Characterizing Change in Argumentation by Using Duality Between Addition and Removal

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Abstract

In this paper, we address a new problem in the field of argumentation theory: the link between two different change operations, namely addition and removal of an argument. We define two concepts of duality reflecting this link. They are used to propose new results about an operation from existing results concerning its dual operation. Finally, the propositions that are obtained are studied for characterizing the change operations.

Keywords: Argumentation, Dynamics in abstract argumentation.

Contents

1 Introduction

Mr Pink knows that a given argument could be fatal to Mr White's argumentation, but this argument is lacking. Another way to win could be to remove one of Mr White's arguments (e.g. by doing an objection). However, Mr Pink does not know all the consequences of this removal. So the study of the connection between addition and removal of an argument is helpful.

The growing field of argumentation is becoming a key approach to deal with incomplete and / or contradictory information, especially for reasoning Dung [1995], Amgoud and Cayrol [2002]. It can moreover represent dialogues between several agents by modeling the exchange of arguments in, for example, negotiation between agents Amgoud et al. [2000].

Argumentation usually consists of a collection of arguments interacting with each other through a relation reflecting conflicts between them, called *attack*. The issue of argumentation is then to determine "acceptable" sets of arguments (*i.e.*, sets able to defend themselves collectively while avoiding internal attacks), called *"extensions"*, and thus to reach a coherent conclusion. Another form of analysis of an argumentation system is the study of the particular status of each argument, status based on the membership (or non-membership) to the extensions.

The creation and use of formal frameworks have greatly eased the modeling and study of argumentation systems. In particular, the formal framework of Dung [1995] allows to completely abstract the "real" meaning of the arguments and relies only on binary interactions that may exist between them. This approach enables the user to focus on other aspects of argumentation, including its dynamic side. Indeed, in the course of a discussion or during the acquisition of new information, an argumentation system can undergo changes such as the addition of a new argument or the removal of an argument considered as illegal. Thus, it is interesting to examine these changes, namely to focus on their characterization, *i.e.*, the new necessary and sufficient conditions that hold when adding or removing an argument. The study of the links between addition and removal through the concept of duality is a way to directly circumscribe the characterization of removal through the work previously done on addition, and conversely.

Although the research on dynamics of argumentation systems is growing Boella et al. [2009b,a], Baumann and Brewka [2010], Moguillansky et al. [2010], Liao et al. [2011], the removal of argument has so far been little considered. An attempt to justify the use of removal may nevertheless be found in Bisquert et al. [2011] (exclusively devoted to removal). *A fortiori*, the relationship between addition and removal of argument has not, to our knowledge, been treated so far. However, these change operations can be considered dual to each other.

In this work, we therefore propose to initiate a theoretical study of the relationship existing between operations of addition and removal of argument and examine the impact this may have on the analysis of the dynamics of an

argumentation system.

The paper is organized as follows: Sect. 2 recalls some key concepts of the theory of abstract argumentation and introduces new definitions relevant to our study. Section 3 displays properties of a change operation reflecting possible modifications of an argumentation system. Various notions of duality, and the results of our study are presented in Sect. 4. Finally, Sect. 5 concludes and suggests perspectives of our work.

2 Formal Framework

Before going further into the subject of this article, we should recall some basic backgrounds.

2.1 Argumentation System

The work presented in this paper falls within the formal framework of Dung [1995].

Definition 1 (**Argumentation System**). *An* argumentation system *is a pair* $\langle \mathbf{A}, \mathbf{R} \rangle$*, where* $\mathbf{A}$ *is a finite nonempty set of arguments and* $\mathbf{R}$ *is a binary relation on* $\mathbf{A}$*, called* attack relation*. Let* $A, B \in \mathbf{A}$*,* $A\mathbf{R}B$ *means that* A *attacks* B*.* $\langle \mathbf{A}, \mathbf{R} \rangle$ *will be represented by an* argumentation graph $\mathcal{G}$ *whose vertices are the arguments and whose edges correspond to* $\mathbf{R}$[1]*.*

In the remainder of this article, we will need an extended notion of the attack, namely the attack of an argument to a set and vice versa.

Definition 2 (**Attack from and to a set**). *Let* $A \in \mathbf{A}$ *and* $\mathcal{S} \subseteq \mathbf{A}$*,*

- $\mathcal{S}$ *attacks* A *iff*[2] $\exists X \in \mathcal{S}$ *such that* $X\mathbf{R}A$*.*
- A *attacks* $\mathcal{S}$ *iff* $\exists X \in \mathcal{S}$ *such that* $A\mathbf{R}X$*.*

The acceptable sets of arguments ("extensions") are determined according to a given semantics whose definition is usually based on the following concepts:

Definition 3 (**Conflict-free set, defense and admissibility**). *Let* $A \in \mathbf{A}$ *and* $\mathcal{S} \subseteq \mathbf{A}$*,*

- $\mathcal{S}$ *is* conflict-free *iff there does not exist* $A, B \in \mathcal{S}$ *such that* $A\mathbf{R}B$*.*
- $\mathcal{S}$ defends *an argument* A *iff each attacker of* A *is attacked by an argument of* $\mathcal{S}$*. The set of the arguments defended by* $\mathcal{S}$ *is denoted by* $\mathcal{F}(\mathcal{S})$*;* $\mathcal{F}$ *is called the* characteristic function *of* $\langle \mathbf{A}, \mathbf{R} \rangle$*. More generally,* $\mathcal{S}$ indirectly defends A *iff* $A \in \bigcup_{i \geq 1} \mathcal{F}^i(\mathcal{S})$*.*

[1] In this work we use freely $\langle \mathbf{A}, \mathbf{R} \rangle$ or $\mathcal{G}$ to refer to an argumentation system. Similarly, if there is no ambiguity, we use without distinction $\mathbf{A}$ and $\mathcal{G}$.

[2] iff = if and only if.

- $\mathcal{S}$ *is an* admissible *set iff it is conflict-free and it defends all its elements.*

The set of extensions of $\langle \mathbf{A}, \mathbf{R} \rangle$ is denoted by $\mathbf{E}$ (with $\mathcal{E}_1, \ldots, \mathcal{E}_n$ standing for the extensions). For instance, for the *grounded semantics*, one of the most traditional semantics proposed by Dung [1995], we have:

Definition 4 (**Grounded semantics**). *Let $\mathcal{E} \subseteq \mathbf{A}$, $\mathcal{E}$ is* the only grounded extension *iff $\mathcal{E}$ is the least fixed point (with respect to $\subseteq$) of the characteristic function $\mathcal{F}$.*

The status of an argument is determined by its presence in the extensions of the selected semantics. For example, an argument can be "skeptically accepted" (resp. "credulously") if it belongs to all the extensions (resp. at least to one extension) and be "rejected" if it does not belong to any extension.

2.2 Change Operations: Addition and Removal

Moreover, since we study the change in an argumentation system, we need to give a definition of a change operation. We rely on the work of Cayrol et al. [2010] which have distinguished four change operations; here are the formal definitions of the two operations of interest in this work, namely the operations of addition and removal of an argument and its interactions.

