Efficient coalitions in Boolean games

Elise Bonzon Marie-Christine Lagasquie-Schiex Jérôme Lang

IRIT / RR- -2007-13- -FR

Abstract

Boolean games are a logical setting for representing strategic games in a succinct way, taking advantage of the expressive power and conciseness of propositional logic. A Boolean game consists of a set of players, each of which controls a set of propositional variables and has a specific goal expressed by a propositional formula. We show here that Boolean games are a very simple setting, yet sophisticated enough, for studying coalitions. Due to the fact that players have dichotomous preferences, the following notion emerges naturally: a coalition in a Boolean game is efficient if it has the power to guarantee that all goals of the members of the coalition are satisfied. We study the properties of efficient coalitions and address computational complexity issues.

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Chapter 1

Introduction

Boolean games [HvdHMW01, Har04, DvdH04, BLSLZ06] are a logical setting for representing strategic games in a succinct way, taking advantage of the expressive power and conciseness of propositional logic. Informally, a Boolean game consists of a set of players, each of which controls a set of propositional variables and has a specific goal expressed by a propositional formula[1]. Thus, a player in a Boolean game has a dichotomous preference relation: either her goal is satisfied or it is not. This restriction may appear at first glance unreasonable. However, many concrete situations can be modelled as games where agents have dichotomous preferences (we give such an example in the paper). Moreover, due to the fact that players have dichotomous preferences, the following simple (yet sophisticated enough) notion emerges naturally: a coalition in a Boolean game is efficient if it has the power to guarantee that all goals of the members of the coalition are satisfied. Our aim in the following is to define and characterize efficient coalitions, and see how they are related to the well-known concept of core.

After recalling some background of Boolean games in Section 2, we study in Section 3 the properties of effectivity functions associated with Boolean games. In Section 4 we study in detail the notion of efficient coalitions. We give an exact characterization of sets of coalitions that can be obtained as the set of efficient coalitions associated with a Boolean game, and we relate coalition efficiency to the notion of core. In Section 5 we address some computational issues. Related work and further issues are discussed in Section 6, and all the proofs are given in the Appendix.

[1] We refer here to the version of Boolean games defined in [BLSLZ06], that generalizes the initial proposal [HvdHMW01].

Chapter 2

n-player Boolean games

For any finite set $V = \{a, b, \ldots\}$ of propositional variables, L_V denotes the propositional language built up from V, the Boolean constants $\top$ and $\bot$, and the usual connectives. Formulas of L_V are denoted by φ, ψ etc. A *literal* is a variable x of V or the negation of a literal. A *term* is a consistent conjunction of literals. A *clause* is a disjunction of literals. If α is a term, then $Lit(\alpha)$ is the set of literals appearing in α. If $\varphi \in L_V$, then $Var(\varphi)$ denotes the set of propositional variables appearing in φ.
2^V is the set of the interpretations for V, with the usual convention that for $M \in 2^V$ and $x \in V$, M gives the value *true* to x if $x \in M$ and *false* otherwise. $\models$ denotes the consequence relation of classical propositional logic.
Let $V' \subseteq V$. A V'-interpretation is a truth assignement to each variable of V', that is, an element of $2^{V'}$. (Note that a V-interpretation is an interpretation.) V'- interpretations are denoted by listing all variables of V', with a $^-$ symbol when the variable is set to false: for instance, let $V' = \{a, b, d\}$, then the V'-interpretation $M = \{a, d\}$ assigning a and d to true and b to false is denoted by $a\bar{b}d$. If $Var(\varphi) \subseteq X$, then $Mod_X(\varphi)$ represents the set of X-interpretations satisfying φ.
If $\{V_1, \ldots, V_p\}$ is a partition of V and $\{M_1, \ldots, M_p\}$ are partial interpretations, where $M_i \in 2^{V_i}$, $(M_1, \ldots, M_p)$ denotes the interpretation $M_1 \cup \ldots \cup M_p$.
Given a set of propositional variables V, a Boolean game on V is an n-player game[1], where the actions available to each player consist in assigning a truth value to each variable in a given subset of V. The preferences of each player i are represented by a propositional formula φ_i formed upon the variables in V.

Definition 1 *An n-**player Boolean game** is a 5-tuple $(N, V, \pi, \Gamma, \Phi)$, where*

- *$N = \{1, 2, \ldots, n\}$ is a set of players (also called agents);*
- *V is a set of propositional variables;*
- *$\pi : N \mapsto V$ is a control assignment function;*
- *$\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ is a set of constraints, where each γ_i is a satisfiable propositional formula of $L_{\pi(i)}$;*
- *$\Phi = \{\varphi_1, \ldots, \varphi_n\}$ is a set of goals, where each φ_i is a satisfiable formula of L_V.*

A 4-tuple (N, V, π, Γ), with N, V, π, Γ defined as above, is called a **pre-Boolean game**.

[1] In the original proposal [HvdHMW01], Boolean games are two-players zero-sum games. However the model can easily be generalized to n players and non necessarily zero-sum games [BLSLZ06].

The control assignment function π maps each player to the variables she controls. For ease of notation, the set of all the variables controlled by i is written π_i instead of $\pi(i)$. Each variable is controlled by one and only one agent, that is, $\{\pi_1, \ldots, \pi_n\}$ forms a partition of V.
For each i, γ_i is a constraint restricting the possible strategy profiles for player i.

Definition 2 *Let $G = (N, V, \pi, \Gamma, \Phi)$ be a Boolean game. A* **strategy**[2] *for player i in G is a π_i-interpretation satisfying γ_i. The set of strategies for player i in G is $S_i = \{s_i \in 2^{\pi_i} \mid s_i \models \gamma_i\}$. A* **strategy profile** *s for G is an n-tuple $s = (s_1, s_2, \ldots, s_n)$ where for all i, $s_i \in S_i$. $S = S_1 \times \ldots \times S_n$ is the set of all strategy profiles.*

Note that since $\{\pi_1, \ldots, \pi_n\}$ forms a partition of V, a strategy profile s is an interpretation for V, i.e., $s \in 2^V$.
The following notations are usual in game theory. Let $s = (s_1, \ldots, s_n)$ a strategy profile. For any nonempty set of players $I \subseteq N$, the projection of s on I is defined by $s_I = (s_i)_{i \in I}$ and $s_{-I} = s_{N \setminus I}$. If $I = \{i\}$, we denote the projection of s on $\{i\}$ by s_i instead of $s_{\{i\}}$; similarly, we note s_{-i} instead of $s_{-\{i\}}$. π_I denotes the set of the variables controlled by I, and $\pi_{-I} = \pi_{N \setminus I}$. The set of strategies for $I \subseteq N$ is $S_I = \times_{i \in I} S_i$, and the set of goals for $I \subseteq N$ is $\Phi_I = \bigwedge_{i \in I} \varphi_i$.
If s and s' are two strategy profiles, (s_{-I}, s'_I) denotes the strategy profile obtained from s by replacing s_i with s'_i for all $i \in I$.
The goal φ_i of player i is a compact representation of a dichotomous preference relation, or equivalently, of a binary utility function $u_i : S \to \{0, 1\}$ defined by $u_i(s) = 0$ if $s \models \neg\varphi_i$ and $u_i(s) = 1$ if $s \models \varphi_i$. s is at least as good as s' for i, denoted by $s \succeq_i s'$, if $u_i(s) \geq u_i(s')$, or equivalently, if $s \models \neg\varphi_i$ implies $s' \models \neg\varphi_i$; s is strictly better than s' for i, denoted by $s \succ_i s'$, if $u_i(s) > u_i(s')$, or, equivalently, $s \models \varphi_i$ and $s' \models \neg\varphi_i$.
Note that this choice of binary utilites clearly implies a loss of generality. However, some interesting problems, as in Example 2, have preferences that are naturally dichotomous. Boolean games allow to represent these problems in a compact way. Furthermore, Boolean games can easily be extended so as to allow for non-dichotomous preferences, represented in some compact language for preference representation (see [BLSL06]).

[2] In this report, only pure strategies are considered.

Chapter 3

Coalitions and effectivity functions in Boolean games

One of the features of Boolean games is the definition of individual strategies as truth assignments of a given set of propositional variables. We might wonder how restrictive this specificity is. In this Section we study Boolean games from the point of view of effectivity functions. Effectivity functions have been developed in social choice to model the ability of coalitions [Mou83, AK91, Pau01]. Clearly, the definition of S_i as $Mod_{\pi_i}(\gamma_i)$ induces some constraints on the power of players. Our aim is to give an exact characterization of effectivity functions induced by Boolean games.
Since in Boolean games the power of an agent i is independent from her goal φ_i, it suffices to consider pre-Boolean games only when dealing with effectivity functions.
As usual, a **coalition** C is any subset of N. N is called the **grand coalition**.

Definition 3 *A* **coalitional effectivity function** *is a function* $\mathrm{Eff}\colon 2^N \to 2^{2^S}$ *which is monotonic: for every coalition* $C \subseteq N$, $X \in \mathrm{Eff}(C)$ *implies* $Y \in \mathrm{Eff}(C)$ *whenever* $X \subseteq Y \subseteq S$.

The function Eff associates to every group of players the set of outcomes for which the group is effective. We usually interpret $X \in \mathrm{Eff}(C)$ as "the players in C have a joint strategy for bringing about an outcome in X". A pre-Boolean game G induces an effectivity function Eff_G as follows:

Definition 4 *Let* $G = (N, V, \pi, \Gamma)$ *be a pre-Boolean game. The coalitional* α*-effectivity function induced by* G *is the function* $\mathrm{Eff}_G : 2^N \to 2^{2^S}$ *defined by: for any* $X \subseteq S$ *and any* $C \subseteq N$, $X \in \mathrm{Eff}_G(C)$ *if there exists* $s_C \in S_C$ *such that for any* $s_{-C} \in S_{-C}$, $(s_C, s_{-C}) \in X$. [1]

This definition is a particular case of the α-effectivity function induced by a strategic game (see [Pau01], chapter 2). Therefore, these functions satisfy the following properties (cf. [Pau01], Theorem 2.27):

(i) $\forall C \subseteq N$, $\varnothing \notin \mathrm{Eff}(C)$;

(ii) $\forall C \subseteq N$, $S \in \mathrm{Eff}(C)$;

(iii) for all $X \subseteq S$, if $\bar{X} \notin \mathrm{Eff}(\varnothing)$ then $X \in \mathrm{Eff}(N)$;

[1] Remark that effectivity functions induced by pre-Boolean games may be equivalently expressed as mappings $\mathrm{Eff} : 2^N \to 2^{L_V}$ from coalitions to sets of logical formulas: $\varphi \in \mathrm{Eff}(I)$ if $Mod_{\pi_I}(\varphi) \in \mathrm{Eff}(I)$. This definition obviously implies syntax-independence, that is, if $\varphi \equiv \psi$ then $\varphi \in \mathrm{Eff}(I)$ iff $\psi \in \mathrm{Eff}(I)$.

