article, we propose practical conditions for tuning a scheme in order to control its artifacts in the vicinity of a mark point. To do so, we look for good behaviour of the limit vertices rather than a good mathematical property of the limit surface. The good behaviour of the limit vertices is characterized with the definition of C^2-convergence of a scheme. We propose necessary explicit conditions for C^2-convergence of a scheme in the vicinity of any mark point being a vertex of valency greater or equal to three. We give some light on the relationship between these conditions with the classical necessary and sufficient conditions for the C^2-continuity of the limit surface. Even though the differences between them are not so large, we stress the fact that our conditions are designed for tuning a scheme leading to less artifacts on the limit surface [2] rather than for tuning a scheme leading to a C^2-continuous limit surface.

References

1. A. A. Ball and D. J. T. Storry. Conditions for tangent plane continuity over recursively generated b-spline surfaces. *ACM Transactions on Graphics*, 7(2):83–102, 1988.
2. L. Barthe and L. Kobbelt. Subdivision scheme tuning around extraordinary vertices. submitted for publication.
3. E. Catmull and J. Clark. Recursively generated b-spline surfaces on arbitrary topological meshes. *Computer Aided Design*, 10(6):350–355, 1978.
4. D. Doo and M. A. Sabin. Behaviour of recursive division surface near extraordinary points. *Computer Aided Design*, 10(6):356–360, 1978.
5. C. Gérot, L. Barthe, N. A. Dodgson, and M. A. Sabin. Subdivision as sequence of sampled c^p-surfaces and conditions for tuning schemes. Technical report, University of Cambridge Computer Laboratory, 2004.
6. C. T. Loop. Smooth subdivision surfaces based on triangles. Master's thesis, University of Utah, 1987.
7. J. Peters and U. Reif. Analysis of algorithms generalizing b-spline subdivision. *SIAM J. Numer. Anal.*, 35(2):728–748, 1998.
8. H. Prautzsch. Smoothness of subdivision surfaces at extraordinary points. *Adv. Comput. Math*, 9:377–389, 1998.
9. H. Prautzsch and G. Umlauf. Improved triangular subdivision schemes. In *Proc Computer Graphics International*, pages 626–632, 1998.
10. U. Reif. A unified approach to subdivision algorithm near extraordinary vertices. *Computer Geometric Aided Design*, 12:153–174, 1995.
11. M. A. Sabin. Eigenanalysis and artifacts of subdivision curves and surfaces. In A. Iske, E. Quak, and M. S. Floater, editors, *Tutorials on multiresolution in geometric modelling*, pages 69–97. Springer-Verlag, 2002.
12. M. A. Sabin. Recent progress in subdivision - a survey. In N. A. Dodgson, M. S. Floater, and M. A. Sabin, editors, *Advances in multiresolution for geometric modeling*. Springer-Verlag, 2004.

13. M. A. Sabin and L. Barthe. Artifacts in recursive subdivision surfaces. In A. Cohen, J.-L. Merrien, and L. L. Schumaker, editors, *Curve and surface fitting*, pages 353–362. Nashboro Press, 2002.
14. J. Stam. Exact evaluation of catmull-clark subdivision surfaces at arbitrary parameter values. *SIGGRAPH 98 Conference Proceddings*, pages 395–404, 1998.
15. D. Zorin. *Stationary subdivision and multiresolution surface representations*. PhD thesis, California Institute of Technology, 1997.