Qualitative integrals and desintegrals

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Abstract

Sugeno integrals and their particular cases such as weighted minimum and maximum have been used in multiple-criteria aggregation when the evaluation scale is qualitative. This paper proposes two new variants of weighted minimum and maximum, where the criteria weights have a role of tolerance threshold. These variants require the use of a residuated structure, equipped with an involutive negation. We propose residuated counterpart of Sugeno integrals, where the weights bear on subsets of criteria, and we study their properties, showing they are analogous to Sugeno integrals to a large extent. Finally we propose dual aggregation operations, we call desintegral, where an item is evaluated in terms of its local defects. Desintegrals are maximal when no defects at all are present, while integrals are maximal when all advantages are sufficiently present. Qualitative integrals and desintegral are ope possible approach to bipolar evaluation processes.

Keywords: Sugeno integral, possibility theory, Gödel implication, bipolarity, multicriteria evaluation

1. Introduction

In multi-criteria decision making, Sugeno integrals are commonly used as qualitative aggregation functions [20]. They are counterparts to quantitative Choquet integrals, but only require an ordered setting. Especially, like a Choquet integral, Sugeno integral delivers a score between the minimum and the maximum of the aggregated partial ratings. The definitions of Sugeno and Choquet integrals are both based on a monotonic set-function named *capacity* or *fuzzy measure*. These set functions are a basic representation tool that can be encountered in many areas, in particular in uncertainty modeling [16], multiple criteria aggregation [18, 20], group decision [22] and game theory [30]. They are used to represent the importance of the sets of possible states of nature, sets of criteria, groups of decision makers, etc. If the range of the capacity is considered as a finite totally ordered scale, then the capacity is said to be *qualitative*, or a q-capacity for short (it includes numerical scales when addition is not used).

In multiple-criteria analysis, the importance of the criteria can be exploited in different ways when aggregating partial evaluations. In weighted averages, the weights are like a number of allowed repetitions of a criterion. In Sugeno integrals, they are just thresholds that restrict the global satisfaction from below or from above. In this paper we consider variants of Sugeno integrals where the importance level is considered as a tolerance threshold such that overcoming it is sufficient to reach the best global rating, or a minimal requirement threshold that, if not reached, leads to downgrading the global evaluation. These variants, we call *qualitative integrals*, use an evaluation scale that is a both a totally ordered Heyting algebra and a Kleene algebra: a finite chain equipped with a residuated implication and an involutive negation [13].

Besides we also focus on the polarity of the evaluation scale. It is worth noticing that when Sugeno integral or these variants are used, the criteria have a *positive* flavor: the higher their values, the better the corresponding evaluation. But sometimes local ratings only reflect the presence or absence of defects, and the global evaluation decreases when the partial ratings increase. Such kinds of criteria are said to be *negative*. In such a case other variants of the Sugeno integrals, we call *qualitative desintegrals*, can be defined. With these new aggregation functions, the more a negative criterion is satisfied, the worse is the global evaluation (that we shall always assume to lie in a positive scale). In the definition of desintegrals, capacities are then replaced by fuzzy *anti-measures*, which are decreasing set functions. They are used to represent a tolerance or a permissiveness level on the negative scales. Similarly to qualitative integrals, such importance levels can be exploited in different ways when aggregating partial evaluations, which induce as many variants of qualitative desintegrals as there are qualitative integrals.

Finally, we exploit the fact that, in a finite setting, a capacity can be represented by a family of possibility distributions, as it is the lower (resp.

upper) bound of the corresponding possibility (resp. necessity) measures [1, 2, 15], and a Sugeno integral is a lower (resp. upper) bound of qualitative integrals with respect to such possibility (resp. necessity) measures, or equivalently lower bounds on prioritized maxima (resp. upper bounds on prioritized minima)[2, 15]. Dually, decreasing set functions can also be represented in terms of decreasing max- or min-decomposable measures. These properties entail a natural question: could we express general qualitative integrals (resp. desintegrals) in terms of the maximum or minimum of finite families of simpler integrals (resp. desintegrals) of the same type with respect to possibility or necessity (resp. guaranteed possibility and weak necessity) measures. This paper addresses this question as well. A positive answer means that in a number of cases the complexity of calculating a qualitative integral can be significantly reduced.

The paper is structured as follows: Section 2 presents the algebraic framework needed in the paper. Section 3 describes two new variants of weighted minimum and maximum. Section **??** presents their extensions in the form of residuation-based variants of Sugeno integrals and studies some of their properties. Section **??** is devoted to the negative counterparts of qualitative integrals, when local evaluations belong to negative scales.

2. Algebraic framework

In this section we first recall the qualitative setting of totally ordered Heyting algebras used as rating scales in multifactorial evaluation.

We consider a finite set of criteria $C = \{1, \cdots, n\}$. Some objects are evaluated using these criteria. The evaluation scale, L, associated to each criterion is assumed to be a totally ordered set. It may be finite or be the interval $[0, 1]$. In both cases the bottom is 0 and the top is 1. The maximum (resp. minimum) will be denoted by $\vee$ (resp. $\wedge$). An object is represented by a vector of ratings on the different criteria, i.e., by $f = (f_1, \cdots, f_n) \in L^n$ where f_i is the rating of f according to the criterion i. In such a context an object is viewed as a function f from L^n to L. In the following without loss of generality when we consider an object f we suppose that $f_1 \leq \cdots \leq f_n$ (we can consider a suitable permutation on the set of criteria induced by the f_i's); and we denote by A_i the set of indices $\{i, \cdots, n\}$ with the convention $A_{n+1} = \emptyset$.

Moreover, on L we can define Gödel implication $\rightarrow_G$ using the residuation

Res as follows

$$a \wedge b \leq c \Leftrightarrow a \leq b \to_G c,$$

so that $b \to_G c = \bigvee\{a : a \wedge b \leq c\} = Res(\wedge)$. In such a context $\langle L, \wedge, \vee, 0, 1, \to_G\rangle$ is a special case of Heyting algebra, i.e., as a special case of complete residuated lattice; indeed $\langle L, \wedge, \vee, 0, 1\rangle$ is a chain and $\langle L, \wedge, 1\rangle$ is a commutative monoid (since $\wedge$ is associative, commutative and for all $a \in L$, $a \wedge 1 = a$).

In the following, we will consider positive criteria and negative criteria. In order to handle the polarity of the evaluation scale, we also need an order-reversing operation on L, denoted by $1-\cdot$, that is decreasing and involutive (a Kleene negation, since the structure $\langle L, \wedge, \vee, 0, 1, 1-\cdot\rangle$ is a Kleene algebra). If L is a positively directed scale (1 means good, 0 means neutral), then $\{1-a : a \in L\}$ is a negatively directed scale (1 means bad, 0 means neutral). On a complete residuated lattice, another negation is defined by $\neg a = a \to 0$ such that $\neg a = 1$ if $a = 0$ and 0 otherwise, hence not involutive. This negation clearly differs from the Kleene negation.

In the structure $\langle L, \wedge, \vee, 0, 1, \to, 1-\cdot\rangle$ there are thus three different implications we are going to use in this paper:

- the Gödel implication defined by $a \to_G b = 1$ if $a \leq b$ and b otherwise;
- the Kleene-Dienes implication: $a \to_D b = (1-a) \vee b$;
- the contrapositive Gödel implication $\to_{GC} = \mathcal{C}(\to_G)$:
  $a \to_{GC} b = (1-b) \to_G (1-a) = 1$ si $a \leq b$ and $1-a$ otherwise.

The contraposition $\mathcal{C}$ applies to any binary operation $\ominus$ in a Kleene algebra, and is such that $a\mathcal{C}(\ominus)b = (1-b) \ominus (1-a)$. Likewise, to each implication $\to$ can be associated a conjunction $\star = \mathcal{S}(\to)$ defined by $a \star b = 1 - a \to (1-b)$ (a transformation $\mathcal{S}$ already introduced in [4]). Via this transformation, one obtains three conjunctions, two of which are non-commutative:

- the Kleene conjunction, already met: $a \star_D b = a \wedge b$;
- the Gödel conjunction : $a \star_G b = 0$ if $a \leq 1-b$ and b otherwise;
- the contrapositive Gödel conjunction: $a \star_{GC} b = 0$ if $b \leq 1-a$ and a otherwise. That is $a \star_{GC} b = b \star_G a$.