Definition 5 (**Change operations**). *Let $\langle \mathbf{A}, \mathbf{R} \rangle$ be an argumentation system, Z be an argument and $\mathcal{I}_z$ be a set of interactions concerning Z.*

- Adding $Z \notin \mathbf{A}$ *and* $\mathcal{I}_z \not\subseteq \mathbf{R}$ *is a change operation, denoted by $\oplus$, providing a new argumentation system such that:*
$$\langle \mathbf{A}, \mathbf{R} \rangle \oplus (Z, \mathcal{I}_z) = \langle \mathbf{A} \cup \{Z\}, \mathbf{R} \cup \mathcal{I}_z \rangle$$
- Removing $Z \in \mathbf{A}$ *and* $\mathcal{I}_z \subseteq \mathbf{R}$ *is a change operation, denoted by $\ominus$, providing a new argumentation system such that:*
$$\langle \mathbf{A}, \mathbf{R} \rangle \ominus (Z, \mathcal{I}_z) = \langle \mathbf{A} \setminus \{Z\}, \mathbf{R} \setminus \mathcal{I}_z \rangle$$

We denote by $\mathcal{O}$ a change operation ($\oplus$ or $\ominus$). The new argumentation system $\langle \mathbf{A}', \mathbf{R}' \rangle$ obtained by the application of $\mathcal{O}$ will be represented by the argumentation graph $\mathcal{G}' = \mathcal{O}(\mathcal{G})$.[3]

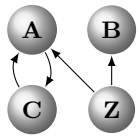
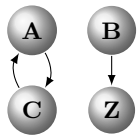
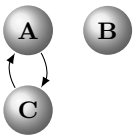
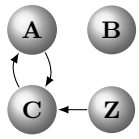
The set of extensions of $\langle \mathbf{A}', \mathbf{R}' \rangle$ is denoted by $\mathbf{E}'$ (with $\mathcal{E}'_1, \ldots, \mathcal{E}'_n$ standing for the extensions). Note that, in the course of this work, we will only consider cases where the semantics remains the same before and after a change.

[3]
- We assume that Z does not attack itself and $\forall (X, Y) \in \mathcal{I}_z$, we have either ($X = Z$ and $Y \neq Z$, $Y \in \mathbf{A}$) or ($Y = Z$ and $X \neq Z$, $X \in \mathbf{A}$).
- In case of removing, $\mathcal{I}_z$ is the set of all the interactions concerning Z in $\langle \mathbf{A}, \mathbf{R} \rangle$.
- The symbols $\oplus$ and $\ominus$ used here correspond to the symbols $\oplus^a_I$ and $\ominus^a_I$ of Cayrol et al. [2010], where a stands for "argument" and I for "interactions", meaning that the operation concerns an argument and its interactions.

It is also important to note that a change operation is a non injective application. Thanks to Definition 5, we know that $\forall \mathcal{G},\ \mathcal{G}' = \mathcal{O}(\mathcal{G})$ is unique. However, for a given $\mathcal{G}'$, there may be several $\mathcal{G}$.

Example 1. *With $\mathcal{O} = \ominus$, three systems can be changed into $\mathcal{G}'$, such that $\mathcal{O}(\mathcal{G}_1) = \mathcal{O}(\mathcal{G}_2) = \mathcal{O}(\mathcal{G}_3) = \mathcal{G}'$ (see Table 1 which also gives the grounded extension of each system).*

Table 1: On the non injective nature of the removal operation.

System before the removal of Z	System after the removal of Z
$\mathcal{G}_1$:  $\mathbf{E}_{\mathcal{G}_1} = \{\{C, Z\}\}$	
$\mathcal{G}_2$:  $\mathbf{E}_{\mathcal{G}_2} = \{\{B\}\}$	$\mathcal{G}'$:  $\mathbf{E}' = \{\{B\}\}$
$\mathcal{G}_3$:  $\mathbf{E}_{\mathcal{G}_3} = \{\{A, B, Z\}\}$	

The impact of a change operation will be studied through the notion of change property. A change property $\mathcal{P}$ can be seen as a set of pairs $(\mathcal{G}, \mathcal{G}')$, where $\mathcal{G}$ and $\mathcal{G}'$ are argumentation graphs.

Example 1 (cont.). *Let $\mathcal{P}$ be the property defined by:*

> *$\mathcal{P}(\mathcal{G}, \mathcal{G}')$ holds iff "Any extension of $\mathcal{G}'$ is included in at least one extension of $\mathcal{G}$."*

Thus, $\mathcal{P}(\mathcal{G}_1, \mathcal{G}')$ does not hold while $\mathcal{P}(\mathcal{G}_2, \mathcal{G}')$ and $\mathcal{P}(\mathcal{G}_3, \mathcal{G}')$ hold.

In the remainder of this work, we will often refer to the fact that an operation *satisfies* a particular property:

Definition 6 (Operation satisfying a property). *A change operation $\mathcal{O}$* satisfies *a property $\mathcal{P}$ iff $\forall \mathcal{G},\ \mathcal{P}(\mathcal{G}, \mathcal{O}(\mathcal{G}))$ holds.*

Example 1 (cont.). *The argumentation system $\mathcal{G}_1$ is such that $\mathcal{P}(\mathcal{G}_1, \mathcal{G}')$ does not hold. Thus, the change operation $\mathcal{O}$ does not satisfy $\mathcal{P}$.*

2.3 Some Propositions About the Addition

The following propositions list the main results obtained for characterizing the operation of addition under the grounded semantics. Proposition 1 is directly taken from Cayrol et al. [2010]:

Proposition 1. *When adding an argument Z under the grounded semantics,*

1. *(Prop. 7) If $X \in \mathcal{E}$ and Z does not indirectly attack X, then $X \in \mathcal{E}'$.*
2. *(Prop. 9, item 3) If $\mathcal{E} = \varnothing$ and Z is attacked by $\mathcal{G}$, then $\mathcal{E}' = \varnothing$.*
3. *(Prop. 9, item 4) If $\mathcal{E} = \varnothing$ and Z is not attacked by $\mathcal{G}$, then $\mathcal{E}' = \{Z\} \cup \bigcup_{i \geq 1} \mathcal{F}'^i(\{Z\})$.*

Proposition 2 is a generalization of some propositions given in Cayrol et al. [2010]. The proof given in Appendix takes into account some properties concerning the addition of the argument Z in the special case where $\mathcal{E} = \varnothing$. These properties allow us to remove the condition $\mathcal{E} \neq \varnothing$ in the initial propositions:

Proposition 2. *When adding an argument Z under the grounded semantics,*

1. *(Prop. 10) If Z does not attack $\mathcal{E}$, then $\mathcal{E} \subseteq \mathcal{E}'$.*
2. *(Prop. 11, item 1) If Z does not attack $\mathcal{E}$ and $\mathcal{E}$ does not defend Z, then $\mathcal{E}' = \mathcal{E}$.*
3. *(Prop. 11, item 2 part 1) If Z does not attack $\mathcal{E}$ and $\mathcal{E}$ defends Z, then $\mathcal{E}' = \mathcal{E} \cup \{Z\} \cup \bigcup_{i \geq 1} \mathcal{F}'^i(\{Z\})$.*
4. *(Prop. 11, item 2 part 2) If Z does not attack $\mathcal{G}$ and $\mathcal{E}$ defends Z, then $\mathcal{E}' = \mathcal{E} \cup \{Z\}$.*
5. *(Prop. 13) $\mathcal{E}' = \varnothing$ iff Z attacks each unattacked argument of $\mathcal{G}$ and Z is attacked by $\mathcal{G}$.*

Proposition 3 is a new proposition about the conservation of the status "rejected" of an argument (see proof in Appendix):

Proposition 3. *When adding an argument Z under the grounded semantics, $\forall X \in \mathcal{G}$, if $X \notin \mathcal{E}$ and Z does not indirectly defend X, then $X \notin \mathcal{E}'$.*

3 Some Change Properties

Change properties express structural modifications of an argumentation system that are caused by a change operation. In this section, we focus on these modifications and try to define them in order to obtain a clear and accurate

classification. For that purpose, a new partition, inspired by the work of Cayrol et al. [2010] and based on three possible cases of evolution of the set of extensions, has been defined:

- the *extensive* case, in which the number of extensions increases,
- the *restrictive* case, in which the number of extensions decreases,
- the *constant* case, in which the number of extensions remains the same.

Note that in this article, due to space limitations, we do not address the *extensive* and *restrictive* cases[4], but we focus exclusively on the *constant* case.

Note also that for the sake of clarity, we will say that a change satisfying a property $\mathcal{P}$ is a "$\mathcal{P}$ *change*"; for example, a change that satisfies the *constant* property is said *constant* change. Here is the formal definition of this change:

Definition 7 (**Constant change**). *The change from $\mathcal{G}$ to $\mathcal{G}'$ is* constant *iff* $|\mathbf{E}| = |\mathbf{E}'|$.

Restricting our scope to the *constant* case allows us to focus on other criteria than the number of extensions of $\mathcal{G}$ and $\mathcal{G}'$, namely inclusions between the various possible extensions ($\mathcal{G}$ to $\mathcal{G}'$ and vice versa), emptiness of these extensions, etc. Here are the definitions of these various sub-cases[5]:

Definition 8. *The change from $\mathcal{G}$ to $\mathcal{G}'$ is:*

1. **c-conservative** *iff* $\mathbf{E} = \mathbf{E}'$.
2. **c-decisive** *iff* $\mathbf{E} = \{\{\}\}$ *and* $\mathbf{E}' = \{\mathcal{E}'\}$, *with* $\mathcal{E}' \neq \varnothing$.
3. **c-destructive** *iff* $\mathbf{E} = \{\mathcal{E}\}$, *with* $\mathcal{E} \neq \varnothing$ *and* $\mathbf{E}' = \{\{\}\}$.
4. **c-expansive** *iff*
   - $\mathbf{E} \neq \varnothing$, $|\mathbf{E}| = |\mathbf{E}'|$,
   - $\forall \mathcal{E}_i \in \mathbf{E}, \exists \mathcal{E}'_j \in \mathbf{E}', \varnothing \neq \mathcal{E}_i \subset \mathcal{E}'_j$ *and*
   - $\forall \mathcal{E}'_j \in \mathbf{E}', \exists \mathcal{E}_i \in \mathbf{E}, \varnothing \neq \mathcal{E}_i \subset \mathcal{E}'_j$.
5. **c-narrowing** *iff*
   - $\mathbf{E} \neq \varnothing$, $|\mathbf{E}| = |\mathbf{E}'|$,
   - $\forall \mathcal{E}_i \in \mathbf{E}, \exists \mathcal{E}'_j \in \mathbf{E}', \varnothing \neq \mathcal{E}'_j \subset \mathcal{E}_i$ *and*
   - $\forall \mathcal{E}'_j \in \mathbf{E}', \exists \mathcal{E}_i \in \mathbf{E}, \varnothing \neq \mathcal{E}'_j \subset \mathcal{E}_i$.
6. **c-altering** *iff* $|\mathbf{E}| = |\mathbf{E}'|$ *and it is neither* c-conservative*, nor* c-decisive*, nor* c-destructive*, nor* c-expansive*, nor* c-narrowing.

[4]For the same reason, we do not address other types of properties, including those related to the status of a specific argument.

[5]Note that the names of these sub-cases are prefixed with the letter c to highlight the fact that they follow from the *constant* property.

Definitions 8.1, 8.2, 8.3 and 8.6 are fairly straightforward. Definition 8.4 states that a *c-expansive* change is a change where all the extensions of $\mathcal{G}$, which are not initially empty, are increased by some arguments. A *c-narrowing* change, according to Definition 8.5, is a change where all the extensions of $\mathcal{G}$ are reduced by some arguments without becoming empty.

4 Usefulness of Duality

4.1 Two Definitions of Duality

As far as we know, the problem of removing an argument and, *a fortiori*, the link between addition and removal of an argument have been little discussed. However, we believe it can be worthy to use the links between these operations for the study of the properties characterizing the changes that may impact an argumentation system. For that purpose, we will rely on the notion of duality.

We focus on two concepts of duality. We will first define a duality at the level of change operations, the *duality based on the notion of inverse*, expressing the opposite nature of two operations, then a duality at the level of change properties, the *duality based on the notion of symmetry*, conveying a correspondence between two properties.

Definition 9 (**Duality based on the notion of inverse**). *Two change operations $\mathcal{O}$ and $\mathcal{O}'$ are the* inverse *of each other iff:*

$$\forall \mathcal{G}, \forall \mathcal{G}',\ \mathcal{O}(\mathcal{G}) = \mathcal{G}' \text{ iff } \mathcal{O}'(\mathcal{G}') = \mathcal{G}.$$

Obviously, following the former definition, it is clear that the operations of addition and removal of an argument defined in Sect. 2 are the inverse of each other.

Definition 10 (**Duality based on the notion of symmetry**). *Two properties $\mathcal{P}$ and $\mathcal{P}'$ are* symmetric *iff:*

$$\forall \mathcal{G},\ \forall \mathcal{G}',\ \mathcal{P}'(\mathcal{G}', \mathcal{G}) \text{ holds iff } \mathcal{P}(\mathcal{G}, \mathcal{G}') \text{ holds.}$$

From these definitions, we can draw a condition for the satisfaction of a property by a change operation.

Proposition 4. *Let $\mathcal{O}$ and $\mathcal{O}'$ two inverse change operations and $\mathcal{P}$ and $\mathcal{P}'$ two symmetric properties. $\mathcal{O}$ satisfies $\mathcal{P}$ iff $\mathcal{O}'$ satisfies $\mathcal{P}'$.*

Both concepts of duality defined above can be used for linking the change properties.