(iv) Eff is superadditive, that is, if for all $C, C' \subseteq N$ and $X, Y \subseteq S$, if $X \in Eff(C)$ and $Y \in Eff(C')$ then $W \cap Y \in Eff(C \cup C')$.

An effectivity function satisfying these four properties is called **strongly playable**. Note that strong playability implies regularity and coalition-monotonicity ([Pau01], Lemma 2.26).
However, pre-Boolean games are a specific case of strategic game forms, therefore we would like to have an exact characterization of those effectivity functions that correspond to a pre-Boolean game. We first have to define two additional properties. Define $At(C)$ as the minimal sets in $\mathrm{Eff}(C)$, that is, $At(C) = \{X \in \mathrm{Eff}(C) |$ there is no $Y \in \mathrm{Eff}(C)$ such that $Y \subseteq X\}$.

Atomicity: Eff satisfies *atomicity* if for every $C \subseteq N$, $At(C)$ forms a partition of S.

Decomposability: Eff satisfies *decomposability* if for every $I, J \subseteq N$ and for every $X \subseteq S$, $X \in \mathrm{Eff}(I \cup J)$ if and only if there exist $Y \in \mathrm{Eff}(I)$ and $Z \in \mathrm{Eff}(J)$ such that $X = Y \cap Z$.

Note that decomposability is a strong property that implies superadditivity.

Proposition 1 *A coalitional effectivity function* Eff *satisfies (1) strong playability, (2) atomicity, (3) decomposability and (4)* $\mathrm{Eff}(N) = 2^S \setminus \varnothing$ *if and only if there exists a Boolean game* $G = (N, V, \pi, \Gamma)$ *and an injective function* $\mu : S \to 2^V$ *such that for every* $C \subseteq N$*:* $\mathrm{Eff}_G(C) = \{\mu(X) | X \in \mathrm{Eff}(C)\}$.

The proof is in the Appendix.

Chapter 4

Efficient coalitions

We now consider full Boolean games and define *efficient coalitions*. Informally, a coalition is efficient in a Boolean game if and only if it has the ability to jointly satisfy the goals of all members of the coalition:

Definition 5 *Let $G = (N, V, \pi, \Gamma, \Phi)$ be a Boolean game. A coalition $C \subseteq N$ is* **efficient** *if and only if $\exists s_C \in S_C$ such that $\forall s_{-C}$, $s_C \models \bigwedge_{i \in C} \varphi_i$. C is a* **minimal efficient coalition** *if and only if $C \neq \varnothing$, C is efficient and for all $D \subset C$, D is not efficient. The set of all efficient coalitions of a game G is denoted by* $\mathrm{EC}(G)$.

Example 1 *Let $G = (N, V, \Gamma, \pi, \Phi)$ where $V = \{a, b, c\}$, $N = \{1, 2, 3\}$, $\gamma_i = \top$ for every i, $\pi_1 = \{a\}$, $\pi_2 = \{b\}$, $\pi_3 = \{c\}$, $\varphi_1 = (\neg a \wedge b)$, $\varphi_2 = (\neg a \vee \neg c)$ and $\varphi_3 = (\neg b \wedge \neg c)$.*
Remark first that $\varphi_1 \wedge \varphi_3$ is inconsistent, therefore no coalition containing $\{1, 3\}$ can be efficient. $\{1\}$ is not efficient, because φ_1 cannot be made true only by fixing the value of a; similarly, $\{2\}$ and $\{3\}$ are not efficient either. $\{1, 2\}$ is efficient, because the joint strategy $s_{\{1,2\}} = \overline{a}b$ is such that $s_{\{1,2\}} \models \varphi_1 \wedge \varphi_2$. $\{2, 3\}$ is efficient, because $s_{\{2,3\}} = \overline{b}\overline{c} \models \varphi_2 \wedge \varphi_3$. Obviously, $\varnothing$ is efficient[1]*, because $\varphi_\varnothing = \bigwedge_{i \in \varnothing} \varphi_i \equiv \top$ is always satisfied. Therefore,* $\mathrm{EC}(G) = \{\varnothing, \{1, 2\}, \{2, 3\}\}$.

From this simple example we see already that EC is neither downward closed nor upward closed, that is, if C is efficient then a subset or a superset of C may not be efficient. We also see that EC is not closed under union or intersection: $\{1, 2\}$ and $\{2, 3\}$ are efficient, but neither $\{1, 2\} \cap \{2, 3\}$ nor $\{1, 2\} \cup \{2, 3\}$ is.

Example 2 (kidney exchange, after [ABS07]) *Consider n pairs of individuals, each consisting of a recipient R_i in urgent need of a kidney transplant, and a donor D_i who is ready to give one of her kidneys to save R_i. As D_i's donor kidney is not necessarily compatible with R_i, a strategy for saving more people consist in considering the graph $\langle \{1, \ldots, n\}, E \rangle$ containing a node $1, \ldots, n$ for each pair (D_i, R_i) and containing the edge (i, j) whenever D_i's kidney is compatible with R_j. A solution is any set of nodes that can be partitioned into disjoint cycles in the graph: in a solution, a donor D_i gives a kidney if and only if R_i is given one. An optimal solution (saving a maximum number of lifes) is a solution with a maximum number of nodes. The problem can be seen as the following Boolean game G:*

- *$N = \{1, \ldots, n\}$;*
- *$V = \{g_{ij} | i, j \in \{1, \ldots, n\}\}$; g_{ij} being true means that D_i gives a kidney to R_j.*

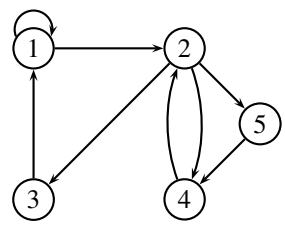
[1] One may argue that it makes little sense to say that the empty coalition is efficient. Anyway, the definition of an efficient coalition could be changed so as to exclude $\varnothing$, Further notions and results would be unchanged.

- $\pi_i = \{g_{ij}; 1 \leq j \leq n\}$;
- *for every i,* $\gamma_i = \bigwedge_{j \neq k} \neg(g_{ij} \wedge g_{ik})$ *expresses that a donor cannot give more than one kidney.*
- *for every i,* $\varphi_i = \bigvee_{(j,i) \in E} g_{ji}$ *expresses that the goal of i is to be given a kidney that is compatible with* R_i.

For example, take $n = 5$ *and* $E = \{(1,1),(1,2),(2,3),(2,4),(2,5),(3,1),(4,2),(5,4)\}$. *Then* $G = (N, V, \Gamma, \pi, \Phi)$, *with*

- $N = \{1,2,3,4,5\}$
- $V = \{g_{ij} \mid 1 \leq i, j \leq 5\}$;
- $\forall i$, $\gamma_i = \bigwedge_{j \neq k} \neg(g_{ij} \wedge g_{ik})$
- $\pi_1 = \{g_{11}, g_{12}, g_{13}, g_{14}, g_{15}\}$, *and similarly for* π_2, *etc.*
- $\varphi_1 = g_{11} \vee g_{31}$; $\varphi_2 = g_{12} \vee g_{42}$; $\varphi_3 = g_{23}$; $\varphi_4 = g_{24} \vee g_{54}$; $\varphi_5 = g_{25}$.

The corresponding graph is depicted below.



Clearly enough, efficient coalitions correspond to solutions. In our example, the efficient coalitions are $\varnothing$, $\{1\}$, $\{2,4\}$, $\{1,2,4\}$, $\{1,2,3\}$, $\{2,4,5\}$ *and* $\{1,2,4,5\}$.

We have seen that the set of efficient coalitions associated with a Boolean game may not be downward closed nor upward closed, nor closed under union or non-empty intersection. We find that it is possible to characterize the efficient coalitions of a Boolean game.

Proposition 2 *Let* $N = \{1,\ldots,n\}$ *be a set of agents and* $\mathcal{SC} \in 2^{2^N}$ *a set of coalitions. There exists a Boolean game G over N such that the set of efficient coalitions for G is* $\mathcal{SC}$ *(i.e.* $\mathrm{EC}(G) = \mathcal{SC}$*) if and only if* $\mathcal{SC}$ *satisfies these two properties:*

(1) $\varnothing \in \mathcal{SC}$.

(2) *for all* $I, J \in \mathcal{SC}$ *such that* $I \cap J = \varnothing$ *then* $I \cup J \in \mathcal{SC}$.

Thus, a set of coalitions corresponds to the set of efficient coalitions for some Boolean game if and only if (a) it contains the emptyset and (b) it is closed by union of disjoint coalitions. The proof is in the Appendix. The left-to-right direction is proven easily; intuitively, when two disjoint coalitions I and J are efficient, each has a strategy guaranteeing its goals to be satisfied, and the joint strategies of I and J guarantee that the goals of all agents in $I \cup J$ is satisfied. As seen in Example 1, this is no longer true when I and J intersect. The right-to-left direction of the proof is more involved and needs a specific Boolean game to be constructed for each set of coalitions $\mathcal{SC}$ satisfying (1) and (2).

The notion of efficient coalition is the same than the one of successful coalition in qualitative coalitional games (QCG) introduced in [WD04].

We now relate the notion of efficient coalitions to the usual notion of core of a coalitional game. In coalitional games with ordinal preferences, the core is usually defined as follows (see e.g. [Aum67, Owe82, Mye91]): a strategy profile s is in the core of a coalitional game if and only if there exists no coalition C with a joint strategy s_C that guarantees that all members of C are better off than with s. Here we consider also a stronger notion of core: a strategy profile s is in the strong core of a coalitional game if and only if there exists no coalition C with a joint strategy s_C that guarantees that all members of C are at least as satisfied as with s, and at least one member of C is strictly better off than with s.