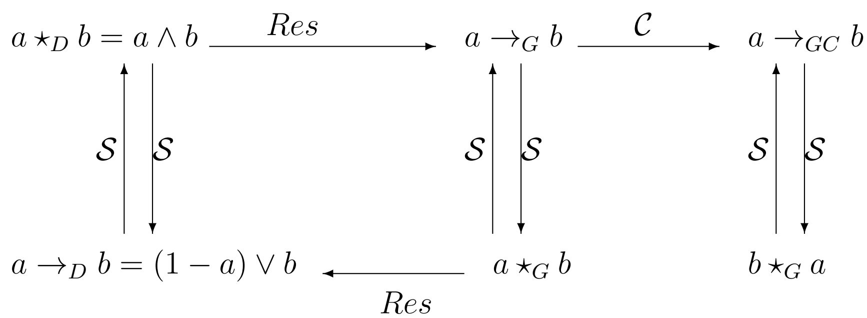

Figure 1: Conjunctions and implications on a finite chain

Moreover in [4] it was proved that the generation process of conjunctions modeled by the triangular norm $\wedge$ is closed as represented on Fig. 1, where $Res(\odot)$ is the residuated operation $bRes(\odot)c = \vee\{a : a \odot b \leq c\}$. Also, $\star = \mathcal{S}(\rightarrow)$ is equivalent to $\rightarrow = \mathcal{S}(\star)$.

Note that the operation $a \star b = (a \star_G b) \wedge (a \star_{GC} b) = a \wedge b$ if $a \geq 1 - b$, and 0 otherwise, is yet another conjunction known as the nilpotent minimum [17, 28]. The corresponding residuated implication is defined by $a \rightarrow_\star b = 1$ if $a \leq b$ and $a \rightarrow_D b$ otherwise. Moreover this implication is self-contrapositive ($a \rightarrow_\star b = (1 - b) \rightarrow_\star (1 - a)$), and $\mathcal{S}(\star) = \rightarrow_\star$.

3. Simple qualitative aggregation schemes on positive scales

This section focuses on the possible elementary qualitative aggegation functions when we consider positive criteria. In such a context the local scales and the global scale are positive.

There are two elementary qualitative aggregation schemes:

- The first one, $\wedge_{i=1}^n f_i$ is pessimistic and very demanding; namely, in order to obtain a good global evaluation, an object needs to satisfy all the criteria.
- The second one, $\vee_{i=1}^n f_i$ is optimistic and very loose; namely, one fulfilled criterion is enough to obtain a good global evaluation.

These two agregation schemes can be generalised by means of importance levels or priorities $\pi_i \in L$, on the criteria i, $i = 1, \cdots, n$. Suppose π_i is all the greater as the criterion i is important. A fully important criterion has importance weight $\pi_i = 1$. In the following, we assume $\pi_i > 0, \forall i$, i.e., there is no useless criterion. In this section, we also assume $\pi_i = 1$, for some criterion i (the most important one). These importance levels can alter each local evaluation f_i in different manners. More precisely, π_i can act as a saturation threshold that blocks the global score under or above a certain value dependent on the importance level of criterion i. Alternatively, π_i can be considered as a threshold above which the decision-maker is perfectly satisfied and under which the local rating is altered or not. There are two such rating modification schemes. All of them use a pair (implication, conjunction) defined previously. Let us present all these cases in details.

3.1. Saturation levels

Here the importance weights act as saturation levels: they reduce the evaluation scale from above or from below. The rating f_i is modified either into $(1 - \pi_i) \vee f_i \in [1 - \pi_i, 1]$, or $\pi_i \wedge f_i \in [0, \pi_i]$. A fully important criterion can affect the whole global score scale.

- In a demanding aggregation, all the important criteria have to be satisfied, which justifies the prioritized minimum [6]:

$$SLMIN_\pi(f) = \wedge_{i=1}^{n}(1 - \pi_i) \vee f_i = \wedge_{i=1}^{n}\pi_i \rightarrow_D f_i. \quad (1)$$

  Hence an important criterion can alone bring the overall score very low and a criterion that is of little importance cannot downgrade the overall score under a certain level $1 - \pi_i$. A fully important criterion ($\pi_i = 1$) acts as a veto as it can lead to a zero global score if violated. This is why under this aggregation scheme, such criteria can actually be viewed as soft constraints [9]. The weights π_i are priorities, that affect the level of acceptance of objects that violate criteria.

- In a loose aggregation we just need to satisfy one important criterion, which justifies of the prioritized maximum[6, 33]:

$$SLMAX_\pi(f) = \vee_{i=1}^{n}\pi_i \wedge f_i = \vee_{i=1}^{n}\pi_i \star_D f_i. \quad (2)$$

  In this case an important criterion is one that alone can bring a good overall score (a maximal one for a fully important criterion) and a not important criterion can never alone bring the overall score higher than π_i.

It is well-known that if the evaluation scale of the local ratings f_i is reduced to $\{0, 1\}$ (Boolean criteria) then letting $A_f = \{i : f_i = 1\}$ be the set of criteria satisfied by object f, $SLMAX_\pi(f) = \vee\{\pi_i : i \in A_f\} = \Pi(A_f)$ is a possibility measure [34] (a maxitive capacity), and $SLMIN_\pi(f) = \wedge\{1-\pi_i : i \notin A_f\} = N(A_f)$ is a necessity measure [7] (a minitive capacity). Note that we have the following De Morgan-like property, that extends the well-known duality $\Pi(A) = 1 - N(A^c)$, where A^c is the complement of A, to graded tuples f:

$$SLMAX_\pi(f) = 1 - SLMIN_\pi(1 - f). \tag{3}$$

3.2. Softening thresholds

The importance weight π_i of a criterion can be considered as a threshold that if passed by the corresponding local rating, is sufficient to obtain a full local score. Namely, if f_i is greater than π_i then the local rating becomes maximal, i.e 1. Otherwise, if f_i is less than π_i, then the local rating remains as it stands. Clearly, the effect of the weight π_i on the original local rating f_i is to turn it into $\pi_i \rightarrow_G f_i$.

- The demanding aggregation is obtained replacing $\rightarrow_D$ by $\rightarrow_G$. We get

  $$STMIN_\pi(f) = \wedge_{i=1}^n \pi_i \rightarrow_G f_i. \tag{4}$$

  The idea is still that the evaluated item should get good grades for all important criteria. In this case, a criterion is all the less important as the required rating for considering it fulfilled is low. A fully important criterion is considered satisfied only if $f_i = 1$. A criterion with a low importance weight π is satisfied even by objects for which f_i is low, provided that this local rating is above π_i.

- We can define the corresponding loose aggregation, changing $\wedge$ into the conjunction $\star_G$ associated to $\rightarrow_G$ in the SLMAX aggregation scheme:

  $$STMAX_\pi(f) = \vee_{i=1}^n \pi_i \star_G f_i. \tag{5}$$

  Since $\pi_i \star_G f_i = 0$ as soon as $f_i \leq 1 - \pi_i$, it means that for a little important criterion, the local rating for criterion i must be very high (at least $1-\pi_i$) to influence the global score and it is eliminated otherwise. On the contrary, an important criterion i may affect the global rating even if the corresponding local rating is low.

In fact, these aggregation schemes are better understood if written as:

$$STMIN_\pi(f) = \bigwedge_{i:f_i<\pi_i} f_i; \quad STMAX_\pi(f) = \bigvee_{i:f_i>1-\pi_i} f_i \tag{6}$$

with the usual conventions $\wedge\emptyset = 1; \vee\emptyset = 0$. It is easy to see that

- $STMIN_\pi(f) = 1$ if and only if $f_i \geq \pi_i, \forall i = 1, \ldots, n$, that is if and only if the local ratings reach at least the levels prescribed by the importance thresholds. Note that is is a rather unsurprising demand.