Proposition 5. *A change is* constant *iff the inverse change is* constant *as well.*

Proposition 6. *A change is* c-destructive *iff the inverse change is* c-decisive.

Proposition 7. *A change is* c-conservative *iff the inverse change is* c-conservative *as well.*

Proposition 8. *A change is* c-narrowing *iff the inverse change is* c-expansive*.*

Proposition 9. *A change is* c-altering *iff the inverse change is* c-altering *as well.*

Figure 1 graphically summarizes the above results; their proofs are in Appendix.

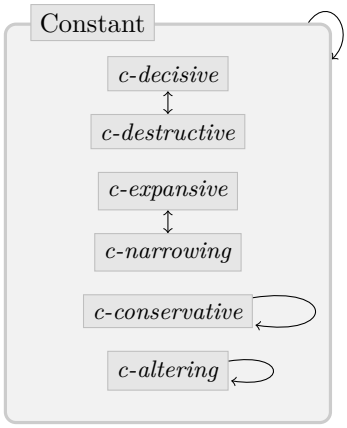


Figure 1: Presentation of the duality of the *constant* property and its sub-cases.

4.2 Methodology for Using Duality

This part describes how to use duality in order to obtain new propositions for the operation of removal, starting from propositions relating to addition[6]. Note first that, in this article, we restrict our study to the grounded semantics (see Definition 4).

Let us describe this methodology: using Proposition 3, we show how duality enables us to obtain a new proposition about the removal of an argument.

Let us first proceed to a renaming in order to clarify the presentation. The graphs and the extensions are going to be indexed by two capital letters - IA, OA, IR and OR - representing respectively the **I**nput system for the **A**ddition, the **O**utput system for the **A**ddition, the **I**nput system for the **R**emoval and the **O**utput system for the **R**emoval. Thus, Proposition 3 can be rewritten as follows:

Proposition 3.1. *When adding an argument Z under the grounded semantics, if $X \notin \mathcal{E}_{IA}$ and Z does not indirectly defend X, then $X \notin \mathcal{E}_{OA}$.*

Let $\mathcal{P}$ be a property and $\mathcal{P}^{-1}$ its symmetric. Thanks to Proposition 4, we can write:

$$\oplus \textit{ satisfies } \mathcal{P} \textit{ iff } \ominus \textit{ satisfies } \mathcal{P}^{-1}$$

[6]This methodology can also be used the other way round from removal to addition.

And thanks to Definition 6, we know that a change operation $\mathcal{O}$ satisfies a property $\mathcal{P}$ if and only if $\forall \mathcal{G}$, it holds that $\mathcal{P}(\mathcal{G}, \mathcal{O}(\mathcal{G}))$. So we can write:

$$\forall \mathcal{G}_{IA},\ \mathcal{P}(\mathcal{G}_{IA}, \oplus(\mathcal{G}_{IA}))\ \textit{holds iff}\ \forall \mathcal{G}_{IR},\ \mathcal{P}^{-1}(\mathcal{G}_{IR}, \ominus(\mathcal{G}_{IR}))\ \textit{holds}$$

Moreover, thanks to Definition 10, we have:

$$\forall \mathcal{G}_{IR},\ \mathcal{P}^{-1}(\mathcal{G}_{IR}, \ominus(\mathcal{G}_{IR}))\ \textit{holds iff}\ \mathcal{P}(\ominus(\mathcal{G}_{IR}), \mathcal{G}_{IR})\ \textit{holds}$$

And so, we have:

$$\forall \mathcal{G}_{IA},\ \forall \mathcal{G}_{IR},\ \mathcal{P}(\mathcal{G}_{IA}, \oplus(\mathcal{G}_{IA}))\ \textit{holds iff}\ \mathcal{P}(\ominus(\mathcal{G}_{IR}), \mathcal{G}_{IR})\ \textit{holds}$$

Let $\mathcal{G}_{OA} = \oplus(\mathcal{G}_{IA})$ and $\mathcal{G}_{OR} = \ominus(\mathcal{G}_{IR})$. Since we know that Property $\mathcal{P}$ holds for the operation of addition, we can rewrite it for the operation of removal:

Proposition 3.2. *When removing an argument Z under the grounded semantics, if $X \notin \mathcal{E}_{OR}$ and Z does not indirectly defend X, then $X \notin \mathcal{E}_{IR}$.*

Which is equivalent to:

Proposition 3.3. *When removing an argument Z under the grounded semantics, if $X \in \mathcal{E}_{IR}$ and Z does not indirectly defend X, then $X \in \mathcal{E}_{OR}$.*

Thus, for the operation of removal, we obtain a proposition analogous to Proposition 3 denoted by Proposition $3^{\ominus}$; in the remainder of this article, the exponent ($^{\oplus}$ or $^{\ominus}$) will represent the correspondence between a proposition and the one obtained by applying the duality methodology.

Proposition $3^{\ominus}$. *When removing an argument Z under the grounded semantics, if $X \in \mathcal{E}$ and Z does not indirectly defend X, then $X \in \mathcal{E}'$.*

So, using the methodology presented here, the propositions of Cayrol et al. [2010] summarized by Proposition 1 and Proposition 2 can be translated.

4.3 Propositions Obtained by Duality

This section is divided into two parts: the first part concerns the propositions on which the application of the methodology is directly meaningful, and the second part concerns the propositions that should be transformed in order to make sense.

4.3.1 Straightforward Application

Here, we deal with the propositions on which our methodology gives analogous propositions that can be used directly.

From Proposition 1.1, we find a proposition that gives a sufficient condition for the conservation of the rejection of an argument X when Z is removed.

Proposition 1.1$^{\ominus}$. *When removing an argument Z under the grounded semantics, if $X \notin \mathcal{E}$ and Z does not indirectly attack X, then $X \notin \mathcal{E}'$.*

Similarly, Propositions 1.2 and 1.3 give sufficient conditions for a non *c-destructive* change.

Proposition 1.2$^{\ominus}$. *When removing an argument Z under the grounded semantics, if $\mathcal{E} \neq \varnothing$ and Z is attacked by $\mathcal{G}$, then $\mathcal{E}' \neq \varnothing$.*

Proposition 1.3$^{\ominus}$. *When removing an argument Z under the grounded semantics, if $\mathcal{E} \neq \{Z\} \cup \bigcup_{i \geq 1} \mathcal{F}^i(\{Z\})$ and Z is not attacked by $\mathcal{G}$, then $\mathcal{E}' \neq \varnothing$.*

The two previous propositions also give a necessary condition for a *c-destructive* change.

Corollary 1. *When removing an argument Z under the grounded semantics, if the change is* c-destructive*, then Z is not attacked by $\mathcal{G}$ and $\mathcal{E} = \{Z\} \cup \bigcup_{i \geq 1} \mathcal{F}^i(\{Z\})$.*

From Proposition 2.5, we obtain a proposition that characterizes a change either *c-conservative* or *c-decisive*.