Definition 6 *Let G be a Boolean game.*
The **(weak) core** *of G, denoted by $WCore(G)$, is the set of strategy profiles $s = (s_1, \ldots, s_n)$ such that there exists no $C \subset N$ and no $s_C \in S_C$ such that for every $i \in C$ and every $s_{-C} \in S_{-C}$, $(s_C, s_{-C}) \succ_i s$.*
The **strong core** *of a Boolean game G, denoted by $SCore(G)$, is the set of strategy profiles $s = (s_1, \ldots, s_n)$ such that there exists no $C \subset N$ and no $s_C \in S_C$ such that for every $i \in C$ and every $s_{-C} \in S_{-C}$, $(s_C, s_{-C}) \succeq_i s$ and there is an $i \in C$ such that for every $s_{-C} \in S_{-C}$, $(s_C, s_{-C}) \succ_i s$.*

This concept of weak core is equivalent to the notion of **strong Nash equilibrium** introduced by [Aum59], where coalitions form in order to correlate the strategies of their members. This notion involves, at least implicitly, the assumption that cooperation necessarily requires that players be able to sign "binding agreements": players have to follow the strategies they have agreed upon, even if some of them, in turn, might profit by deviating. However, if players of a coalition C agreed for a strategy s_C, at least one player $i \in C$ is satisfied by this strategy: we have $\exists i \in C$ such that $s \models \varphi_i$.

The relationship between the (weak) core of a Boolean game and its set of efficient coalitions is expressed by the following simple result:

Proposition 3 *Let $G = (N, V, \Gamma, \pi, \Phi)$ be a Boolean game. $s = (s_1, \ldots, s_n) \in WCore(G)$ if and only if s satisfies at least one member of every efficient coalition, that is, for every $C \in \mathrm{EC}(G)$, $s \models \bigvee_{i \in C} \varphi_i$.*

In particular, when no coalition of a Boolean game G is efficient, then all strategy profiles are in $WCore(G)$. Moreover, the weak core of a Boolean game cannot be empty:

Proposition 4 *For any Boolean game G, $WCore(G) \neq \varnothing$.*

The strong core of a Boolean game is harder to characterize in terms of efficient coalitions. We only have the following implication.

Proposition 5 *Let $G = (N, V, \Gamma, \pi, \Phi)$ be a Boolean game, and s be a strategy profile. If $s \in SCore(G)$ then for every $C \in \mathrm{EC}(G)$ and every $i \in C$, $s \models \varphi_i$.*

Thus, a strategy in the strong core of G satisfies the goal of every member of every efficient coalition. The following counterexample shows that the converse does not hold.

Example 3 *Let $G = (N, V, \Gamma, \pi, \Phi)$ be a boolean game. We have: $V = \{a, b, c, d, e, f\}$, $N = \{1, 2, 3, 4, 5, 6\}$, $\Gamma = \top$, $\pi_1 = \{a\}$, $\pi_2 = \{b\}$, $\pi_3 = \{c\}$, $\pi_4 = \{d\}$, $\pi_5 = \{e\}$, $\pi_6 = \{f\}$, $\varphi_1 = b \vee d$, $\varphi_2 = a \vee c$, $\varphi_3 = \neg b \vee d$, $\varphi_4 = e$, $\varphi_5 = \neg a \wedge \neg b \wedge \neg c$ and $\varphi_6 = \neg a \wedge \neg c \wedge \neg d$.*
This game has two efficient coalitions: $\{1, 2\}$ and $\{2, 3\}$.

Let $s = abcd\overline{e}f$*. We have* $s \models \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \neg\varphi_4 \wedge \neg\varphi_5 \wedge \neg\varphi_6$*. So,* $\forall C \in \mathrm{EC}(G)$*,* $\forall i \in C$*,* $s \models \varphi_i$*.*
However, $s \notin SCore(G)$*:* $\exists C' = \{1,2,3,4,5\} \subset N$ *such that* $\exists s_C = abcde \models \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4 \wedge \neg\varphi_5$*. So,* $\forall s_{-C}$*,* $(s_C, s_{-C}) \succeq_1 s$*,* $(s_C, s_{-C}) \succeq_2 s$*,* $(s_C, s_{-C}) \succeq_3 s$*,* $(s_C, s_{-C}) \succeq_5 s$*, and* $(s_C, s_{-C}) \succ_4 s$*.* $s \notin SCore(G)$*.*

Note that the strong core of a Boolean game can be empty: in Example 1, the set of efficient coalitions is $\{\varnothing, \{1,2\}, \{2,3\}\}$, therefore there is no $s \in S$ such that for all $C \in \mathrm{EC}(G)$, for all $i \in C$, $s \models \varphi_i$, therefore, $SCore(G) = \varnothing$. However, we can show than the non-emptyness of the strong core is equivalent to the following simple condition on efficient coalitions.

Proposition 6 *Let* $G = (N, V, \Gamma, \pi, \Phi)$ *be a Boolean game. We have the following:*
$SCore(G) \neq \emptyset$ *if and only if* $\bigcup\{C \subseteq N | C \in \mathrm{EC}(G)\} \in \mathrm{EC}(G)$*, that is, if and only if the union of all efficient coalitions is efficient.*

Chapter 5

Computational issues

We start by identifying the complexity of the key decision problems related to efficient coalitions.

Proposition 7 *Deciding whether there exists a non-empty efficient coalition in a Boolean game is Σ_2^p-complete, and is Σ_2^p-hard even if $n = 2$.*

Proposition 8

- *deciding whether an agent i belongs to at least one efficient coalition for G is Σ_2^p-complete, and is Σ_2^p-hard even if $n = 2$.*
- *deciding whether an agent i belongs to at all non-empty efficient coalitions for G is Π_2^p-complete, and is Π_2^p-hard even if $n = 2$.*

Proposition 3 leads the following result:

Proposition 9 *Deciding whether a strategy profile s is in the weak core of a Boolean game G is Co-NP-complete.*

We now focus on simple results that may help us identifying efficient coalitions, using the syntactical nature of goals. Intuitively, if the goal φ_i of player i does not depend on any variable controlled by player j, then the satisfaction of i does not depend directly on j. This is only a sufficient condition: it may be the case that the syntactical expression of φ_i does mention a variable controlled by i, but that this variable plays no role whatsoever in the satisfaction of φ_i, as variable y in $\varphi_i = x \wedge (y \vee \neg y)$. We therefore use a stronger notion of formula-variable independence [LLM03].

Definition 7 *Let φ a propositional formula of L_V and $x \in V$. φ is* independent from *x if there exists a formula ψ being logically equivalent to φ and in which x does not appear.*[1]

Definition 8 *Let $G = (N, V, \Gamma, \pi, \Phi)$ be a Boolean game. The set of* **relevant variables** *for a player i, denoted by $RV_G(i)$, is the set of all variables $v \in V$ such that φ_i is not independent from v.*

[1] We have this equivalent semantical characterization of formula-variable independence [LLM03]: φ is independent from x if there exists an interpretation s such that $s \models \varphi$ and switch$(s,x) \models \neg\varphi_i$, where switch$(s,x)$ is obtained by switching the value of v in s, and leaving the values of other variables unchanged.

For the sake of notation, the set of relevant variables for a given Boolean game G will be denoted by RV_i instead of $RV_G(i)$.

We now easily define the *relevant players* for a given player i as the set of players controlling at least one variable of RV_i.

Definition 9 *Let $G = (N, V, \Gamma, \pi, \Phi)$ be a Boolean game. The set of* **relevant players** *for a player i, denoted by RP_i, is the set of agents $j \in N$ such as j controls at least one relevant variable of i: $RP_i = \bigcup_{v \in RV_i} \pi^{-1}(v)$.*

Again, the set of relevant players for a Boolean game G is denoted by $RP_G(i)$: for the sake of notation we simply write RP_i.
Let us illustrate these two notions by means of an example.

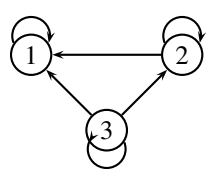
Example 4 *Let $G = (N, V, \Gamma, \pi, \Phi)$ be the Boolean game defined by $V = \{a, b, c\}$, $N = \{1, 2, 3\}$, $\Gamma = \varnothing$, $\pi_1 = \{a\}$, $\pi_2 = \{b\}$, $\pi_3 = \{c\}$, $\varphi_1 = \neg a$, $\varphi_2 = a \leftrightarrow b$ and $\varphi_3 = a \wedge \neg b \wedge \neg c$.*
We have: $RV_1 = \{a\}$, $RV_2 = \{a, b\}$, $RV_3 = \{a, b, c\}$, $RP_1 = \{1\}$, $RP_2 = \{1, 2\}$, $RP_3 = \{1, 2, 3\}$.

This relation between players can be seen as a directed graph containing a vertex for each player, and an edge from i to j whenever $j \in RP_i$, i.e. if j is a relevant player of i.

Definition 10 *Let $G = (N, V, \pi, \Phi)$ be a Boolean game. The* **dependency graph** *of G is the directed graph $\mathcal{P} = \langle N, R \rangle$, with $\forall i \in N$, $(i, j) \in R$ (denoted by $R(i, j)$) if $j \in RP_i$.*

Thus, $R(i)$ is the set of players from which i may need some action in order to be satisfied: $j \in R(i)$ if and only if $j \in RP_i$. Remark however that $j \in R(i)$ does not imply that i *needs* some action by j to see her goal satisfied. For instance, if $\pi_1 = \{a\}$, $\pi_2 = \{b\}$ and $\varphi_1 = a \vee b$, then $1 \in R(2)$; however, 1 has a strategy for satisfying her goal (setting a to true) and therefore does not have to rely on 2.

Example 4, continued: *The dependence graph $\mathcal{P}$ induced by G is depicted as follows:*



For instance we have $R^{-1}(1) = \{1, 2, 3\}$, $R^{-1}(2) = \{2, 3\}$ and $R^{-1}(3) = \{3\}$.

We already know that if two disjoint coalitions I and J are efficient then their union is efficient. The converse does not hold in the general case, that is, there may exist two disjoint sets I and J such that $I \cup J$ is efficient and neither I or J is. However, the converse holds in this specific case:

Proposition 10 *Let $G = (N, V, \Gamma, \pi, \Phi)$ be a Boolean game. Let I and J be be two coalitions such that $I \cap J = \varnothing$, $I \cup J$ is efficient, $R(I) \cap J = \varnothing$[2] and $R(J) \cap I = \varnothing$. Then, I and J are both efficient.*

Definition 11 *Let $G = (N, V, \Gamma, \pi, \Phi)$ be a Boolean game and R its associated dependence relation. $B \subseteq N$ is* **stable** *for R if and only if $R(B) \subseteq B$ that is, $\forall j \in B$, $\forall i$ such that $i \in R(j)$, then $i \in B$.*

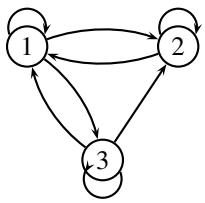
The following proposition is then straightforward:

Proposition 11 *Let $G = (N, V, \Gamma, \pi, \Phi)$ be a Boolean game. If $B \subseteq N$ is stable for R, then B is an efficient coalition of G ($B \subseteq \mathrm{EC}(G)$) if and only if $\varphi_B = \bigwedge_{i \in B} \varphi_i$ is consistent.*

[2] $R(I) = \bigcup_{i \in I} R(i)$.