- $STMIN_\pi(f) = 0$ if and only if $\exists i, f_i = 0$ and $\pi_i > 0$[1], that is if some criterion is totally violated

- $STMAX_\pi(f) = 1$ if and only if $\exists i, f_i = 1$ and $\pi_i > 0$, that is if some criterion is totally satisfied

- $STMAX_\pi(f) = 0$ if and only if $\forall i, f_i \leq 1 - \pi_i$, that is if no criterion passes the rating threshold $1 - \pi_i$.

In fact, with $STMIN_\pi(f)$, the weights select violated criteria that alone are enough to eliminate f, and with $STMAXN_\pi(f)$, the weights select satisfied criteria that alone are enough to accept f.

We have again the following De Morgan-like duality:

$$STMAX_\pi(f) = 1 - STMIN_\pi(1 - f). \tag{7}$$

However, $STMIN_\pi(f)$ and $STMAX_\pi(f)$ cannot be considered as a proper generalization to fuzzy events of possibility and necessity measures, since when the f_i's belong to $\{0, 1\}$ and is the characteristic function μ_B of set B, we do not get $STMIN_\pi(\mu_B) = N(B)$, nor $STMAX_\pi(\mu_B) = \Pi(B)$.

Indeed, in that case $STMIN_\pi(f) \in \{0, 1\}$ and $STMAX_\pi(f) \in \{0, 1\}$ as well. Namely $STMIN_\pi(\mu_B) = 1$ if $B = \mathcal{C}$, 0 otherwise, and $STMAX_\pi(\mu_B) = 1$ if $B \neq \emptyset$, and 0 otherwise. In other words, everything happens as if weights were all equal to 1, $STMIN_\pi$ being a standard conjunction, and $STMAX_\pi$ a standard disjunction. It is known [11] that the residuation-based extension of necessity measures to fuzzy events is not based on Gödel implication, but on its contrapositive form.

[1]The latter condition is assumed for all criteria.

3.3. Drastic thresholdings

Another way of handling importance weights is to downgrade or upgrade local ratings to a fixed value when they fail to reach the importance thresholds π_i, this prescribed value being all the lower as the criterion is important. Namely, if $f_i < \pi_i$ then we set the rating to $1 - \pi_i$. As a consequence, the modified rating is modelled by $\pi_i \rightarrow_{GC} f_i$, so that the local evaluation scale of criterion i is reduced to pseudo-Boolean values in the set $\{1-\pi_i, 1\}$, which is a drastic way of handling graded ratings. Again we shall have demanding and loose aggregations.

- The demanding aggregation will be

$$DTMIN_\pi(f) = \wedge_{i=1}^n \pi_i \rightarrow_{GC} f_i \tag{8}$$

  When violated (i.e. the threshold π_i is missed), an important criterion alone may drastically downgrade the overall score, while the local rating according to an unimportant criterion may be upgraded (in each case, to $1 - \pi_i$, which is low in the first case and high in the second case).

- The loose counterpart will be

$$DTMAX_\pi(f) = \vee_{i=1}^n \pi_i \star_{GC} f_i. \tag{9}$$

  An important criterion, if satisfied, can alone bring the overall score to a high value but an unimportant criterion, even if satisfied, cannot bring the overall score to a high value (π_i in each case).

These aggregation schemes are better understood if expressed as follows:

$$DTMIN_\pi(f) = \bigwedge_{i:f_i<\pi_i} 1 - \pi_i; \quad DTMAX_\pi(f) = \bigvee_{i:f_i>1-\pi_i} \pi_i \tag{10}$$

Letting $A_f^\downarrow = \{i : f_i < \pi_i\}$, we observe that $DTMIN_\pi(f) = N(A_f^\downarrow)$. Likewise denoting $A_f^\uparrow = \{i : f_i > 1 - \pi_i\}$, we observe that $DTMAX_\pi(f) = \Pi(A_f^\uparrow)$, so that when $f_i \in \{0, 1\}$, we do get necessity and possibility measures (and then, $A_f^\downarrow = (A_f^\uparrow)^c$). We have again the following duality:

$$DTMAX_\pi(f) = 1 - DTMIN_\pi(1 - f). \tag{11}$$

Remark 1. *There are alternative ways of handling importance weights in the qualitative setting. In the scope of a loose aggregation, one way would be to downgrade to 0 ratings that do not pass the importance threshold* π_i*, keeping them as such otherwise. This operation is a kind of residuated subtraction* $f_i \ominus \pi_i = \inf\{x : \pi_i \vee x \geq f_i\} = f_i$ *if* $\pi_i < f_i$ *and 0 otherwise.*

It would lead to consider aggregation schemes of the form:

$$SUMAX_\pi(f) = \bigvee_{i:f_i>\pi_i} f_i; \quad SUMIN_\pi(f) = \bigwedge_{i:f_i>1-\pi_i} f_i$$

The last equation is obtained from the first by duality. Pseudo-Boolean counterparts of such aggregation methods are

$$DUMAX_\pi(f) = \bigvee_{i:f_i>\pi_i} \pi_i; \quad DUMIN_\pi(f) = \bigwedge_{i:f_i<1-\pi_i} 1-\pi_i$$

The study of such methods is a topic for further research.

4. Variants of qualitative integrals

This part focuses on the generalisation of the qualitative weighted aggregation schemes presented in the previous part to the case where weights are directly assigned to subsets of criteria rather than to individual ones only. This kind of approach enables various kinds of interactions between criteria to be taken into account. Note that in the demanding aggregation schemes using $SLMIN, STMIN$ and $DTMIN$, there is a synergy between criteria (they need to be all fullfilled) while in the loose aggregation schemes, using $SLMAX, STMAX$ and $DTMAX$, the criteria are more or less redundant. We consider more general forms of interaction here.

4.1. Sugeno integral

Importance levels can be assigned to sets of criteria (instead of single ones) by means of a capacity which is a mapping $\gamma : 2^C \to L$ such that $\gamma(\emptyset) = 0$, $\gamma(C) = 1$, and if $A \subseteq B$ then $\gamma(A) \leq \gamma(B)$. The conjugate $\gamma^c(A)$ of capacity γ is a capacity defined by $\gamma^c(A) = 1 - \gamma(\overline{A}), \forall A \subseteq C$, where $\overline{A}$ is the complement of subset A. This generalised importance assignment enables dependencies between criteria to be accounted for; namely, redundant criteria in a set A are such that $\gamma(A) = \max_{i\in A} \gamma(\{i\})$ while a synergy between them is expressed when $\gamma(A) > \max_{i\in A} \gamma(\{i\})$.

A special case of capacity is a possibility measure [34, 7] which is a maxitive capacity, i.e., a capacity Π such that $\Pi(A \cup B) = \Pi(A) \vee \Pi(B)$. Since the set of criteria is finite, the possibility distribution $\pi : \pi(s) = \Pi(\{s\})$ is enough to recover the set-function: $\forall A \subseteq C, \Pi(A) = \vee_{s \in A} \pi(s)$. When modeling uncertainty, the value $\pi(s)$ is understood as the possibility that s be the actual state of the world: $\exists s \in S : \pi(s) = 1$. When modeling priorities, $\pi(s)$ is the importance of criterion s, and normalization means that there is at least one criterion that is fully important.

The conjugate of a possibility measure Π is a necessity measure $N(A) = 1 - \Pi(A^c)$, and then N is a minitive capacity i.e $N(A \cap B) = N(A) \wedge N(B)$. Moreover, $N(A) = \wedge_{s \notin A} \iota(s)$ where $\iota(s) = N(C \setminus \{s\})$ (this is the degree of impossibility of s when dealing with uncertainty), and $\iota(s) = 1 - \pi(s)$, where π defines the conjugate possibility measure $\Pi = N^c$.