Proposition 2.5$^{\ominus}$. *When removing an argument Z under the grounded semantics, $\mathcal{E} = \varnothing$ iff Z attacks each unattacked argument of $\mathcal{G} \setminus \{Z\}$ and Z is attacked by $\mathcal{G} \setminus \{Z\}$.*

4.3.2 Not So Straightforward Application

Now, we deal with propositions requiring additional work to be usable; for example when the proposition uses a condition on the output system, we must consider what it means on the input system.

From Proposition 2.1, we obtain:

Proposition 2.1$^{\ominus}$. *When removing an argument Z under the grounded semantics, if Z does not attack $\mathcal{E}'$ in $\mathcal{G}$, then $\mathcal{E}' \subseteq \mathcal{E}$.*

This proposition uses a condition on the output argumentation system (Z *does not attack* $\mathcal{E}'$). Lemma 1 expresses the meaning of this condition for the input argumentation system in the case of a removal. For this lemma, we need a new notation.

Notation 1. *Let $\mathcal{U} \subseteq \mathcal{G}$, $\mathcal{U}$ is the set of unattacked arguments in $\mathcal{G} \setminus \{Z\}$.*

Informally, Lemma 1 means that if an argument X is attacked by Z, X is also attacked by another argument $Y \neq Z$ which prevents X to belong to the grounded extension $\mathcal{E}'$.

Lemma 1. *When removing an argument Z under the grounded semantics, Z does not attack $\mathcal{E}'$ in $\mathcal{G}$ iff $\forall X \in \mathcal{G}'$, if Z attacks X then (X is attacked by $\mathcal{G} \setminus \{Z\}$ and X is not indirectly defended by $\mathcal{U}$ in $\mathcal{G} \setminus \{Z\}$).*

Using Lemma 1, we can rewrite the proposition, which gives us a sufficient condition for the fact that no argument non accepted before the change is accepted after. Hence, the change is either *c-conservative*, *c-destructive* or *c-narrowing*.

Proposition 2.1$^{\ominus}$ (v2). *When removing an argument Z under the grounded semantics, if $\forall X \in \mathcal{G}$, if Z attacks X then (X is attacked by $\mathcal{G} \setminus \{Z\}$ and X is not indirectly defended by $\mathcal{U}$ in $\mathcal{G} \setminus \{Z\}$), then $\mathcal{E}' \subseteq \mathcal{E}$.*

From Proposition 2.2, we obtain:

Proposition 2.2$^{\ominus}$. *When removing an argument Z under the grounded semantics, if Z does not attack $\mathcal{E}'$ in $\mathcal{G}$ and $\mathcal{E}'$ does not defend Z in $\mathcal{G}$, then $\mathcal{E} = \mathcal{E}'$.*

Similarly, we need to express the condition *$\mathcal{E}'$ does not defend Z in $\mathcal{G}$* by a condition for the input argumentation system. Such a condition is given by Lemma 2.

Lemma 2. *When removing an argument Z under the grounded semantics, if Z does not attack $\mathcal{E}'$, then the following equivalence holds:*

$$Z \in \bigcup_{i \geq 1} \mathcal{F}^i(\mathcal{U}) \text{ iff } \mathcal{E}' \text{ defends } Z \text{ in } \mathcal{G}.$$

Thanks to Lemmata 1 and 2, we can rewrite the proposition, which gives us a sufficient condition for the *c-conservative* change:

Proposition 2.2$^{\ominus}$ (v2). *When removing an argument Z under the grounded semantics, if*

- *$\forall X \in \mathcal{G}$, if Z attacks X then (X is attacked by $\mathcal{G} \setminus \{Z\}$ and X is not indirectly defended by $\mathcal{U}$ in $\mathcal{G} \setminus \{Z\}$) and*
- *$Z \notin \bigcup_{i \geq 1} \mathcal{F}^i(\mathcal{U})$,*

then $\mathcal{E} = \mathcal{E}'$.

From Proposition 2.3, we obtain:

Proposition 2.3$^{\ominus}$. *When removing an argument Z under the grounded semantics, if Z does not attack $\mathcal{E}'$ in $\mathcal{G}$ and $\mathcal{E}'$ defends Z in $\mathcal{G}$, then $\mathcal{E} = \mathcal{E}' \cup \{Z\} \cup \bigcup_{i \geq 1} \mathcal{F}^i(\{Z\})$.*

Let $N_Z = (\bigcup_{i \geq 1} \mathcal{F}^i(\{Z\}) \cup \{Z\}) \setminus \mathcal{E}'$; thus, we have $N_Z \subseteq \mathcal{E}$. N_Z contains Z and the arguments of $\mathcal{G} \setminus \{Z\}$ which could not be defended without using Z. In other words, if $X \neq Z$, $X \in N_Z$ if and only if Z is required for proving that $X \in \mathcal{E}$. Obviously, the pair $(\mathcal{E}', N_Z)$ constitutes a partition of $\mathcal{E}$. So, $\mathcal{E}' = \mathcal{E} \setminus N_Z$.

Thanks to Lemmata 1 and 2, we can rewrite the proposition, which gives us a sufficient condition for the *c-narrowing* change:

Proposition 2.3$^{\ominus}$ (v2). *When removing an argument Z under the grounded semantics, if*

- *$\forall X \in \mathcal{G}$, if Z attacks X then (X is attacked by $\mathcal{G} \setminus \{Z\}$ and X is not indirectly defended by $\mathcal{U}$ in $\mathcal{G} \setminus \{Z\}$) and*
- *$Z \in \bigcup_{i \geq 1} \mathcal{F}^i(\mathcal{U})$,*

then $\mathcal{E}' = \mathcal{E} \setminus N_Z$.

From Proposition 2.4, we obtain:

Proposition 2.4$^{\ominus}$. *When removing an argument Z under the grounded semantics, if Z does not attack $\mathcal{G}'$ and $\mathcal{E}'$ defends Z in $\mathcal{G}$, then $\mathcal{E} = \mathcal{E}' \cup \{Z\}$.*

Thanks to Lemma 2, we can rewrite the proposition, which also gives us a sufficient condition for the *c-narrowing* change:

Proposition 2.4$^{\ominus}$ (v2). *When removing an argument Z under the grounded semantics, if*

- *$Z \in \bigcup_{i \geq 1} \mathcal{F}^i(\mathcal{U})$ and*
- *Z does not attack $\mathcal{G} \setminus \{Z\}$,*

then $Z \in \mathcal{E}$ and $\mathcal{E}' = \mathcal{E} \setminus \{Z\}$.