The converse is not necessarily true, as we can see on the following example:

Example 5 *Let $G = (N, V, \Gamma, \pi, \Phi)$ be the Boolean game defined by $V = \{a, b, c\}$, $N = \{1, 2, 3\}$, $\pi_1 = \{a\}$, $\pi_2 = \{b\}$, $\pi_3 = \{c\}$, $\varphi_1 = (a \vee c) \wedge \neg b$, $\varphi_2 = a \wedge b$ and $\varphi_3 = a \wedge b \wedge c$.*
The players' dependence graph $\mathcal{P}$ of G is the following:



We have: $RV_1 = \{a, b, c\}$, $RV_2 = \{a, b\}$, $RV_3 = \{a, b, c\}$, $RP_1 = \{1, 2, 3\}$, $RP_2 = \{1, 2\}$, $RP_3 = \{1, 2, 3\}$.
The coalition $\{1, 2\}$ is efficient, but is not stable for R: $R(\{1, 2\}) = \{1, 2, 3\} \not\subseteq \{1, 2\}$.

However, the converse can be true under the very restrictive condition that the satisfaction of the goal of a player depends only on the actions of *one* player, that is, if RP_i is a singleton for every $i \in B$.

Proposition 12 *Let $G = (N, V, \Gamma, \pi, \Phi)$ be a Boolean game. If $B \subseteq N$ is an efficient coalition of G ($B \subseteq \mathrm{EC}(G)$) such that $\forall i \in B$, $|RP_i| = 1$, then B is stable for R. In this case, a coalition B such that $\varphi_B = \bigwedge_{i \in B} \varphi_i \not\models \bot$ is efficient if and only if B is stable for R.*

In this specific case where RP_i is a singleton for every $i \in B$, we have furthermore this intuitive graph-theoretic characterization of efficient coalitions:

Proposition 13 *Let $G = (N, V, \Gamma, \pi, \Phi)$ be a Boolean game such that $\forall i \in N$, $|RP_i| = 1$. For any coalition $C \subseteq N$, C forms a cycle in the dependence graph if and only if C is stable for R and constitutes a minimum efficient coalition.*

Another interesting issue is the study of efficient coalitions in Boolean games where goals have a specific syntactical structure. As it stands, when goals are either positive clauses (resp. positive terms), that is, clauses (resp. terms) containing only positive literals, then we have the following intuitive characterization of efficient coalitions:

Proposition 14 *Let $G = (N, V, \Gamma, \pi, \Phi)$ be a Boolean game.*

1. *if for every $i \in N$, φ_i is a positive term, then for any $B \subseteq N$, B is efficient if and only if B is stable for R.*

2. *if for every $i \in N$, φ_i is a positive clause, then for any $B \subseteq N$, B is efficient if and only if there exists a cycle in the dependence graph associated G whose nodes are exactly the members of B.*

Chapter 6

Conclusion

We have shown that Boolean games can be used as a compact representation setting for coalitional games where players have dichotomous preferences. This specificity lead us to define an interesting notion of efficient coalitions. We gave an exact characterization of sets of coalitions that correspond to the set of efficient coalitions for a Boolean game, and we gave several results concerning the computation of efficient coalitions.

Note that some of our notions and results do not explicitly rely on the use of propositional logic. For instance, efficient coalitions can be defined in a more general setting where goals are simply expressed as nonempty sets of states. However, many notions (in particular, the control assignment function π) become much less clear when abstracting from the propositional representation.

Clearly, a limitation of our results is that they apply to dichotomous preferences only. However, as illustrated on Example 2, some problems are naturally expressed with dichotomous goals. Moreover, it is always worth starting by studying simple cases, especially when they already raise complex notions.

As Boolean games, qualitative coalitional games (QCG), introduced in [WD04], are games in which agents are not assigned utility values over outcomes, but are satisfied if their goals are achieved. A first difference between QCG and Boolean games is that there is no control assignment function in QCG. A second one is that each agent in QCG can has a set of goals, and is satisfied if at least one of her goal is satisfied, whereas each agent in Boolean games has a unique goal. However, QCG's characteristic function, which associates to each coalition C the sets of goals that members of C can achieve, corresponds in Boolean games to the set $W(C) = \{X \subseteq \{\varphi_1, \ldots, \varphi_n\}$ such that $\exists s_C \in S_C : s_C \models \varphi_i\}$[1].

Coalition logic [Pau01] allows to express, for any coalition C and any formula φ, the ability of C to ensure that φ hold (which writes $[C]\varphi$). In Boolean games, the power of agents, expressed by the control assignment function π, is still in the metalanguage. Expressing π within coalition logic would however be possible, probably using ideas from [vdHW05]. The next step would then consist in introducing goals into coalition logic. This is something we plan to do in a near future.

[1] For instance, we have for Example 1 : $W(\{1\}) = W(\{3\}) = W(\{1,3\}) = \{\varphi_2\}$, $W(\{2\}) = \varnothing$, $W(\{1,2\}) = \{\varphi_1, \varphi_2\}$, $W(\{2,3\}) = \{\varphi_2, \varphi_3\}$, $W(\{1,2,3\}) = \{\{\varphi_1, \varphi_2\}, \{\varphi_2, \varphi_3\}\}$.

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Appendix A

Proofs

Proposition 1 *A coalitional effectivity function* Eff *satisfies (1) strong playability, (2) atomicity, (3) decomposability and (4)* $\mathrm{Eff}(N) = 2^S \setminus \varnothing$ *if and only if there exists a pre-Boolean game* $G = (N, V, \pi, \Gamma)$ *and an injective function* $\mu : S \to 2^V$ *such that for every* $C \subseteq N$*:* $\mathrm{Eff}_G(C) = \{\mu(X) | X \in \mathrm{Eff}(C)\}$.

If G is a pre-Boolean game, the set of atoms for the effectivity functions Eff_G will be denoted by At_G.

Lemma 1 *For any pre-Boolean game G,* Eff_G *satisfies strong playability, atomicity, decomposability, and (4).*

Proof: Eff_G is a (specific) α-effectivity function, therefore by Theorem 2.27 in [Pau01], Eff_G satisfies strong playability (and, a fortiori, superaddivity). For atomicity, remark first that $X \in At_G(C)$ if and only if X is the set of all π_C-interpretations satisfying $\gamma_C = \bigwedge_i \gamma_i$, which clearly implies that any two distinct subsets in $At_G(C)$ are disjoint. Then remark that $\bigcup_{s_C \in S_C} \{s \mid s \supset s_C\} = S$. Therefore, $At_G(C)$ forms a partition of S. For decomposability, from left to right: let $X \in \mathrm{Eff}_G(I \cup J)$. Then there exists a joint strategy $s_{I \cup J}$ such that if $W = \{s \in S | s \supseteq s_{I \cup J}\}$, then $W \subseteq X$. Consider now $Y = \{s \in S | s \supseteq s_I\}$ and $Z = \{s \in S | s \supseteq s_J\}$. We have $Y \in \mathrm{Eff}_G(I)$, $Z \in \mathrm{Eff}_G(J)$ and $X = Y \cap Z$. From right to left: let $Y \in \mathrm{Eff}_G(I)$ and $Z \in \mathrm{Eff}_G(J)$, then by superaddivity, $Y \cap Z \in \mathrm{Eff}_G(I \cup J)$. Lastly, it is readily checked that $\mathrm{Eff}_G(N) = 2^S \setminus \varnothing$. ■

Lemma 2 *If there exists a pre-Boolean game* $G = (N, V, \pi, \Gamma)$ *and an injective function* $\mu : S \to 2^V$ *such that for every* $C \subseteq N$*:* $\mathrm{Eff}_G(C) = \{\mu(X) | X \in \mathrm{Eff}(C)\}$*, then* Eff *satisfies strong playability, atomicity, decomposability and* $\mathrm{Eff}(N) = 2^S \setminus \varnothing$.

Proof: Eff_G satisfies these properties and μ is a bijection between S and $\mu(S)$, therefore these properties transfer to Eff. ■

Lemma 3 *Let G be a pre-Boolean game and* T_i *be a minimal subset of* $\mathrm{Eff}_G(i)$*. Then* $T_i = \{s | s \supseteq s_i\}$ *for some* $s_i \in S_i$.

Proof: i can only enforce a subset of S_i, that is, $X \in \mathrm{Eff}_G(i)$ if X contains $S_1 \times \ldots \times S_{i-1} \times S_i^* \times S_{i+1} \times \ldots S_n$ for some $S_i^* \subseteq S_i$. Therefore the minimal subsets of $\mathrm{Eff}_G(i)$ are exactly those of the form $S_1 \times \ldots \times S_{i-1} \times \{s_i\} \times S_{i+1} \times \ldots S_n$, that is, of the form $\{s | s \supseteq s_i\}$. ■

From now on, let Eff be a coalitional effecitivity function satisfying strong playability, atomicity, decomposability and $\text{Eff}(N) = 2^S \setminus \varnothing$.
Remark first that, due to decomposability, Eff is entirely determined by $\{At(i), i \in N\}$.