The usual generalisation of the prioritized maximum $SLMAX_\pi$ and the prioritized minimum $SLMIN_\pi$ is the well-known Sugeno integral widely used to aggregate qualitative local evaluations in multiattribute evaluation [31]:

$$\oint_\gamma (f) = \bigvee_{A \subseteq \mathcal{C}} \gamma(A) \wedge \wedge_{i \in A} f_i \tag{12}$$

The notation $\oint_\gamma$, letting the capacity symbol appear as a subscript, is unusual for integrals. It is conveniently concise for this paper where the domain plays no particular role.

It is easy to see (and well-known [3, 21]) that if the capacity is a possibility measure, $\oint_\Pi (f) = SLMAX_\pi(f)$. Indeed, letting $j \in A$ be such that $\pi_j = \Pi(A)$, it is obvious that $\Pi(A) \wedge \wedge_{i \in A} f_i \leq \pi_j \wedge f_j$.

There are alternative expressions of Sugeno integral as follows [25]:

$$\oint_\gamma (f) = \bigvee_{A \subseteq \mathcal{C}} \gamma(A) \wedge \wedge_{i \in A} f_i = \bigwedge_{A \subseteq \mathcal{C}} \gamma(\overline{A}) \vee \vee_{i \in A} f_i) \tag{13}$$

$$= \bigvee_{i=1}^{n} f_i \wedge \gamma(\{i, \ldots, n\}) = \bigwedge_{i=1}^{n} f_i \vee \gamma(\{i+1, \ldots, n\}). \tag{14}$$

$$= \bigvee_{a \in L} a \wedge \gamma(\{i : f_i \geq a\}) = \bigwedge_{a \in L} a \vee \gamma(\{i : f_i > a\}). \tag{15}$$

where we have supposed $f_1 \leq \cdots \leq f_n$ as it is explained at the beginning of the second section.

To make the rest of the paper easier to read it is useful to recall how these properties are justified, as well as some other related properties. Sugeno integral has exponential complexity in terms of the number of criteria, but can be reduced to a expression of linear size [31, 32, 12, 25, 24]:

Lemma 1. $\bigvee_{A \subseteq \mathcal{C}} \gamma(A) \wedge \wedge_{i \in A} f_i = \bigvee_{i=1}^{n} f_i \wedge \gamma(\{i, \ldots, n\})$

Proof If $f_j = \wedge_{i \in A} f_i$, then $A \subseteq \{j, \ldots, n\}$, so $\forall A \subseteq \mathcal{C}, f_j \wedge \gamma(\{j, \ldots, n\}) \geq \gamma(A) \wedge \wedge_{i \in A} f_i$. ■

Lemma 2. $\bigwedge_{A \subseteq \mathcal{C}} \gamma(\overline{A}) \vee \vee_{i \in A} f_i = \bigwedge_{i=1}^{n} f_i \vee \gamma(\{i+1, \ldots, n\}))$

Proof If $f_j = \vee_{i \in A} f_i$, then $A \subseteq \{1, \ldots, j\}$, i.e., $\{j+1, \ldots, n\} \subseteq \overline{A}$. So, $\forall A \subseteq \mathcal{C}, f_j \vee \gamma(\{j+1, \ldots, n\}) \leq \gamma(\overline{A}) \vee \vee_{i \in A} f_i$. ■

Lemma 3. $\oint_{\gamma}(f) = \bigvee_{i=1}^{n} f_i \wedge \gamma(\{i, \ldots, n\}) = \bigwedge_{i=1}^{n} f_i \vee \gamma(\{i+1, \ldots, n\})$

Proof The f_i form an increasing sequence, and $g_i = \gamma(\{i, \ldots, n\})$ form a decreasing sequence. Since $g_1 = 1, \bigvee_{i=1}^{n} f_i \wedge g_i$ is the median of $\{f_1, \ldots f_n\} \cup \{g_2, \ldots g_n\}$ ([6], Proposition 1). Likewise, since $g_{n+1} = 0, \bigwedge_{i=1}^{n} f_i \vee g_{i+1}$ is the median of the same set of numbers ([6], Proposition 2). ■

These results make it easy to realize that [21, 25]:

Corollary 1. *For a necessity measure N based on possibility distribution π:* $\oint_{N}(f) = SLMIN_{\pi}(f)$.

Proof $\oint_{N}(f) = \bigwedge_{i=1}^{n} f_i \vee N(\{i+1, \ldots, n\}) = \bigwedge_{i=1}^{n} f_i \vee (\wedge_{j \leq i} 1 - \pi_j)$. If the minimum were reached for $i > j$, one would have $\oint_{N}(f) = f_i \vee (1 - \pi_j)$, but note that $f_i \vee (1 - \pi_j) \geq f_j \vee (1 - \pi_j)$. So the minimum is reached for $i = j$. ■

Sugeno integral can be rewritten using the Kleene implication $\to_D$ and conjunction $\star_D$, which highlights the connection between the two forms of Sugeno integral and the two families of optimistic and pessimistic aggregation operations laid bare in the previous section. Consider the following expressions:

$$\oint_{\gamma}^{\star D}(f) = \bigvee_{A \subseteq \mathcal{C}} \gamma(A) \star_D \wedge_{i\in A} f_i \text{ and } \oint_{\gamma}^{\uparrow D}(f) = \bigwedge_{A \subseteq \mathcal{C}} \gamma^c(A) \to_D \vee_{i\in A} f_i.$$

As recalled above, these are two forms of Sugeno integral that satisfy the following equalities:

$$\oint_{\gamma}(f) = \oint_{\gamma}^{\star D}(f) = \oint_{\gamma}^{\uparrow D}(f) \tag{16}$$

Finally there is a duality relation between Sugeno integrals with respect to conjugate capacities:

Proposition 1. $\oint_{\gamma}(f) = 1 - \oint_{\gamma^c}(1-f)$

Proof:
$1 - \oint_{\gamma^c}(1-f) = 1 - \bigwedge_{A\subseteq\mathcal{C}} \gamma^c(\overline{A}) \vee (\vee_{i\in A} 1 - f_i) = \bigvee_{A\subseteq\mathcal{C}} 1 - \gamma^c(\overline{A}) \wedge (\wedge_{i\in A} f_i)$. ■

4.2. Residuation-based integrals and their common properties

We are now in a position to propose generalisations of other weighted aggregations in a similar way as above, changing Kleene implication into Gödel implication and its contrapositive form, as well as the associated conjunctions induced by the Kleene negation:

Definition 1.

Soft integrals

$$\oint_{\gamma}^{\uparrow G}(f) = \bigwedge_{A\subseteq\mathcal{C}} \gamma^c(A) \to_G \vee_{i\in A} f_i = \bigwedge_{A\subseteq\mathcal{C}: \vee_{i\in A} f_i < \gamma^c(A)} \vee_{i\in A} f_i;$$

$$\oint_{\gamma}^{\star G}(f) = \bigvee_{A\subseteq\mathcal{C}} \gamma(A) \star_G \wedge_{i\in A} f_i = \bigvee_{A\subseteq\mathcal{C}: \wedge_{i\in A} f_i > 1-\gamma(A)} \wedge_{i\in A} f_i.$$

Drastic integrals

$$\oint_{\gamma}^{\uparrow GC}(f) = \bigwedge_{A\subseteq\mathcal{C}} \gamma^c(A) \to_{GC} \vee_{i\in A} f_i = \bigwedge_{A\subseteq\mathcal{C}:\vee_{i\in A} f_i < 1-\gamma^c(A)} (1-\gamma^c(A));$$

$$\oint_{\gamma}^{\star GC}(f) = \bigvee_{A\subseteq\mathcal{C}} \gamma(A) \star_{GC} \wedge_{i\in A} f_i = \bigvee_{A\subseteq\mathcal{C}:\wedge_{i\in A} f_i > 1-\gamma(A)} \gamma(A).$$

Note that the drastic integrals can be written more directly in terms of Gödel connectives as:

$$\oint_{\gamma}^{\uparrow GC}(f) = \bigwedge_{A\subseteq\mathcal{C}} \wedge_{i\in A}(1-f_i) \to_G \gamma(A^c)$$

and

$$\oint_{\gamma}^{\star GC}(f) = \bigvee_{A\subseteq\mathcal{C}} \wedge_{i\in A} f_i \star_G \gamma(A).$$

The soft integrals are supposed to be a generalisation of $STMAX_\pi$ and $STMIN_\pi$, while drastic integrals are supposed to be a generalisation of $DTMIN_\pi$ and $DTMAX_\pi$. We shall then show that these residuation-based expressions can be simplified in terms of equivalent forms in a way similar to Sugeno integral.