5 Discussion and Conclusion

In this paper, we studied the link between addition and removal of an argument. To this end, after recalling the basis of abstract argumentation theory, we first took up and refined the change properties of Cayrol et al. [2010] into a clear partition for a special case (the cardinality of the set of extensions remains unchanged). We then defined two notions of duality, namely the *duality based on the notion of inverse* and the *duality based on the notion of symmetry*, in order to link these change properties and change operations. This allowed us, after describing a particular methodology based on the two dualities, to discover propositions for an operation thanks to the propositions already known for its dual operation. More specifically, we obtained propositions characterizing the removal operation thanks to propositions for the addition operation; they are organized in Table 2. Some of these new propositions have however revealed a difficulty preventing a naive application of our methodology: indeed, the proposition obtained by the application of our methodology may contain conditions on the output argumentation system and can thus not be directly used. We then look for equivalent conditions on the input argumentation system.

Thus, despite the interest of such a methodology allowing to get new propositions for an operation in a easy way, it is important to note that a "post-processing" is sometimes necessary in order to ensure that the result makes sense.

Table 2: Synthesis of the necessary and sufficient conditions for *constant* properties under the grounded semantics.

Propositions	**Change properties**
Prop. $1.2^{\ominus}$	CS for non *c-destructive*
Prop. $1.3^{\ominus}$	CS for non *c-destructive*
Corollary 1	CN for *c-destructive*
Prop. $2.1^{\ominus}$	CS for • *c-conservative* **or** • *c-destructive* **or** • *c-narrowing*
Prop. $2.2^{\ominus}$	CS for *c-conservative*
Prop. $2.3^{\ominus}$	CS for *c-narrowing*
Prop. $2.4^{\ominus}$	CS for *c-narrowing*
Prop. $2.5^{\ominus}$	CNS for • *c-conservative* **or** • *c-decisive*

Let us come back to the example presented in the introduction. For Mister Pink, adding a new argument attacking a specific argument of Mister White without threatening his own accepted arguments corresponds to Proposition 1.1. Proposition $3^{\ominus}$, on the other hand, allows him to ensure that the removal of his opponent's argument achieves the same result if this argument is not giving assistance to any of his own accepted arguments. Thereby, instead of using Proposition 1.1, Mister Pink can benefit from Proposition $3^{\ominus}$ thanks to our methodology.

This kind of work deals with a facet of the argumentation theory that has not been studied so far. Hence, many points are to be deepened or explored further; here are some issues that seem to be of short-term importance:

- In this work, we have focused on the grounded semantics and studied only two of the four operations of Cayrol et al. [2010]. A first issue is to extend our work to the two missing operations, addition and removal of an interaction, and also to other semantics.
- Moreover, we could consider the addition or removal of a set of arguments. These special operations may be seen as a sequence of change operations and their study seems essential in order to approach minimal change problems.
- Due to lack of space, we have outlined here a small subset of the possible change properties. It would be interesting to study and evaluate all the remaining properties through the duality methodology.

- The obligation to perform a post-processing of some of the propositions obtained by our approach is a sore point. We should find new ways to avoid this post-processing or at least to find criteria for identifying propositions that would require transformations.

A Proofs

Proof of Proposition 2. Proposition 2 has already been proven in Cayrol et al. [2010] under the condition $\mathcal{E} \neq \varnothing$: when adding an argument Z under the grounded semantics,

- (Prop. 10) If $\mathcal{E} \neq \varnothing$ and Z does not attack $\mathcal{E}$, then $\mathcal{E} \subseteq \mathcal{E}'$.
- (Prop. 11, item 1) If $\mathcal{E} \neq \varnothing$ and Z does not attack $\mathcal{E}$ and $\mathcal{E}$ does not defend Z, then $\mathcal{E}' = \mathcal{E}$.
- (Prop. 11, item 2 part 1) If $\mathcal{E} \neq \varnothing$ and Z does not attack $\mathcal{E}$ and $\mathcal{E}$ defends Z, then $\mathcal{E}' = \mathcal{E} \cup \{Z\} \cup \bigcup_{i \geq 1} \mathcal{F}'^i(\{Z\})$.
- (Prop. 11, item 2 part 2) If $\mathcal{E} \neq \varnothing$ and Z does not attack $\mathcal{G}$ and $\mathcal{E}$ defends Z, then $\mathcal{E}' = \mathcal{E} \cup \{Z\}$.
- (Prop. 13) If $\mathcal{E} \neq \varnothing$, then $\mathcal{E}' = \varnothing$ iff Z attacks each unattacked argument of $\mathcal{G}$ and Z is attacked by $\mathcal{G}$.

It only remains to prove this propositions under the condition $\mathcal{E} = \varnothing$:

- If $\mathcal{E} = \varnothing$ then we trivially have $\mathcal{E} \subseteq \mathcal{E}'$.
- If $\mathcal{E}$ does not defend Z, then Z is attacked by $\mathcal{G}$. Moreover, by Proposition 1.2, we know that if $\mathcal{E} = \varnothing$ and Z is attacked by $\mathcal{G}$, then $\mathcal{E}' = \varnothing$ and thus $\mathcal{E} = \mathcal{E}'$.
- If $\mathcal{E} = \varnothing$, then it always holds that Z does not attack $\mathcal{E}$. Moreover, if $\mathcal{E} = \varnothing$ defends Z, then Z is not attacked by $\mathcal{G}$. Thus, when adding Z, we trivially have $\mathcal{E}' = \mathcal{E} \cup \{Z\} \cup \bigcup_{i \geq 1} \mathcal{F}'^i(\{Z\})$. Furthermore, if Z does not attack $\mathcal{G}$, then Z does not defend any argument so we have $\mathcal{E}' = \mathcal{E} \cup \{Z\}$.
- If $\mathcal{E} = \varnothing$ then there is no unattacked argument in $\mathcal{G}$. Then, if we add an argument Z such that Z is attacked by $\mathcal{G}$, we can conclude that there is also no unattacked argument in $\mathcal{G}'$ and $\mathcal{E}' = \varnothing$.

  Moreover, if $\mathcal{E} = \varnothing$ and $\mathcal{E}' = \varnothing$, then the added argument Z is inevitably attacked by $\mathcal{G}$. And since there is no unattacked argument in $\mathcal{G}$, Z trivially attacks each unattacked argument of $\mathcal{G}$.

So we can generalize the previous propositions by removing the condition $\mathcal{E} \neq \varnothing$. □

Proof of Proposition 3. To prove this proposition, we will focus on its contraposition: when adding an argument Z under grounded semantics, if $X \in \mathcal{E}'$ and $X \neq Z$ and Z does not indirectly defend X, then $X \in \mathcal{E}$.

We know, thanks to Definition 1, that $\mathbf{R}$ is finite and thus, according to Dung [1995], we have $\mathcal{E} = \bigcup_{i \geq 1} \mathcal{F}^i(\varnothing)$ et $\mathcal{E}' = \bigcup_{i \geq 1} \mathcal{F}'^i(\varnothing)$. Let us prove by induction on $i \geq 1$ that if $X \in \mathcal{F}'^i(\varnothing)$ and $X \neq Z$ and Z does not indirectly defend X, then $X \in \mathcal{F}^i(\varnothing)$.