Lemma 4 *For every $s \in S$ there exist a unique $(Z_1, \ldots, Z_n)$ such that for every i, $Z_i \in At(i)$ and $Z_1 \cap \ldots \cap Z_n = \{s\}$.*

Proof: Let $s \in S$. Because $\text{Eff}(N) = 2^S \setminus \{\varnothing\}$, we have $\{s\} \in \text{Eff}(N)$, and by decomposability, there exist $(T_1, \ldots, T_n)$ such that for every i, $T_i \in \text{Eff}(i)$ and $T_1 \cap \ldots \cap T_n = \{s\}$. Let $i \in N$. By definition of $At(i)$, there exists $Z_i \in At(i)$ such that $s \in Z_i$ and $Z_i \subseteq T_i$. Suppose there exists $Z'_i \in At(i)$ such that $s \in Z'_i$ and $Z'_i \subseteq T_i$. $Z_i \cap Z'_i \neq \varnothing$, since s belongs to both Z_i and Z'_i. Therefore, by atomicity, $Z_i = Z'_i$, and this holds for every i. ■

Lemma 4 allows us to write $Z_i(s)$ for every s and i to be the unique subset in $At(i)$ containing s. For any non-empty coalition C, let us write $Z_C(s) = \bigcap_{i \in C} Z_i(s)$.
In order to prove Proposition 1, we use the following construction. Let Eff satisfies (1) strong playability, (2) atomicity and (3) decomposability. Let us build $G^* = G(\text{Eff})$ as follows:

- for every i, number $At(i)$: let r_i be a bijective mapping from $At(i)$ to $\{0, 1, \ldots, |At(i)| - 1\}$. Then create $p_i = \lceil \log_2 |At(i)| \rceil$ propositional variables $x_i^1, \ldots, x_i^{p_i}$. Finally, let $V = \{x_i^j | i \in N, 1 \leq j \leq p_i\}$;
- for each i: $\pi_i = \{x_i^1, \ldots, x_i^{p_i}\}$;
- for each i and each $j \leq p_i$, let $\varepsilon_{i,j}$ be the jth digit in the binary representation of p_i. Note that $\varepsilon_{i,p_i} = 1$ by definition of p_i. If x is a propositional variable then we use the following notation: $0.x = \neg x$ and $1.x = x$. Then define
$$\gamma_i = \bigwedge_{j \in \{2,\ldots,p_i\}, \varepsilon_{i,j}=0} \left(\bigwedge_{1 \leq k \leq j-1} \varepsilon_{i,j}.x_i^k \rightarrow \neg x_i^j \right)$$
- finally, for each $s \in S$, let $\mu(s) \in 2^V$ defined by: $x_i^j \in \mu(s)$ if and only if the jth digit of the binary representation of $r_i(Z_i(s))$ is 1.

For every $i \in N$ and every $Z \in At(i)$, let $k = r_i(Z)$ and $s_i(Z)$ the strategy of player in i in G corresponding to the binary representation of k using $\{x_i^1, \ldots, x_i^{p_i}\}$, x_i^1 being the most significant bit. For instance, if $p_i = 3$ and $r(Z_i) = 6$ then $s_i(Z) = (x_i^1, x_i^2, \neg x_i^3)$. ■

We denote by S_{G^*} the set of states associated with G^*. States of S_{G^*} are denoted by s_{G^*}. The set of atoms of Eff_{G^*} is noted $At_{G^*}(i)$.
Now we have to show that for every C, $\text{Eff}_{G^*}(C) = \mu(\text{Eff}(C))$.

Note: To follow the proof, it may be helpful to see how this construction works on an example. Let $N = \{1,2,3\}$, $s = \{1,2,3,4,5,6,7,8,9,A,B,C\}$, $At(1) = \{1234, 5678, 9ABC\}$, $At(2) = \{13579B, 2468AC\}$, $At(3) = \{12569C, 3478AB\}$ (parentheses for subsets of S are omitted – 1234 means $\{1,2,3,4\}$ and so on). By decomposability,we have $At(12) = \{13, 24, 57, 68, 9B, AC\}$, $At(13) = \{12, 34, 56, 78, 9C, AB\}$, and $At(23) = \{159, 37B, 26C, 48A\}$. $|At(1)| = 3$, therefore $p_1 = 2$. $|At(2)| = |At(3)| = 2$, therefore $p_2 = p_3 = 1$. Thus, $V = \{x_1^1, x_1^2, x_2^1, x_3^1\}$. Let $At(1) = \{Z_0, Z_1, Z_2\}$, that is, $r_1(1234) = 0$, $r_1(5678) = 1$ and $r_1(9ABC) = 2$.

Likewise, $r_2(13579B) = 0$, $r_2(2468AC) = 1$, $r_3(12569C) = 0$ and $r_3(3478AB) = 1$. Consider $s = 6$. We have $s = 5678 \cap 2468AC \cap 12569C$, therefore $s_G = \mu(s) = (x_1^1, \neg x_1^2, x_2^1, \neg x_3^1)$. The constraints are $\gamma_1 = (x_1^1 \to \neg x_1^2)$, $\gamma_2 = \gamma_3 = \top$.

Lemma 5 *For every $i \in N$ and $Z \in At(i)$: $\mu(Z) = \{s_{G^*} \in S_{G^*} | s_{G^*} \supseteq s_i(Z)\}$*

Proof: Let $i \in N$ and $Z \in At(i)$. Let $s_{G^*} \in \mu(Z)$; by definition of $\mu(Z)$, there exists an $s \in S$ such that $\mu(s) = s_{G^*}$. Consider the decomposition of s into atoms, that is, $\{s\} = Z_1(s) \cap \ldots \cap Z_n(s)$ (cf. Lemma 4). By construction of μ, the projection of $\mu(s)$ on $\{x_i^1, \ldots, x_i^{p_i}\}$ corresponds to the binary representation of $r_i(s)$. Therefore, $\mu(s) = s_{G^*}$ extends $s_i(Z)$.
Conversely, let s_{G^*} such that $s_{G^*} \supseteq s_i(Z)$. For every $j \leq n$, let k_j be the number whose binary representation in $\{x_j^1, \ldots, x_j^{p_j}\}$ is the projection of s_{G^*} on $\{x_j^1, \ldots, x_j^{p_j}\}$. Let s be defined by $\{s\} = Z_1(k_1) \cap \ldots \cap Z_n(k_n)$. By construction of μ, we have $\mu(s) = s_{G^*}$. Moreover, $Z_i(k_i) = Z$ by atomicity, that is, $s \in Z$. Therefore $s_{G^*} \in \mu(Z)$. ■

Proof: (Proposition 1, continued)
We still have to prove than for every $C \subseteq N$ and every $X \subseteq S$, $X \in \mathrm{Eff}(C)$ holds if and only if $\mu(X) \in \mathrm{Eff}_{G^*}(C)$.
Decomposability of both Eff and Eff_{G^*} imply that it is enough to show that for every i and every $X \subseteq S$, $X \in \mathrm{Eff}(i)$ if and only if $\mu(X) \in \mathrm{Eff}_{G^*}(i)$. Because both Eff and Eff_{G^*} satisfy coalition monotonicity, it is enough to show that for every i, $Z_i \in At_{G^*}(i)$ implies $\mu(Z_i) \in \mathrm{Eff}_{G^*}(i)$ and $T_i \in At_{G^*}(i)$ implies $\mu^{-1}(T_i) \in \mathrm{Eff}(i)$.
Let $Z_i \in At(i)$. Because $r(Z_i) \leq p_i$, we have $s_i(Z_i) \models \gamma_i$, therefore $s_i(Z_i) \in \mathrm{Eff}_{G^*}(i)$. By Lemma 5, $\mu(Z_i) = \{s_{G^*} | s_{G^*} \supseteq s_i(Z_i)\}$. Therefore, $\mu(Z_i) \in \mathrm{Eff}_{G^*}(i)$.
Conversely, let $T_i \in At_{G^*}(i)$. By Lemma 3, $T_i = \{s | s \supseteq s_i\}$ for some $s_i \in S_i$. Let $s_i = (\varepsilon_1.x_1^1, \ldots, \varepsilon_{p_i}.x_i^{p_i})$ and $q(s_i) = \sum_{k=1}^{p_i} 2^{p_i - k}.\varepsilon_k$. Note that $q(s_i) \leq p_i$, because $s_i \in S_i$ implies $s_i \models \gamma_i$. Now, let $j = r_i^{-1}(q(s_i))$. Let $Z_i^j \in At(i)$ such that $r_i(Z_i^j) = j$. We have $\mu(Z_i^j) = \{s | s \supseteq s_i\} = T_i$. Now, $Z_i^j \in \mathrm{Eff}(i)$, because $Z_i^j \in At(i)$. Therefore, $\mu^{-1}(T_i) \in \mathrm{Eff}(i)$.
We have now proven that for $C \subseteq N$ and every $X \subseteq S$, $X \in \mathrm{Eff}(C)$ holds if and only if $\mu(X) \in \mathrm{Eff}_{G^*}(C)$. We can now conclude that if Eff satisfies strong playability, atomicity, decomposability, and $\mathrm{Eff}(N) = 2^S \setminus \varnothing$, then there exists a game $G (= G^*)$ and an injective function $\mu : S \to 2^V$ such that for every $C \subseteq N$: $\mathrm{Eff}_G(C) = \{\mu(X) | X \in \mathrm{Eff}(C)\}$. ■

Proposition 2 *Let $N = \{1, \ldots, n\}$ be a set of agents and $\mathcal{SC} \in 2^{2^N}$ a set of coalitions. There exists a Boolean game G over N such that the set of efficient coalitions for G is $\mathcal{SC}$ (i.e. $\mathrm{EC}(G) = \mathcal{SC}$) if and only if $\mathcal{SC}$ satisfies these two properties:*

(1) $\varnothing \in \mathcal{SC}$.

(2) *for all $I, J \in \mathcal{SC}$ such that $I \cap J = \varnothing$ then $I \cup J \in \mathcal{SC}$.*

Lemma 6 *Let I, J be two coalitions of a Boolean game G. If I and J are efficient and $I \cap J = \varnothing$, then $I \cup J$ is efficient.*

Proof: If I is efficient, then we know than $\exists s_I \in S_I$ such that $s_I \models \bigwedge_{i \in I} \varphi_i$, and the same for J: $\exists s_J \in S_J$ such that $s_J \models \bigwedge_{j \in J} \varphi_j$. Moreover, as $I \cap J = \varnothing$, we have $(s_I, s_J) \models \bigwedge_{i \in I \cup J} \varphi_i$, so $I \cup J$ is an efficient coalition.

■

Lemma 6 proves the ($\Rightarrow$) direction of Proposition 2. For the ($\Leftarrow$) direction, we define the following construction.
Let $\mathcal{SC}$ be a set of coalitions satisfying (1) and (2). Define the following Boolean game G as follows:

- $V = \{connect(i,j) | i,j \in N\}$ (all possible connections between players);
- $\Gamma = \top$;
- $\pi_i = \{connect(i,j) | j \in N\}$ (all connections from player i);
- $\varphi_i = \bigvee_{I \in \mathcal{SC}, i \in I} F_I$,

  where
  $$F_I = \left(\bigwedge_{j,k \in I} connect(j,k) \right) \bigwedge \left(\bigwedge_{j \in I, k \notin I} \neg connect(j,k) \right)$$
  (player i wants that all the players of her coalition are interconnected and that there is no connection from the coalition to the "outside" of the coalition)

We want to show that the set $\mathrm{EC}_G = \mathcal{SC}$ (where EC_G is the set of efficient coalitions for G).