A generalized version of property of duality between the loose and the demanding aggregation schemes holds for all the integrals:

Proposition 2.
$\oint_{\gamma}^{\star}(f) = 1 - \oint_{\gamma^c}^{\uparrow}(1-f)$ *where* $(\to, \star) \in \{(\to_G, \star_G), (\to_{GC}, \star_{GC})\}$.

Proof: $1 - \oint_{\gamma^c}^{\uparrow}(1-f) =$

$1 - \wedge_{A\subseteq\mathcal{C}}\gamma(A) \to \vee_{i\in A} 1 - f_i = \vee_{A\subseteq\mathcal{C}} 1 - (\gamma(A) \to 1 - \wedge_{i\in A} f_i) = \oint_{\gamma}^{\star}(f)$. ■

Like Sugeno integral, the residuation-based integrals have exponential complexity in terms of the number of criteria, but can be reduced to a expression of linear size similar to (15):

Proposition 3.
$$\oint_\gamma^{\uparrow}(f) = \bigwedge_{i=1}^{n}(\gamma^c(\overline{A_{i+1}}) \to f_i) \text{ and } \oint_\gamma^{\star}(f) = \bigvee_{i=1}^{n}\gamma(A_i) \star f_i$$
where $(\to, \star) \in \{(\to_G, \star_G), (\to_{GC}, \star_{GC})\}$.

Proof:
$$\oint_\gamma^{\uparrow}(f) = \bigwedge_{i=1}^{n}(\gamma^c(\overline{A_{i+1}}) \to f_i) \wedge \bigwedge_{A \notin \{\overline{A_2}, \cdots, \overline{A_{n+1}}\}}(\gamma^c(A) \to \vee_{i\in A} f_i).$$
We consider $A \notin \{\overline{A_2}, \cdots, \overline{A_{n+1}}\}$ and let $f_k = \vee_{i\in A} f_i$.
Now, $A \subseteq \overline{A_{k+1}}$, for index k. Then clearly $\gamma^c(A) \leq \gamma^c(\overline{A_{k+1}})$.
So $\gamma^c(A) \to \vee_{i\in A} f_i \geq \gamma^c(\overline{A_{k+1}}) \to f_k \geq \wedge_{i=1}^{n}(\gamma^c(\overline{A_{i+1}}) \to f_i)$.

For the integral using the conjunction, we denote $\wedge_{i\in A} f_i = f_k$. Then $\wedge_{i\in A} f_i = f_k$ is maximal for $A = A_k$, and so is $\gamma(A)$ among all A such that $\wedge_{i\in A} f_i = f_k$. ■

Sugeno integral also has the following expression (16): $\oint_\gamma(f) = \bigvee_{a\in L} a \wedge \gamma(\{i : f_i \geq a\})$. The soft integrals and the drastic integrals satisfy similar properties:

Proposition 4. $\oint_\gamma^{\star}(f) = \bigvee_{a\in L}\gamma(\{f \geq a\}) \star a,$
$$\oint_\gamma^{\uparrow}(f) = \bigwedge_{a\in L}(1 - \gamma(\{f > a\})) \to a = \bigwedge_{a\in L}\gamma^c(\{f \leq a\})) \to a,$$
where $(\to, \star) \in \{(\to_G, \star_G), (\to_{GC}, \star_{GC})\}$.

Proof: We have $\oint_\gamma^{\star}(f) = \vee_{i=1}^{n}\gamma(A_i) \star f_i$. Let us consider $a \in L$.

- If $\exists i$ such that $a = f_i$ then $\gamma(\{f \geq a\}) \star a = \gamma(A_i) \star f_i$ (we take the least index i such that $f_i = a$).
- If $a > f_n$ then $\gamma(\{f \geq a\}) = \gamma(\emptyset) = 0$ hence $\gamma(\{f \geq a\}) \star a = 0$.
- Otherwise let us denote i the index such that $f_{i-1} < a < f_i$. In such a context we have $\gamma(\{f \geq a\}) \star a = \gamma(A_i) \star a$ which entails $\gamma(\{f \geq a\}) \star a \leq \gamma(A_i) \star f_i$.

So we have $\vee_{a\in L}\gamma(\{f\geq a\})\star a=\vee_{i=1}^{n}\gamma(A_i)\star f_i$.
Using Proposition 2 we have

$$\oint_{\gamma}^{\uparrow}(f)=1-\oint_{\gamma^c}^{\star}(1-f)=1-\bigvee_{a\in L}\gamma^c(\{1-f\geq a\}\star a.$$

So $\displaystyle\oint_{\gamma}^{\uparrow}(f)=\bigwedge_{a\in L}(1-\gamma^c(\{1-f\geq a\})\star a)$
$=\bigwedge_{a\in L}[1-(1-\gamma^c(\{1-f\geq a\})\to(1-a))]$
$=\bigwedge_{a\in L}\gamma^c(\{1-f\geq a\})\to(1-a)=\bigwedge_{a\in L}(1-\gamma(\{1-f<a\}))\to(1-a)$. ■

As recalled above, he weighted aggregations $SLMIN_\pi$ and $SLMAX_\pi$ are particular cases of the Sugeno integral with respect to a necessity and a possibility measure, respectively. Similarly there is a connection between the soft (resp. drastic) integrals and aggregations $STMIN_\pi(f)$, $STMAX_\pi(f)$ (resp. $DTMIN_\pi(f)$, $DTMAX_\pi(f)$).

Proposition 5.

- *If γ is a necessity measure N, then* $\displaystyle\oint_{N}^{\uparrow G}=STMIN_\pi$ *and* $\displaystyle\oint_{N}^{\uparrow GC}=DTMIN_\pi$.
- *If γ is a possibility measure Π then* $\displaystyle\oint_{\Pi}^{\star G}=STMAX_\pi$ *and* $\displaystyle\oint_{\Pi}^{\star GC}=DTMAX_\pi$.

Proof: If γ is a necessity measure then γ^c is a possibility measure Π based on possibility degrees $\pi_i, i=1,\ldots,n$.

- For each index i, there exists a subset B_i such that $\pi_i=\Pi(B_i)$. It is then obvious that $\displaystyle\oint_{N}^{\uparrow G}(f)\leq STMIN_\pi(f)$ and $\displaystyle\oint_{N}^{\uparrow GC}(f)\leq DTMIN_\pi(f)$ as the integrals consider the minimum over many more situations.

- Now let A be a set such that $\displaystyle\oint_{N}^{\uparrow G}(f)=\vee_{j\in A}\pi_j\to_G\vee_{i\in A}f_i$. Let k,ℓ such that $\displaystyle\oint_{N}^{\uparrow G}(f)=\pi_k\to_G f_\ell$. If $\pi_k\to_G f_\ell=1$, then $\displaystyle\oint_{N}^{\uparrow G}(f)\geq STMIN_\pi(f)$ is obvious. Otherwise $\pi_k\to_G f_\ell=f_\ell<1$. But by construction $f_\ell\geq f_k$. Hence $\displaystyle\oint_{N}^{\uparrow G}(f)=f_\ell\geq\pi_k\to_G f_k\geq STMIN_\pi(f)$.

Similarly for the drastic integral let us denote $\oint_N^{\uparrow GC}(f) = \pi_k \to_{GC} f_l$. By construction $f_1 \leq \cdots \leq f_k \leq \cdots \leq f_l$ and $\pi_l \leq \pi_k$.