- Basic case ($i = 1$): if $X \in \mathcal{F}'(\varnothing)$ then X is not attacked by $\mathcal{G}'$ and if $X \neq Z$ then $X \in \mathcal{G}$. Thus, X is not attacked by $\mathcal{G}$ and thus $X \in \mathcal{F}(\varnothing)$.
- Induction hypothesis (for $1 \leq i \leq p$, the proposition holds): let $X \in \mathcal{F}'^{p+1}(\varnothing) = \mathcal{F}'(\mathcal{F}'^p(\varnothing))$. In order to prove that $X \in \mathcal{F}^{p+1}(\varnothing) = \mathcal{F}(\mathcal{F}^p(\varnothing))$, we have to prove that $\mathcal{F}^p(\varnothing)$ defends X in $\mathcal{G}$. Assume that X is attacked by Y in $\mathcal{G}$. Since we are within the context of the addition of the argument Z, X is attacked by Y in $\mathcal{G}'$. As $X \in \mathcal{F}'(\mathcal{F}'^p(\varnothing))$, $\mathcal{F}'^p(\varnothing)$ defends X against Y. So, there exists an argument $W \in \mathcal{F}'^p(\varnothing)$ such that W attacks Y. We know that Z does not indirectly defend X, so $W \neq Z$ and Z does not indirectly defend W. Using the induction hypothesis for W, we have $W \in \mathcal{F}^p(\varnothing)$, thus $\mathcal{F}^p(\varnothing)$ defends X against Y, and thus $X \in \mathcal{F}^{p+1}(\varnothing)$.

□

Proof of Proposition 4. It follows directly from Definitions 6, 9 and 10. □

Proof of Proposition 5. Following its definition, the *constant* property is the set of pairs $(\mathcal{G}, \mathcal{G}')$, where $\mathcal{G}$ and $\mathcal{G}'$ are argumentation graphs such that $|\mathbf{E}| = |\mathbf{E}'|$. According to Proposition 4, a change $\mathcal{O}$ satisfies the *constant* property if and only if its inverse change $\mathcal{O}'$ satisfies the symmetric of the *constant* property. And, according to Definitions 10 and 7, the symmetric of the *constant* property is the *constant* property itself. So a change is *constant* if and only if its inverse change is *constant* as well. □

The proofs of Propositions 6 to 8 are similar to the one of Proposition 5.

Proof of Proposition 9. Knowing that the different change properties that we have defined constitute a partition of all possible changes (in the *constant* case), and relying on previous propositions, a change is *c-altering* if and only if its inverse change is *c-altering* as well. □

Proof of Lemma 1. The fact that Z does not attack $\mathcal{E}'$ is equivalent to the fact that if Z attacks an argument $X \in \mathcal{G}'$ then $X \notin \mathcal{E}'$. Let $\mathcal{U}$ be the set of unattacked arguments of $\mathcal{G}'$, *i.e.* the set of arguments different from Z unattacked by $\mathcal{G} \setminus \{Z\}$. We have $\mathcal{E}' = \mathcal{U} \cup \bigcup_{i \geq 1} \mathcal{F}'^i(\mathcal{U})$. Thus, we have $X \notin \mathcal{E}'$ if and only if $X \notin \mathcal{U}$ and $X \notin \bigcup_{i \geq 1} \mathcal{F}'^i(\mathcal{U})$, that is to say if and only if X is attacked by $\mathcal{G} \setminus \{Z\}$ and X is not indirectly defended by $\mathcal{U}$ in $\mathcal{G} \setminus \{Z\}$. □

The proof of Lemma 2 needs some intermediary results that are given in Lemmata 3 to 7.

Lemma 3. *When removing an argument Z under the grounded semantics, if Z does not attack $\mathcal{U}$, then $\mathcal{U} \subseteq \mathcal{F}(\mathcal{U})$ and so for each $i \geq 1$, $\mathcal{F}^i(\mathcal{U}) \subseteq \mathcal{F}^{i+1}(\mathcal{U})$.*

Proof of Lemma 3. By definition, $\mathcal{U}$ is the set of the unattacked arguments in $\mathcal{G} \setminus \{Z\}$. So, if Z does not attack $\mathcal{U}$, no argument of $\mathcal{U}$ is attacked by $\mathcal{G}$. Thus, $\mathcal{U} \subseteq \mathcal{F}(\mathcal{U})$. Moreover, since $\mathcal{F}$ is monotonic, for each $i \geq 1$, $\mathcal{F}^i(\mathcal{U}) \subseteq \mathcal{F}^{i+1}(\mathcal{U})$. □

Lemma 4. *Let $\mathcal{S} \subseteq \mathcal{G} \setminus \{Z\}$. When removing an argument Z under the grounded semantics, $\mathcal{F}(\mathcal{S}) \setminus \{Z\} \subseteq \mathcal{F}'(\mathcal{S})$.*

Proof of Lemma 4. Let $Y \in \mathcal{F}(\mathcal{S}) \setminus \{Z\}$. Two cases are possible. First, if Y is unattacked by $\mathcal{G}'$, Y trivially belongs to $\mathcal{F}'(\mathcal{S})$. Secondly, suppose that Y is attacked by $\mathcal{G}'$. $\forall W \in \mathcal{G}'$ such that W attacks Y in $\mathcal{G}'$, as $Y \in \mathcal{F}(\mathcal{S})$, $\mathcal{S}$ attacks W in $\mathcal{G}$. Since $\mathcal{S} \subseteq \mathcal{G} \setminus \{Z\}$, $\mathcal{S}$ defends Y in $\mathcal{G}'$ and thus $Y \in \mathcal{F}'(\mathcal{S})$. □

Lemma 5. *When removing an argument Z under the grounded semantics, if Z does not attack $\mathcal{U}$, then $\forall i \geq 1$, if $Z \notin \mathcal{F}^i(\mathcal{U})$, then $\mathcal{F}^i(\mathcal{U}) \subseteq \mathcal{F}'^i(\mathcal{U})$.*

Proof of Lemma 5. Let us prove by induction on $i \geq 1$ that if Z does not attack $\mathcal{U}$ and $Z \notin \mathcal{F}^i(\mathcal{U})$, then $\mathcal{F}^i(\mathcal{U}) \subseteq \mathcal{F}'^i(\mathcal{U})$.

- Basic case ($i = 1$): $Z \notin \mathcal{F}(\mathcal{U})$. Using Lemma 4 with $\mathcal{S} = \mathcal{U}$, we obtain $\mathcal{F}(\mathcal{U}) \subseteq \mathcal{F}'(\mathcal{U})$.
- Induction hypothesis (for $1 \leq i \leq p$, the proposition holds): let $Z \notin \mathcal{F}^{p+1}(\mathcal{U})$, due to Lemma 3, $Z \notin \mathcal{F}^p(\mathcal{U})$. Moreover, $\mathcal{F}^{p+1}(\mathcal{U}) = \mathcal{F}(\mathcal{F}^p(\mathcal{U}))$. Using Lemma 4 with $\mathcal{S} = \mathcal{F}^p(\mathcal{U})$, we obtain $\mathcal{F}^{p+1}(\mathcal{U}) \subseteq \mathcal{F}'(\mathcal{F}^p(\mathcal{U}))$. Using the induction hypothesis (since $Z \notin \mathcal{F}^p(\mathcal{U})$), we have $\mathcal{F}'(\mathcal{F}^p(\mathcal{U})) \subseteq \mathcal{F}'(\mathcal{F}'^p(\mathcal{U}))$, so by transitivity $\mathcal{F}^{p+1}(\mathcal{U}) \subseteq \mathcal{F}'^{p+1}(\mathcal{U})$.