We first show that $\mathcal{SC} \subseteq \mathrm{EC}_G$. Let $I \in \mathcal{SC}$. If every agent $i \in I$ plays $\left(\bigwedge_{j \in I} connect(i,j)\right) \bigwedge \left(\bigwedge_{k \notin I} \neg connect(i,k)\right)$, then φ_i is satisfied for every $i \in I$. Hence, I is an efficient coalition for G and $\mathcal{SC}$ is included in $\mathrm{EC}(G)$.
■

Before proving that $\mathrm{EC}_G \subseteq \mathcal{SC}$, we establish the following lemmas:

Lemma 7 *For any collection $\mathcal{SC} = \{C_i, i = 1, \ldots, q\} \subseteq 2^{2^A}$, $\bigwedge_{1 \leq i \leq q} F_{C_i}$ is satisfiable if and only if for any $i,j \in \{1, \ldots, q\}$, either $C_i = C_j$ or $C_i \cap C_j = \varnothing$.*

Proof:
(a) If $C_1 = \ldots = C_q = C$ then $\bigwedge_{1 \leq i \leq q} F_{C_i} \equiv F_C$. F_C is consistent, as it is satisfied by any interpretation assigning each $connect(i,j)$ such that $i,j \in C$ to true, each $connect(i,j)$ such that $i \in C$ and $j \notin C$ to false (the value assigned to $connect(i,j)$ such that $i,j \notin C$ being irrelevant).
(b) Assume now that for any $i,j \in \{1, \ldots, q\}$, either $C_i = C_j$ or $C_i \cap C_j = \varnothing$.
Then $\bigwedge_{1 \leq i \leq q} F_{C_i}$ is equivalent to

$$\left(\bigwedge_{1 \leq i \leq q} \bigwedge_{j,k \in C_i} connect(j,k) \right) \bigwedge \left(\bigwedge_{1 \leq i \leq q} \bigwedge_{j \in C_i, k \notin C_i} \neg connect(j,k) \right)$$

and $\bigwedge_{1 \leq i \leq q} F_{C_i}$ is satisfied by any interpretation assigning each $connect(j,k)$ such that j,k belong to the same C_i to true, each $connect(j,k)$ such that $j \in C_i$ for some i and $k \notin C_i$ to false. Hence $\bigwedge_{1 \leq i \leq q} F_{C_i}$ is satisfiable.
(c) Assume that for some $i,j \in \{1, \ldots, q\}$, we have $C_i \cap C_j \neq \varnothing$ and $C_i \neq C_j$. Let $k \in C_i \cap C_j$ and (without loss of generality) $l \in C_i \setminus C_j$. Then $F_{C_i} \models connect(k,l)$ and $F_{C_j} \models \neg connect(k,l)$, hence $F_{C_i} \wedge F_{C_j}$ is unsatisfiable, and a fortiori, so is $\bigwedge_{1 \leq i \leq q} F_{C_i}$. ■

We now define a *covering of a coalition I by disjoint subsets of $\mathcal{SC}$* as a tuple $\vec{C} = \langle C_i | i \in I \rangle$ of coalitions such that:
(a) for every $k \in I$, $C_k \in \mathcal{SC}$;
(b) for all $C_j, C_k \in \vec{C}$, either $C_j = C_k$ or $C_j \cap C_k = \varnothing$;
(c) for every $i \in I$, $i \in C_i$.
Let $Cov(\mathcal{SC}, I)$ be the set of all covering of I by disjoint subsets of $\mathcal{SC}$.
For instance, if $\mathcal{SC} = \{1, 24, 123, 124\}$ then $Cov(\mathcal{SC}, 12) = \{\langle 1, 24\rangle, \langle 123, 123\rangle, \langle 124, 124\rangle\}$[1]; $Cov(\mathcal{SC}, 123) = \{\langle 123, 123, 123\rangle\}$, $Cov(\mathcal{SC}, 124) = \{\langle 1, 24, 24\rangle, \langle 124, 124, 124\rangle\}$ and $Cov(\mathcal{SC}, 234) = Cov(\mathcal{SC}, 1234) = \varnothing$.

Lemma 8 *For any $I \neq \emptyset$, Φ_I is equivalent to $\bigvee_{\vec{C} \in Cov(\mathcal{SC}, I)} \bigwedge_{i \in I} F_{C_i}$.*

Proof:

$$\begin{array}{ll} & \Phi_I \\ \equiv & \bigwedge_{i \in I} \varphi_i \\ \equiv & \bigwedge_{i \in I} \bigvee_{I \in \mathcal{SC}, i \in I} F_I \\ \equiv & \bigvee_{\langle C_i, i \in I \rangle \in \mathcal{SC}^{|I|} \text{ such that } i \in C_i \text{ for every } i \in I} \bigwedge_{i \in I} F_{C_i} \end{array}$$

Now, by Lemma 7, $\bigwedge_{i \in I} F_{C_i}$ is satisfiable if and only if for all $i, j \in I$, either $C_i = C_j$ or $C_i \cap C_j = \varnothing$. Therefore, $\Phi_I \equiv \bigvee_{\vec{C} \in Cov(\mathcal{SC}, I)} \bigwedge_{i \in I} F_{C_i}$. ■

For instance, if $\mathcal{SC} = \{1, 24, 123, 124\}$ then $\Phi_{12} \equiv (F_1 \wedge F_{24}) \vee F_{123} \vee F_{124}$; $\Phi_{123} \equiv F_{123}$; $\Phi_{124} \equiv (F_1 \wedge F_{24}) \vee F_{124}$; $\Phi_{234} \equiv \bot$.

Lemma 9 *Let $I \subseteq 2^A$. Φ_I is satisfiable if and only if there exists $J \in \mathcal{SC}$ such that $I \subseteq J$.*

Proof: The case $I = \emptyset$ is straightforward: $\Phi \equiv \top$ is satisfiable, and $\emptyset \in \mathcal{SC}$ by assumption, therefore there exists $J \in \mathcal{SC}$ ($J = \emptyset$) such that $I \subseteq J$.

Now, let $I \neq \emptyset$.

$\Rightarrow$ Assume Φ_I is satisfiable. By Lemma 8, Φ_I is equivalent to $\bigvee_{\vec{C} \in Cov(\mathcal{SC}, I)} \bigwedge_{i \in I} F_{C_i}$, therefore there exists a $\vec{C}$ in $Cov(\mathcal{SC}, I)$ such that $\bigwedge_{i \in I} F_{C_i}$ is satisfiable, therefore $Cov(\mathcal{SC}, I)$ is not empty. Now, $\vec{C} \in Cov(\mathcal{SC}, I)$ implies that:
(a) for every $i \in I$, $C_i \in \mathcal{SC}$;
(b) $I \subseteq \bigcup_{i \in I} C_i$
and (c) for every $i, j \in I$, either $C_i = C_j$ or $C_i \cap C_j = \varnothing$.
Now, (a), (c) and property (2) (assumption of Proposition 2) imply that $\bigcup_{i \in I} C_i \in \mathcal{SC}$, which together with (b) proves that there exists a $J \in \mathcal{SC}$ (namely $J = \bigcup_{i \in I} C_i$) such that $I \subseteq J$).

$\Leftarrow$ Assume that there is a $J \in \mathcal{SC}$ such that $I \subseteq J$. Then $\Phi_J \models \Phi_I$, and Φ_J is consistent (consider the interpretation assigning each $connect(i, j)$ such that $i, j \in J$ to true). ■

[1] There are 2 players in $I = \{1, 2\}$, therefore each $\vec{C}$ in $Cov(\mathcal{SC}, 12)$ contains 2 coalitions, one for each player, satisfying (a), (b) and (c).

Proof: (Proposition 2, continued)
It remains to be shown that $\mathrm{EC}_G \subseteq \mathcal{SC}$. Let I be a coalition such that $I \notin \mathcal{SC}$ (which implies $I \neq \emptyset$, because by assumption $\emptyset \in \mathcal{SC}$).

- If $I = N$ then there is no $J \in \mathcal{SC}$ such that $I \subseteq J$ (because $I \notin \mathcal{SC}$), and then Lemma 9 implies that Φ_I is unsatisfiable, therefore I cannot be efficient for G.

- Assume now that $I \neq N$ and define the following $\bar{I}$-strategy $S_{\bar{I}}$ ($\bar{I} = N \setminus I$): for every $i \in \bar{I}$, $s_i = \{\neg connect(i,j) | j \in I\}$ (plus whatever on the variables $connect(i,j)$ such that $j \notin I$).

  Let $\vec{C} = \langle C_i, i \in I \rangle \in Cov(\mathcal{SC}, I)$.

  We first claim that there is a $i^* \in I$ such that C_{i^*} is not contained in I. Indeed, suppose that for every $i \in I$, $C_i \subseteq I$. Then, because $i \in C_i$ holds for every i, we have $\bigcup_{i \in I} C_i = I$. Now, $C_i \in \mathcal{SC}$ for all i, and any two distinct C_i, C_j are disjoint, therefore, by property (2) we get $I \in \mathcal{SC}$, which by assumption is false.

  Now, let $k \in C_{i^*} \setminus I$ (such a k exists because C_{i^*} is not contained in I). Because $k \in \bar{I}$ and $i^* \in I$, player k assigns $connect(k, i^*)$ to false in s_k. Now, the satisfaction of F_{C_i} requires $connect(k, i^*)$ to be true, because both i and k are in C_i. Therefore $s_k \models \neg F_{C_i}$, and a fortiori $s_{\bar{I}} \models \neg F_{C_i}$, which entails $s_{\bar{I}} \models \neg \bigwedge_{i \in I} F_{C_i}$.

  This being true for any $\vec{C} \in Cov(\mathcal{SC}, I)$, we have $s_{\bar{I}} \models \bigwedge_{\vec{C} \in Cov(\mathcal{SC}, I)} \neg \bigwedge_{i \in I} F_{C_i}$. that is, $s_{\bar{I}} \models \neg \bigvee_{\vec{C} \in Cov(\mathcal{SC}, I)} \bigwedge_{i \in I} F_{C_i}$. Together with Lemma 8, this entails $s_{\bar{I}} \models \neg \Phi_I$. Hence, I does not control Φ_I and I cannot be efficient for G.