- If $\oint_N^{\uparrow GC}(f) = 1$ then $\oint_N^{\uparrow GC}(f) \geq DTMIN_\pi(f)$.
- If $\oint_N^{\uparrow GC}(f) = 1 - \pi_k$ hence $\pi_k > f_l \geq f_k$ so $\oint_N^{\uparrow GC}(f) = \pi_k \to_{GC} f_k \geq DTMIN_\pi(f)$.

If γ is a possibility measure, using the relation between the implication and the conjunction we have $\oint_\Pi^{\star}(f) = 1 - \oint_N^{\uparrow}(1-f)$ where $N = \Pi^c$ is a necessity measure. Hence we conclude using the relation between the simple weigthed aggregations. ■

4.3. Properties specific to residuation-based integrals

There is a major difference between Sugeno integrals and its residuation-based variant: the counterpart of equality (17) satisfied by the Sugeno integral is not true for the soft and the drastic desintegrals. More precisely $\oint_\gamma^{\uparrow G} \neq \oint_\gamma^{\star G}$ and $\oint_\gamma^{\uparrow GC} \neq \oint_\gamma^{\star GC}$.

In fact, we can prove inequalities only as a by-product of Proposition 3:

Corollary 2. *$\oint_\gamma^{\star G}(f) \geq \oint_\gamma^{\uparrow G}(f)$ and $\oint_\gamma^{\star GC}(f) \geq \oint_\gamma^{\uparrow GC}(f)$.*

Proof: First write the expressions in the forms $\oint_\gamma^{\star G}(f) = \bigvee_{i:f_i > 1-\gamma(A_i)} f_i$ and $\oint_\gamma^{\uparrow G}(f) = \bigwedge_{i=:\gamma^c(\overline{A_{i+1}}) > f_i} f_i$, where $\gamma^c(\overline{A_{i+1}}) = 1 - \gamma(A_{i+1})$. By definition, $1 - \gamma(A_{i+1}) \geq 1 - \gamma(A_i)$. Hence $\vee_{i:f_i > 1-\gamma(A_i)} f_i \geq \wedge_{i:f_i < 1-\gamma(A_{i+1})} f_i$.

We can write $\oint_\gamma^{\star GC}(f) = \bigvee_{i:\gamma(A_i) > 1-f_i} \gamma(A_i)$ and $\oint_\gamma^{\uparrow GC}(f) = \bigwedge_{i=:\gamma(A_{i+1}) < 1-f_i} \gamma(A_{i+1})$.

Let i be an index such that $\oint_\gamma^{\star GC}(f) = \gamma(A_i)$. Hence $\gamma(A_i) \leq \gamma(A_{i-1}) \leq$

Integrals	attain 0	attain 1
$\oint_\gamma^{\uparrow D}(f)$	$\forall A, \gamma(A) = 0$ or $\exists i \in A, f_i = 0$	$\exists A, \gamma(A) = 1$ and $\forall i \in A, f_i = 1$
$\oint_\gamma^{\uparrow G}(f)$	$\exists A, \forall i \in A, f_i = 0 < \gamma^c(A)$	$\forall A, \exists i \in A, f_i \geq \gamma^c(A)$
$\oint_\gamma^{\star G}(f)$	$\forall A, \exists i \in A, f_i \leq 1 - \gamma(A)$	$\exists A, \gamma(A) > 0$ and $\forall i \in A, f_i = 1$
$\oint_\gamma^{\uparrow GC}(f)$	$\exists A, \forall i \in A, f_i < 1 = \gamma^c(A)$	$\forall A, \exists i \in A, f_i \geq \gamma^c(A)$
$\oint_\gamma^{\star GC}(f)$	$\forall A, \exists i \in A, f_i \leq 1 - \gamma(A)$	$\exists A, \gamma(A) = 1$ and $\forall i \in A, f_i > 0$

Table 1: Discrepancies between conjunctive and implicative forms of residuation-based agregations

$1 - f_{i-1}$ so we have $\gamma(A_i) \geq \oint_\gamma^{\uparrow GC}(f)$. ■

So the conjunctive expressions are more liberal than their implicative counterparts. The difference between $\oint_\gamma^{\uparrow G}$ and $\oint_\gamma^{\star G}$, as well as between $\oint_\gamma^{\star GC}(f)$ and $\oint_\gamma^{\uparrow GC}(f)$ can be extreme, as indicated on Table 1 by the cases when these expressions take values 0 or 1, which correspond to different conditions.

The cases where $\oint_\gamma^{\uparrow GC}(f) = 1$ and $\oint_\gamma^{\star GC}(f) = 0$ are the same as their counterparts for the soft thresholding integrals. The drastic nature of $\oint_\gamma^{\uparrow G}$ (resp. $\oint_\gamma^{\star G}(f)$) can be seen by the weak condition under which it vanishes (resp: it is maximal). This table sheds some light on the intuitive meanings of these aggregation operations.

- For $\oint_\gamma^{\uparrow G}(f)$ to be large, you need to have in each subset of criteria one that is satisfied at least at degree $\gamma^c(A)$. The same requirement holds for $\oint_{\gamma}^{\uparrow GC}(f)$ to be large. This requirement may look more natural than the one (first line) that ensures that Sugeno integral is high. Besides $\oint_{\gamma}^{\uparrow G}(f)$ vanishes if the local ratings are very bad on all criteria in a group of dual positive importance $\gamma^c(A)$. This condition brings the global evaluation to zero more often that the one that brings Sugeno integral to zero. Note that for this aggregation, the thresholds are determined by γ^c, because the form of the expression is implicative.

- For $\oint_{\gamma}^{\star G}(f)$ to be large, you only need to find one set of criteria where all local ratings pass the threshold $1 - \gamma(A)$ (it is low for important groups of criteria); this is much less demanding than for Sugeno integral. In contrast $\oint_{\gamma}^{\star G}(f)$ is low as soon as for all subsets of criteria, the local rating pertaining to one of them fails to pass this threshold. The same condition keeps $\oint_{\gamma}^{\star GC}(f)$ at a low value.

- $\oint_{\gamma}^{\uparrow GC}(f)$ is low whenever there is a fully important group of criteria ($\gamma^c(A) = 1$) for which no local rating is maximal, which is drastic indeed. On the other hand, $\oint_{\gamma}^{\star GC}(f)$ is large as soon as all local ratings are positive for a group of criteria with maximal importance ($\gamma(A) = 1$).

Extreme discrepancies between the implicative and conjunctive forms can be observed on very simple examples:

Example 1. • *Let us consider $\mathcal{C} = \{1, 2\}$, a capacity γ such that $\gamma(2) > 0$, and an object f such that $f_1 = 0$ and $f_2 = 1$.*
Hence $\oint_{\gamma}^{\uparrow G}(f) \leq \gamma^c(1) \to_G f_1 = 0$ and $\oint_{\gamma}^{\star G} \geq 1 - \gamma(2) \to_G (1 - f_2) = 1$. So the implicative expression judges f to be very bad because it has a very bad local rating for criterion 1 which matters since its weight is assessed using γ^c (even if $\gamma(1) = 0$). The conjunctive expression considers f very good as its local rating on criterion 2 is maximal and criterion 2 is of positive importance (according to γ).

- *Let us consider* $\mathcal{C} = \{1, 2\}$, *a capacity* γ *such that* $\gamma(2) = 0$, $\gamma(1) = 1$ *and an object* f *such that* $0 < f_1 < 1$.
  Then $\oint_{\gamma}^{\uparrow GC}(f) \leq \gamma^c(1) \to_{GC} f_1 = 0$ *and* $\oint_{\gamma}^{\star GC}(f) \geq \gamma(1) \star_{GC} f_1 = 1$.
  Here the implicative expression finds f *very bad because the local rating on a maximally important criterion is not maximal. While the conjunctive expression finds* f *excellent because the local rating on a maximally important criterion is not zero (even if both conditions are simultaneously verified here).*

The first case in the example spots the reason for the discrepancy: conjunctive and disjunctive expressions do not use the same thresholds to test the local ratings. The second case in the example uses a Boolean capacity so that it shows an extreme discordance between the implicative and conjunctive drastic criteria even in this case.