□

Lemma 6. *Let $\mathcal{S} \subseteq \mathcal{G}$. When removing an argument Z under the grounded semantics, if Z does not attack $\mathcal{F}'(\mathcal{S})$, then $\mathcal{F}'(\mathcal{S}) \subseteq \mathcal{F}(\mathcal{S})$.*

Proof of Lemma 6. Let $X \in \mathcal{F}'(\mathcal{S})$. Since we are within the context of the removal of the argument Z, we know that $X \neq Z$. We have to prove that $X \in \mathcal{F}(\mathcal{S})$. Let $Y \in \mathcal{G}$ be an argument that attacks X in $\mathcal{G}$. Since Z does not attack $\mathcal{F}'(\mathcal{S})$, we know that $Y \neq Z$, hence Y attacks X in $\mathcal{G}'$. So there exists an argument $W \in \mathcal{S}$ such that W attacks Y in $\mathcal{G}'$, so $W \neq Z$ and W also attacks Y in $\mathcal{G}$, and thus $X \in \mathcal{F}(\mathcal{S})$. □

Lemma 7. *When removing an argument Z under the grounded semantics, if Z does not attack $\mathcal{E}'$, then $\forall k \geq 1$, $\mathcal{F}'^k(\mathcal{U}) \subseteq \mathcal{F}^k(\mathcal{U})$.*

Proof of Lemma 7. Let us prove by induction on $k \geq 1$ that if Z does not attack $\mathcal{E}'$, then $\mathcal{F}'^k(\mathcal{U}) \subseteq \mathcal{F}^k(\mathcal{U})$.

- Basic case ($k = 1$): since Z does not attack $\mathcal{E}'$, so Z does not attack $\mathcal{F}'(\mathcal{U})$ (since $\mathcal{U} \subseteq \mathcal{E}'$ and $\mathcal{F}'(\mathcal{U}) \subseteq \mathcal{F}'(\mathcal{E}') = \mathcal{E}'$). So using Lemma 7, we have $\mathcal{F}'(\mathcal{U}) \subseteq \mathcal{F}(\mathcal{U})$.
- Induction hypothesis (for $1 \leq k \leq p$, the proposition holds): we have to prove that $\mathcal{F}'^{p+1}(\mathcal{U}) \subseteq \mathcal{F}^{p+1}(\mathcal{U})$. By definition, $\mathcal{F}'^{p+1}(\mathcal{U}) = \mathcal{F}'(\mathcal{F}'^{p}(\mathcal{U}))$. Using Lemma 6, we have $\mathcal{F}'(\mathcal{F}'^{p}(\mathcal{U})) \subseteq \mathcal{F}(\mathcal{F}'^{p}(\mathcal{U}))$ (with $\mathcal{S} = \mathcal{F}'^{p}(\mathcal{U})$). Using the induction hypothesis, we have $\mathcal{F}'^{p}(\mathcal{U}) \subseteq \mathcal{F}^{p}(\mathcal{U})$. Hence, $\mathcal{F}'^{p+1}(\mathcal{U}) \subseteq \mathcal{F}^{p+1}(\mathcal{U})$.

□

Proof of Lemma 2.

(⇒) Let us show that if Z does not attack $\mathcal{E}'$ and $Z \in \bigcup_{i\geq 1} \mathcal{F}^i(\mathcal{U})$, then $\mathcal{E}'$ defends Z in $\mathcal{G}$: let $Y \in \mathcal{G}$ be an argument that attacks Z in $\mathcal{G}$. Let us recall that, by Definition 5, an argument can not attack itself, so $Y \neq Z$. Using Lemma 3, since Z does not attack $\mathcal{E}'$ and thus Z does not attack $\mathcal{U}$, we have $\mathcal{U} \subseteq \mathcal{F}(\mathcal{U})$ and thus the $\mathcal{F}^i(\mathcal{U})$ are nested. Let i be the smallest index ≥ 1 such that $Z \in \mathcal{F}^i(\mathcal{U})$, hence $Z \notin \mathcal{F}^{i-1}(\mathcal{U})$.

- $i = 1$: $Z \in \mathcal{F}(\mathcal{U})$. By definition, $\mathcal{U}$ defends Z, and $\mathcal{U} \subseteq \mathcal{E}'$ thus $\mathcal{E}'$ defends Z in $\mathcal{G}$.
- $i > 1$: $Z \in \mathcal{F}^i(\mathcal{U}) = \mathcal{F}(\mathcal{F}^{i-1}(\mathcal{U}))$ and $Z \notin \mathcal{F}^{i-1}(\mathcal{U})$. Using Lemma 5, we have $\mathcal{F}^{i-1}(\mathcal{U}) \subseteq \mathcal{E}'$. Hence, $Z \in \mathcal{F}(\mathcal{E}')$, so $\mathcal{E}'$ defends Z in $\mathcal{G}$.

(⇐) Let us show that if Z does not attack $\mathcal{E}'$ and $\mathcal{E}'$ defends Z in $\mathcal{G}$, then $Z \in \bigcup_{i\geq 1} \mathcal{F}^i(\mathcal{U})$:

- If Z is not attacked by $\mathcal{G}$, then we trivially have $Z \in \bigcup_{i\geq 1} \mathcal{F}^i(\mathcal{U})$.
- Let us suppose that Z is attacked by $\mathcal{G}$. By definition, we have $\mathcal{E}' = \mathcal{U} \cup \bigcup_{i\geq 1} \mathcal{F}'^{i}(\mathcal{U})$. Hence, $\exists i \geq 0$ such that $\mathcal{F}'^{i}(\mathcal{U})$ defends Z in $\mathcal{G}$.
  - $i = 0$: $\mathcal{U}$ defends Z in $\mathcal{G}$ and so $Z \in \bigcup_{i\geq 1} \mathcal{F}^i(\mathcal{U})$.
  - $i \geq 1$: We know that Z does not attack $\mathcal{E}'$. Using Lemma 7, we deduce that $\mathcal{F}'^{i}(\mathcal{U}) \subseteq \mathcal{F}^i(\mathcal{U})$. Since $\mathcal{F}'^{i}(\mathcal{U})$ defends Z, $\mathcal{F}^i(\mathcal{U})$ defends Z. Thus, $Z \in \bigcup_{j\geq 1} \mathcal{F}^j(\mathcal{U})$.

□

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