■

Proposition 3 *Let $G = (N, V, \Gamma, \pi, \Phi)$ be a Boolean game. $s = (s_1, \ldots, s_n) \in WCore(G)$ if and only if s satisfies at least one member of every efficient coalition, that is, for every $C \in \mathrm{EC}(G)$ then $s \models \bigvee_{i \in C} \varphi_i$.*

Proof:
$s = (s_1, \ldots, s_n) \in WCore(G)$ if and only if

$$\begin{aligned}
& \nexists C \subset N \text{ such that } \exists s_C \in S_C,\ \forall i \in C,\ \forall s_{-C} \in S_{-C},\ (s_C, s_{-C}) \succ_i s \\
\Leftrightarrow\ & \nexists C \subset N \text{ such that } \exists s_C \in S_C,\ \forall i \in C,\ \forall s_{-C} \in S_{-C},\ ((s_C, s_{-C}) \models \varphi_i) \wedge (s \models \neg\varphi_i) \\
\Leftrightarrow\ & \nexists C \subset N \text{ such that } \exists s_C \in S_C,\ \forall i \in C,\ (s_C \models \varphi_i) \wedge (s \models \neg\varphi_i) \\
\Leftrightarrow\ & \nexists C \subset N \text{ such that } \exists s_C \in S_C,\ (s_C \models \textstyle\bigwedge_{i \in C} \varphi_i) \wedge (s \models \bigwedge_{i \in C} \neg\varphi_i) \\
\Leftrightarrow\ & \nexists C \subset N \text{ such that } (s \models \textstyle\bigwedge_{i \in C} \neg\varphi_i) \wedge (\exists s_C \in S_C, s_C \models \bigwedge_{i \in C} \varphi_i) \\
\Leftrightarrow\ & \forall C \subset N,\ (s \models \textstyle\bigvee_{i \in C} \varphi_i) \vee (\forall s_C \in S_C, s_C \models \bigvee_{i \in C} \neg\varphi_i) \\
\Leftrightarrow\ & \forall C \subset N, \text{ if } \exists s_C \in S_C : s_C \models \textstyle\bigwedge_{i \in C} \varphi_i \text{ then } s \models \bigvee_{i \in C} \varphi_i
\end{aligned}$$

$\exists s_C \in S_C : s_C \models \bigwedge_{i \in C} \varphi_i$ means than coalition C is efficient. So, we have $s = (s_1, \ldots, s_n) \in WCore(G)$ if and only if $\forall C \subset N$, if $C \in \mathrm{EC}(G)$ then $s \models \bigvee_{i \in C} \varphi_i$.

■

Proposition 4 *For any Boolean game G, $WCore(G) \neq \varnothing$.*

Proof: We construct the following set of coalitions E as follows. First, initialize E to $\emptyset$. Then, while there exists a coalition C in $\mathrm{EC}(G)$ such that $C \cap C' = \emptyset$ holds for every $C' \in E$, pick such a C and add it to E (that is, $E := E \cup \{C\}$). At the end of the algorithm, E is a set of disjoint efficient coalitions $\{C_I, i \in I\}$, therefore, by Proposition 2, $\cup_{i \in I} C_i$ is efficient. Therefore, there exists a $s_E \in S_E$ such that $s_E \models \bigwedge_{i \in E} \varphi_i$, and E contains at least one element of every efficient coalition (if this were not the case, there would remain an efficient coalition C that intersects none of the C_i's, and the algorithm would have continued and incorporated C into E). Let s extending s_C. s satisfies at least one member of every efficient coalition, therefore, by Proposition 3, $WCore(G) \neq \varnothing$.

■

Proposition 5 *Let $G = (N, V, \Gamma, \pi, \Phi)$ be a Boolean game, and s be a strategy profile. If $s \in SCore(G)$ then for every $C \in \mathrm{EC}(G)$ and every $i \in C$, $s \models \varphi_i$.*

Proof: We prove the converse: if exists $C \in \mathrm{EC}(G)$ and exists $i \in C$ such that $s \models \neg\varphi_i$, then $s \in SCore(G)$.

$$\begin{aligned}
& \exists C \in \mathrm{EC}(G), \exists i \in C \text{ st } s \models \neg\varphi_i \\
\Leftrightarrow\ & \exists C \subseteq N, \exists s_C \in S_c, \exists i \in C \text{ st } (s_C \models \textstyle\bigwedge_{j \in C} \varphi_j) \wedge (s \models \neg\varphi_i) \\
\Rightarrow\ & \exists C \subseteq N, \exists s'_C \in S_c, \forall s_{-C} \in S_{-C}, \exists i \in C \text{ st } (((s_C, s_{-C}) \models \varphi_i) \wedge (s \models \neg\varphi_i)) \wedge (s_C \models \textstyle\bigwedge_{j \in C} \varphi_j) \\
\Rightarrow\ & \exists C \subseteq N, \exists s_C \in S_c, \forall s_{-C} \in S_{-C}, \exists i \in C \text{ st } (((s_C, s_{-C}) \succ_i S) \wedge (\forall i \in C (s_C, s_{-C}) \succeq_i s)
\end{aligned}$$

So, we know than if $\exists C \in \mathrm{EC}(G)$, $\exists i \in C$ such that $s \models \neg\varphi_i$, then $s \notin SCore(G)$.

■

Proposition 6 *Let $G = (N, V, \Gamma, \pi, \Phi)$ be a Boolean game. We have the following:*
$SCore(G) \neq \emptyset$ if and only if $\bigcup\{C \subseteq N | C \in \mathrm{EC}(G)\} \in \mathrm{EC}(G)$, that is, if and only if the union of all efficient coalitions is efficient.

Proof: Let $MEC(G) = \bigcup_{C \subseteq N}\{C \in \mathrm{EC}(G)\}$.

$\Leftarrow$ $Score(G) \neq \varnothing$. Let $s \in Score(G)$. From Proposition 5, we know than $\forall C \in \mathrm{EC}(G)$, $\forall i \in C$, $s \models \varphi_i$. So, $\forall i \in MEC(G)$, $s \models \varphi_i$. So, $MEC(G) \in \mathrm{EC}(G)$.

$\Rightarrow$ $MEC(G) \in \mathrm{EC}(G)$. Let $s_{MEC(G)} \in S_{MEC(G)}$ such that $\forall s_{-MEC(G)}$, $s_{MEC(G)} \models \Phi_{MEC(G)}$. We are looking for s such that $s \in Score(G)$.

Let $s_{-MEC(G)} \in S_{-MEC(G)}$ such that $MAX = \{i | s = (s_{MEC(G)}, s_{-MEC(G)}) \models \varphi_i\}$ be maximal for $\subseteq$. $s_{-MEC(G)}$ exists, in worst case $s \models \Phi_{MEC(G)}$. As MAX is maximal, we cannot find a $C \subseteq N$ such that $\exists s_C \in S_C$, such that $\forall s_{-C} \in S_{-C}$, $\forall i \in C$, $(s_C, s_{-C}) \succeq_i s$, and $\exists i \in C$, $(s_C, s_{-C}) \succ_i s$. Indeed, if we assume that this C exists, then $\forall i \in N$ such that $s \models \varphi_i$, we have $s_C \models \varphi_i$, and $\exists i \in N$ such that $s \not\models \varphi_i$ and $s_C \models \varphi_i$. In this case, MAX is not maximal for $\subseteq$.

■

Proposition 7 *Deciding whether there exists a non-empty efficient coalition in a Boolean game is Σ_2^p-complete, and is Σ_2^p-hard even if $n = 2$.*

Proof: Membership to Σ_2^p is immediate. To show that deciding whether there is a non-empty efficient coalition in a Boolean game is Σ_2^p-hard (even with 2 agents), consider the following polynomial reduction from $\text{QBF}_{2,\exists}$. To each instance $I = \exists a_1 \dots a_p \forall b_1 \dots b_q \varphi$ of $\text{QBF}_{2,\exists}$, let us consider the following Boolean game $G_I = \langle N, V, \pi, \Gamma, \Phi \rangle$, where where $N = \{1,2\}$, $\gamma_1 = \gamma_2 = \top$, $V = \{a_1 \dots a_p, b_1 \dots b_q, x\}$, $\pi_1 = \{a_1, \dots, a_p, x\}$, $\pi_2 = \{b_1, \dots, b_q\}$, $\varphi_1 = \varphi$ and $\varphi_2 = \neg\varphi \wedge x$. Neither $\{2\}$ nor $\{1,2\}$ can be efficient; therefore, the only possible non-empty efficient coalition is $\{1\}$. Now, it is easily seen that $\{1\}$ is efficient if and only if I is a valid instance of $\text{QBF}_{2,\exists}$.

Proposition 8

- *deciding whether an agent i belongs to at least one efficient coalition for G is Σ_2^p-complete, and is Σ_2^p-hard even if $n = 2$.*
- *deciding whether an agent i belongs to at all non-empty efficient coalitions for G is Π_2^p-complete, and is Π_2^p-hard even if $n = 2$.*

Proof:
For both problems, the membership part of the proof is easy.
To show that deciding whether an agent i belongs to at least one efficient coalition for G is Σ_2^p-hard (even with 2 agents), consider the following polynomial reduction from $\text{QBF}_{2,\exists}$. To each instance $I = \exists a_1 \dots a_p \forall b_1 \dots b_q \varphi$ of $\text{QBF}_{2,\exists}$, let us consider the following Boolean game $G_I = \langle N, V, \pi, \Gamma, \Phi \rangle$, where $N = \{1,2\}$, $\gamma_1 = \gamma_2 = \top$, $V = \{a_1, \dots, a_p, b_1, \dots, b_q\}$, $\pi_1 = \{a_1, \dots, a_p\}$, $\pi_2 = \{b_1, \dots, b_q\}$, $\varphi_1 = \varphi$ and $\varphi_2 = \neg\varphi$. $\{1\}$ is efficient if and only if there exists a strategy s_1 such that $s_1 \models \varphi$, that is, if and only if I is valid. Now, $\{1,2\}$ cannot be efficient, because $\varphi_1 \wedge \varphi_2 = \top$. therefore, 1 belongs to an efficient coalition if and only if $\{1\}$ is efficient, that is, if and only if I is valid.
To show that deciding whether an agent i belongs to at all non-empty efficient coalitions for G is Π_2^p-hard (even with 2 agents), consider the following polynomial reduction from $\text{QBF}_{2,\forall}$. To each instance $I = \exists a_1 \dots a_p \forall b_1 \dots b_q \varphi$ of $\text{QBF}_{2,\forall}$, let us consider the following Boolean game $G_I = \langle N, V, \pi, \Gamma, \Phi \rangle$, where $N = \{1,2\}$, $\gamma_1 = \gamma_2 = \top$, $V = \{a_1 \dots a_p, b_1 \dots b_q, x\}$, $\pi_1 = \{a_1, \dots, a_p, x\}$, $\pi_2 = \{b_1, \dots, b_q\}$, $\varphi_1 = \neg\varphi$ and $\varphi_2 = \varphi \wedge x$. Neither $\{2\}$ nor $\{1,2\}$ can be efficient; therefore, 2 belongs no all non-empty efficient coalitions if and only if $\{1\}$ is not efficient, that is, if $\exists a_1 \dots a_p \forall b_1 \dots b_q \neg\varphi$ is not valid, or equivalently, if $\forall a_1 \dots a_p \exists b_1 \dots b_q \varphi$ is valid. ■

Proposition 9 *Deciding whether a strategy profile s is in the weak core of a Boolean game G is Co-NP-complete.*

Proof: Membership to Co-NP is immediate: the problem to decide if a strategy profile s is not in the weak core of G is in NP: we first have to find a coalition $C \subseteq N$, then we have to guess a strategy $s'_C \in S_C$ for this coalition such that $\forall i \in C$, $(s'_C, s_{-C}) \succ_i s$. So we have to check than $(s'_C, s_{-C}) \models \varphi_i$ and $s \models \neg\varphi_i$, which is in P.