Remark 2. *In the case of Sugeno integral, written in implicative form, the condition for* $\oint_{\gamma}(f) = 1$ *reads :* $\forall B$, *if* $\gamma(B) < 1$ *then* $\exists i \in \overline{B}, f_i = 1$. *which is not obviously equivalent to the condition obtained from the conjunctive form in Table 1:* $\exists A, \gamma(A) = 1$ *and* $\forall i \in A, f_i = 1$. *Proving the equivalence requires some elaboration:*

- *From conjunctive to implicative: suppose* $\exists A, \gamma(A) = 1$ *and* $\forall i \in A, f_i = 1$. *Consider a set* B. *If* $B = A$, *the pre-condition* $\gamma(B) < 1$ *does not apply. As* γ *is a capacity, we can dispense with the case when* B *contains* A. *Then we can restrict to the case when* $\exists i \in A \setminus B \neq \emptyset$; *by construction* $f_i = 1$.

- *From implicative to conjunctive: suppose* $\forall B$, *if* $\gamma(B) < 1$ *then* $\exists i \in \overline{B}, f_i = 1$. *Let* $A = \{i : f_i = 1\}$. *This set is not empty since* $\gamma(\emptyset) = 0$. *Now it is clear that* $\gamma(A) = 1$ *as* $f_i < 1$ *whenever* $i \notin A$, *by construction.*

For Boolean capacities ($\beta(A) \in \{0, 1\}$), the conditions in Table 1 reduce to:

- $\oint_{\beta}^{\uparrow G}(f) = \oint_{\beta}^{\uparrow GC}(f) = 1$ if and only if for all A such that $\beta^c(A) = 1, \exists i \in A, f_i = 1$;

- $\oint_{\beta}^{\star G}(f) = \oint_{\beta}^{\star GC}(f) = 1$ if and only if $f_i > 0, \forall i \in A$, for some A for which $\beta(A) = 1$.

- $\oint_{\beta}^{\star G}(f) = \oint_{\beta}^{\star GC}(f) = 0$ if and only if for all A such that $\beta(A) = 1, \exists i \in A, f_i = 0$;

- $\oint_{\beta}^{\uparrow G}(f) = 0$ if and only if $\forall i \in A, f_i = 0$ for some A with $\beta^c(A) = 1$, while $\oint_{\beta}^{\uparrow GC}(f) = 0$ if and only if for some A with $\beta^c(A) = 1, \forall i \in A, f_i < 1$.

The first condition is violated in the second part of the example because $f_1 < 1$ while the second condition is satisfied because $f_1 > 0$. We shall always observe this discrepancy in this case.

It is worth noticing that Equality (17) between implicative and conjunctive forms only holds for the drastic integrals when f is the characteristic function μ_B of a subset B. Then

$$\oint_{\gamma}^{\star GC}(\mu_B) = \oint_{\gamma}^{\uparrow GC}(\mu_B) = \gamma(B).$$

However, we have the following result for soft integrals:

Proposition 6. $\oint_{\gamma}^{\uparrow G}(\mu_B) = \begin{cases} 1 \text{ if } \gamma(B) = 1 \\ 0 \text{ otherwise} \end{cases}$ *and* $\oint_{\gamma}^{\star G}(\mu_B) = \begin{cases} 1 \text{ if } \gamma(B) > 0 \\ 0 \text{ otherwise.} \end{cases}$

Proof: If $A \cap B \neq \emptyset, \exists i \in A, f_i = 1 \geq \gamma^c(A)$, and $\gamma^c(A) \to_G \vee_{i \in A} f_i = 1$; if $A \cap B = \emptyset$, then $f_i = 0, \forall i \in A$ so you need $\gamma^c(A) = 0$ to get $\gamma^c(A) \to_G \vee_{i \in A} f_i = 1$. Now the condition reads $\gamma^c(A) = 0, \forall A : A \cap B = \emptyset$. It can also read $\gamma(A^c) = 1, \forall A^c : B \subseteq A^c$; but since γ is monotonic, this is equivalent to $\gamma(B) = 1$.

For the second expression, if $A \not\subseteq B$, then $\wedge_{i \in A} f_i = 0$. Otherwise, $\wedge_{i \in A} f_i = 1 = \oint_{\gamma}^{\star G}(\mu_B)$, provided that $\gamma(A) > 0$. So $\oint_{\gamma}^{\star G}(\mu_B) = 1$ if and only if $\exists B \subseteq A : \gamma(A) > 0$. This is equivalent to $\gamma(B) > 0$ from monotonicity. ∎

In fact the above result also shows the following invariance property: given a capacity γ, define the Boolean capacity $\check{\gamma}$ such that $\check{\gamma}(A) = 1$ if $\gamma(A) = 1$ and 0 otherwise. Likewise define $\hat{\gamma}$ such that $\forall A \subseteq \mathcal{C}, \hat{\gamma}(A) = 1$ if $\gamma(A) > 0$ and 0 otherwise. Then it is easy to see that, $\forall B \in \mathcal{C}$,

$$\oint_{\gamma}^{\uparrow G}(\mu_B) = \oint_{\check{\gamma}}^{\uparrow G}(\mu_B) = \check{\gamma}(B); \quad \oint_{\gamma}^{\star G}(\mu_B) = \oint_{\hat{\gamma}}^{\star G}(\mu_B) = \hat{\gamma}(B).$$

Since $\hat{\gamma} \geq \check{\gamma}$, the above proposition actually confirms that $\oint_{\gamma}^{\uparrow G}(\mu_B) \leq \oint_{\gamma}^{\star G}(\mu_B)$, that is, the former is more demanding than the latter. We have seen by Corollary 2 that this inequality always holds for general functions f. It also confirms the lack of equality between $\oint_{\gamma}^{\star G}(f)$ and $\oint_{\gamma}^{\uparrow G}(f)$. However note that if f reduces to a crisp set B, $\oint_{\gamma}^{\star G}(\mu_B) = \hat{\gamma}(B)$, $\oint_{\gamma^c}^{\uparrow G}(\mu_B) = \check{\gamma^c}(B)$, and $\hat{\gamma}$ is conjugate to $\check{\gamma^c}$ (i.e. $(\check{\gamma^c})^c = \hat{\gamma}$): $\hat{\gamma}(B) = 1 - \check{\gamma^c}(B^c)$. The connection between and $\oint_{\gamma}^{\star G}(f)$ and $\oint_{\gamma^c}^{\uparrow G}(f)$ for general functions remains to be studied. In particular, contrary to the case of Sugeno integrals expressions of $\oint_{N}^{\star G}(f)$ and $\oint_{\Pi}^{\uparrow G}(f)$ are not obvious to simplify, while $\oint_{\Pi}^{\star G}(f)$ and $\oint_{N}^{\uparrow G}(f)$ reduce to simple weighted aggregations.

As, in general, $\oint_{\gamma}^{\uparrow G}(\mu_B)$ and $\oint_{\gamma}^{\star G}(\mu_B)$ are not equal to $\gamma(B)$, none of these"integrals" extends the capacity from Boolean to non-Boolean events. Hence, $\oint_{\gamma}^{\uparrow G}$ nor $\oint_{\gamma}^{\star G}$ is not a universal integral in the sense defined in [23]. $\oint_{\gamma}^{\uparrow GC}$ and $\oint_{\gamma}^{\star GC}$ are not universal integrals either since in general

$$\oint_{\gamma}^{\uparrow GC}(c \wedge \mu_B) \neq c \wedge \gamma(B) \text{ and } \oint_{\gamma}^{\star GC}(c \wedge \mu_B) \neq c \wedge \gamma(B)$$

Example 2. *We consider* $C = \{1, 2\}$, $L = [0, 1]$, *the capacity* γ *such that* $\gamma(\{1\}) = \gamma(\{2\}) = 0.5$ *and* $c = 0.2$. *Hence*

- $\oint_{\gamma}^{\uparrow GC}(0.2 \wedge \mu_{\{1\}}) = 0$ *and* $0.2 \wedge \oint_{\gamma}^{\uparrow GC}(\mu_{\{1\}}) = 0.2.$

- $\oint_{\gamma}^{\star GC} (0.2 \wedge \mu_{\{1\}}) = 0$ *and* $0.2 \wedge \oint_{\gamma}^{\star GC} (\mu_{\{1\}}) = 0.2.$

4.4. Residuation-based integrals as upper and lower possibilistic aggregations

It has been noticed [1, 2] that the set $\{\pi : \Pi(A) \geq \gamma(A), \forall A \subseteq \ S \ \}$ of possibility distributions whose associated possibility measures Π dominate a given capacity γ is never empty. We call this set the possibilistic core of γ [15], which, in this paper we denote by $S(\gamma)$, by similarity with game theory [30], where the core of a capacity is the (possibly empty) set of probability measures that dominate it.