Hardness is obtained by a reduction from the problem of deciding the validity of a $\texttt{QBF}_{2,\exists}$. Given $Q = \exists N, \forall B, \Phi$, where A and B are disjointed sets of variables and Φ is a formula of $L_{A \cup B}$, we define a two-players Boolean game by $\varphi_1 = a \wedge \Phi$, $\varphi_2 = b$ where a, b are new variables ($a, b \notin N \cup B$), $\pi_1 = N \cup \{a\}$ and $\pi_2 = B \cup \{b\}$. Obviously, this game can be built in polynomial time given Q.
Let M_A be any A-interpretation, s_1 be (M_A, a), M_B be any B-interpretation, and s_2 be $(M_B, \overline{b})$. $s = (s_1, s_2)$.

Assume Q is valid. Thus, $M_A \in 2^A$ is a witness of Φ, (M_A, a) is a winning strategy for 1, and $s \models \varphi_1$. $s \in WCore(G)$.
Conversely, if Q is not valid, then 2 has a winning strategy, but $s \models \neg\varphi_2$. So, $\exists C = \{2\} \subset N$ such as $PI_{\pi_2}(\varphi_2) \neq \varnothing$ and $s \models \neg\varphi_2$. Then, from Proposition 3, $s \notin WCore(G)$.
Finally, $s \in WCore(G)$ if and only if Q is valid. ■

Proposition 10 *Let $G = (N, V, \Gamma, \pi, \Phi)$ be a Boolean game. Let I and J be be two coalitions such that $I \cap J = \varnothing$, $I \cup J$ is efficient, $R(I) \cap J = \varnothing$ and $R(J) \cap I = \varnothing$. Then, I and J are both efficient.*

In order to prove this proposition, we introduce the following definitions: Let ψ be a propositional formula. A term α is an *implicant* of ψ if and only if $\alpha \models \psi$ holds. α is a *prime implicant* of ψ if and only if α is an implicant of ψ and for every implicant α' of ψ, if $\alpha \models \alpha'$ holds, then $\alpha' \models \alpha$ holds. $PI(\psi)$ denotes the set of all the prime implicants of ψ.
We can now prove Proposition 10.
Proof: We know than $I \cup J$ is efficient, so $\exists s_{I\cup J} \in S_{I\cup J}$ such that $s_{I\cup J} \models (\bigwedge_{i\in I\cup J} \varphi_i)$.
As $I \cap J = \varnothing$, we can write than:

$$\begin{aligned}
& \exists s_I \in S_I, \exists s_J \in S_J : \quad (s_I, s_J) \models (\bigwedge_{i\in I\cup J} \varphi_i) \\
\Leftrightarrow \quad & \exists s_I \in S_I, \exists s_J \in S_J : \quad (s_I, s_J) \models (\bigwedge_{i\in I} \varphi_i \wedge \bigwedge_{j\in J} \varphi_j) \\
\Leftrightarrow \quad & \exists s_I \in S_I, \exists s_J \in S_J : \quad ((s_I, s_J) \models (\bigwedge_{i\in I} \varphi_i)) \wedge \Gamma \wedge ((s_I, s_J) \models (\bigwedge_{j\in J} \varphi_j \wedge \Gamma))
\end{aligned}$$

Moreover, we know than $\forall i \in I, j \in J$, $j \notin RP_i$ (resp. $i \notin RPj$). So, $\forall i \in I, j \in J$, $\forall v \in Var(PI(\varphi_i))$, $v \notin \pi_j$ (resp. $\forall w \in Var(PI(\varphi_j))$, $w \notin \pi_i$). We know than no player in J controls a variable of a goal of a player in I (and vice versa).
As we have $\exists s_I \in S_I$, $((s_I, s_J) \models (\bigwedge_{i\in I} \varphi_i))$ and $\forall i \in I, j \in J$, $\forall v \in Var(PI(\varphi_i))$, $v \notin \pi_j$, we have: $s_I \models (\bigwedge_{i\in I} \varphi_i)$ (resp. $s_J \models (\bigwedge_{j\in J} \varphi_j)$.
So, both I and J are efficient. ■

Proposition 11 *Let $G = (N, V, \Gamma, \pi, \Phi)$ be a Boolean game. If $B \subseteq N$ is stable for R, then B is an efficient coalition of G ($B \subseteq \mathrm{EC}(G)$) if and only if $\varphi_B = \bigwedge_{i\in B} \varphi_i$ is consistent.*

Proof: Let B a stable set for R. Then, we have:

$$\begin{aligned}
& \forall i \in B, \forall j \text{ such that } j \in R(i), j \in B \\
\Leftrightarrow \quad & \forall i \in B, RP_i \subseteq B \\
\Rightarrow \quad & \forall i \in B, \exists s_B \in S_B \text{ such that } s_C \models \varphi_i \\
\Leftrightarrow \quad & \exists s_B \in S_B \text{ such that } s_B \models \bigwedge_{i\in B} \varphi_i \text{ if and only if } \bigwedge_{i\in B} \varphi_i \not\models \bot
\end{aligned}$$

■

Proposition 12 *Let $G = (N, V, \Gamma, \pi, \Phi)$ be a Boolean game. If $B \subseteq N$ an efficient coalition of G ($B \subseteq \mathrm{EC}(G)$) such that $\forall i \in B$, $|RP_i| = 1$, then B is stable for R.*

Proof: B is an effective coalition, so $\bigwedge_{i\in B}\varphi_i$ is consistent, and $\exists s_B \in S_B$ such that $s_B \models \bigwedge_{i\in B}\varphi_i \wedge \Gamma$.
We know than $\forall i \in B$, $|RP_i| = 1$. So, $\exists j \in N$ such that $RP_i = \{j\}$, *i.e.* $\forall v \in Var(PI(\varphi_i))$, $v \in \pi_j$. As we have $s_B \models \varphi_i$, with $s_B \in S_B$, we know than B controls at least one variable in φ_i. So, $j \in B$, and thus B is stable for R.

■

Proposition 13 *Let $G = (N, V, \Gamma, \pi, \Phi)$ be a Boolean game such that $\forall i \in N$, $|RP_i| = 1$. For any coalition $C \subseteq N$, C forms a cycle in the dependence graph if and only if C is stable for R and constitutes a minimum efficient coalition.*

Proof:

$\Rightarrow$ As $\forall i$, $|RP_i| = 1$, only one edge can go out for each player. So, if there is a cycle between p players, and if we rename these players with respect to the topological order, we have $RP_1 = \{2\}$, $RP_2 = \{3\}$, ..., $RP_{p-1} = \{p\}$, $RP_p = \{1\}$. Let $C = \{1, \ldots, p\}$. As we obviously have $R(C) = C$ ($\forall i \in C, RP_i = \{(i+1) mod p\} \in C$), C is stable for R.
Moreover, we know than $\forall i, j \in C$, $RP_i \neq RP_j$, so $\forall i, j \in C$, $\varphi_i \wedge \varphi_j \not\models \bot$. So, from Proposition 11, C is efficient.
Assume than $\exists I \subset C$ efficient. So, $\exists s_I$ such that $s_I \models \bigwedge_{i\in I}\varphi_i$. So, as $|RP_i| = 1$, $\forall i \in I$, $RP_i \in I$. So, $I = C$. C is minimum efficient.

$\Leftarrow$ If C is stable for R, then $\forall i \in C$, $\exists j \in C$ such that $RP_i = \{j\}$. So, if $C = \{1, \ldots, p\}$, we can rename these players in order to have $RP_1 = \{2\}$, $RP_2 = \{3\}$, ..., $RP_{p-1} = \{p\}$, $RP_p = \{1\}$, and C forms a cycle in the dependance graph.

■

Proposition 14 *Let $G = (N, V, \Gamma, \pi, \Phi)$ be a Boolean game.*

1. *if for every $i \in N$, φ_i is a positive term, then for any $B \subseteq N$, B is efficient if and only if B is stable for R.*

2. *if for every $i \in N$, φ_i is a positive clause, then for any $B \subseteq N$, B is efficient if and only if there exists a cycle in the dependence graph associated G whose nodes are exactly the members of B.*

Proof:

1. $\Rightarrow$ Let B be a stable set for R. As $\forall i$, φ_i is a positive term, we know than $\bigwedge_{i\in B}\varphi_i$ is consistant. So, from Proposition 11, B is an efficient coalition.

   $\Leftarrow$ Let B be an efficient coalition. So $\exists s_B \in S_B$ such that $s_B \models \bigwedge_{i\in B}\varphi_i$. As $\forall i$, $\varphi_i = \bigwedge_{v\in Lit(\varphi_i)} v$, $s_B \models \bigwedge_{v\in\bigcup_{i\in B} Lit(\varphi_i)} v$. So, $\forall v \in \bigcup_{i\in B} Lit(\varphi_i)$, $v \in \pi_B$, and then $\forall i \in B$, $RP_i \subseteq B$. B is stable for R.

2. Let $C = \{1, \ldots, p\}$ be a cycle between p players. So, $\forall i \in C$, $\exists j \in C$ such that $j \in RP_i$. As $\forall i \in N$, $\varphi_i = \bigvee_{v\in Lit(\varphi)} v$, we know than $\forall i \in C$, $\exists j \in C$, $\exists s_j \in S_j$ such that $s_j \models \varphi_j$. Moreover, we know than all the φ_i are positive clauses, so $\bigwedge_{i\in N}\varphi_i \not\models \bot$. Then, $\exists s_C \in S_C$ such that $s_C \models \bigwedge_{i\in C}\varphi_i$.

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