There is always at least one possibility measure that dominates any capacity: the vacuous possibility measure, based on the distribution $\pi^?$ expressing ignorance, since then $\forall A \neq \emptyset \subset C, \Pi^?(A) = 1 \geq \gamma(A), \forall$ capacity γ, and $\Pi^?(\emptyset) = \gamma(\emptyset) = 0$.

Some possibility distributions in the core can be generated by permtuations of elements. Let σ be a permutation of the $n = |C|$ elements in C. The ith element of the permutation is denoted by $\sigma(i)$. Moreover let $C^i_\sigma = \{\sigma(i), \ldots, \sigma(n)\}$. Define the possibility distribution π^γ_σ as follows:

$$\forall i = 1 \ldots, n, \pi^\gamma_\sigma(\sigma(i)) = \gamma(C^i_\sigma) \tag{17}$$

There are at most $n!$ (number of permutations) such possibility distributions which are named the marginals of γ. It can be checked that the possibility distribution π^γ_σ lies in $S(\gamma)$ and that the $n!$ such possibility distributions enable γ to be reconstructed (as already pointed out by Banon [1]). More precisely,

$$\forall A \subseteq C, \gamma(A) = \bigwedge_\sigma \Pi^\gamma_\sigma(A).$$

$\forall \pi \in S(\gamma), \pi(s) \geq \pi^\gamma_\sigma(s), \forall s \in C$ for some permutation σ of C.

A possibility measure Π_1 is said to be more specific than another possibility measure Π_2 if $\forall A \subset C, \Pi_1(A) \leq \Pi_2(A)$ (equivalently $\forall s \in C, \pi_1(s) \leq \pi_2(s)$). In fact, $\pi^?$ is the unique maximal element of $S(\gamma)$ for this ordering.

In the qualitative case, $S(\gamma)$ is closed under the qualitative counterpart of a convex combination or mixture: namely, if $\pi_1, \pi_2 \in S(\gamma)$ then $\forall a, b \in L$, such that $a \vee b = 1$, it holds that $(a \wedge \pi_1) \vee (b \wedge \pi_2) \in S(\gamma)$, and $(a \wedge \Pi_1) \vee (b \wedge \Pi_2)$ is a possibility measure too[8]. In fact, $S(\gamma)$ is an upper semi-lattice. Let $\mathcal{S}_*(\gamma) = \min S(\gamma)$ be the set of minimal elements in $S(\gamma)$.

Besides, it follows from the definition of the possibilistic core that $\gamma(A) = \bigwedge_{\Pi \in S(\gamma)} \Pi(A)$, and thus any capacity can be viewed either as a lower possibility measure or as an upper necessity measure, defined on the minimal possibility distributions in the core:

Proposition 7. *[2, 15]*

$$\gamma(A) = \bigwedge_{\pi \in \mathcal{C}_*(\gamma)} \Pi(A) = \bigvee_{\pi \in \mathcal{C}_*(\gamma^c)} N(A)$$

The second result can be obtained by applying the first one to γ^c.

Note that Sugeno integral can be written as a prioritized maximum. Let π^f be the marginal of γ obtained from a permutation induced by the function f. Namely, as $f_1 \leq \ldots \leq f_n$, define $\pi_i^f = \gamma(A_i)$, where $A_i = \{i, \ldots, n\}$. Then it is clear that $\Pi^f(A_i) = \gamma(A_i)$, and Sugeno integral, in the form (15):

$$\oint_\gamma (f) = \oint_{\Pi^f} (f) = \bigvee_{i=1}^{n} f_i \wedge \pi_i^f = SLMAX_{\pi^f}(f) \tag{18}$$

Likewise, letting $\bar{\pi}_i^f = 1 - \gamma(\{i+1, \ldots, n\}) = 1 - \pi_{i+1}^f$ denote the degree of possibility of i induced by the opposite permutation, Sugeno integral after the right-hand side of (15) can be written as a prioritized minimum:

$$\oint_\gamma (f) = \oint_{N^f} (f) = \bigwedge_{i=1}^{n} f_i \vee (1 - \bar{\pi}_i^f) = SLMIN_{\bar{\pi}^f}(f). \tag{19}$$

As a consequence of this result, it was proved in [2] that Sugeno integral is a lower prioritized maximum, as well as an upper prioritized minimum:

Proposition 8. $\oint_\gamma (f) = \bigwedge_{\pi \in \mathcal{C}*(\gamma)} \oint_\Pi (f)$ *and* $\oint_\gamma (f) = \bigvee_{\pi \in \mathcal{C}_*(\gamma^c)} \oint_N (f)$, *where* $\oint_N (f) = \bigwedge_{s \in S} (1 - \pi(s)) \vee f(s)$.

Proof: Viewing γ as a lower possibility, it comes (with $f_A = \wedge_{s \in A} f(s)$):
$\oint_\gamma (f) = \bigvee_{A \subseteq S} (\wedge_{\pi \in \mathcal{C}_*(\gamma)} \Pi(A)) \wedge f_A = \bigvee_{A \subseteq S} \bigwedge_{\pi \in \mathcal{C}*(\gamma)} (\Pi(A) \wedge f_A)$
$\leq \bigwedge_{\pi \in \mathcal{C}_*(\gamma)} \bigvee_{A \subseteq S} (\Pi(A) \wedge f_A)$, hence $\oint_\gamma (f) \leq \bigwedge_{\pi \in \mathcal{C}*(\gamma)} \oint_\Pi (f)$. Conversely, let π^f

be the marginal of γ obtained from a permutation induced by the function f, which satisfies $\oint_{\gamma}(f) = \oint_{\Pi_f}(f)$ (equation (19)). As $\pi^f \in \mathcal{S}(f)$, $\oint_{\Pi_f}(f) \geq \bigwedge_{\pi\in\mathcal{C}_*(\gamma)} \mathcal{S}_{\Pi}(f)$.

Using conjugacy properties, especially Proposition 1, one can prove the second equality. ■

Note that in the numerical case, the same feature occurs, namely, lower expectations with respect to a convex probability set are sometimes Choquet integrals with respect to the capacity equal to the lower probability constructed from this probability set (for instance convex capacities, and belief functions). However, this is not true for any capacity and any convex probability set.

We obtain the same results for the residuation-based qualitative integrals.

Proposition 9. $\oint_{\gamma}^{\uparrow}(f) = \bigvee_{\pi\in\mathcal{C}_*(\gamma^c)} \oint_{N}^{\uparrow}(f)$ *and* $\oint_{\gamma}^{*}(f) = \bigwedge_{\pi\in\mathcal{C}_*(\gamma)} \oint_{\Pi}^{*}(f)$ *where* $(\rightarrow, \star) \in \{(\rightarrow_G, \star_G), (\rightarrow_{GC}, \star_{GC})\}$.

Proof: Consider $\pi \in \mathcal{C}_*(\gamma^c)$; then $\Pi(A) \rightarrow \vee_{i\in A} f_i \leq \gamma^c(A) \rightarrow \vee_{i\in A} f_i$ and $\oint_{N}^{\uparrow}(f) \leq \oint_{\gamma}^{\uparrow}(f)$. So we have $\oint_{\gamma}^{\uparrow}(f) \geq \bigvee_{\pi\in\mathcal{C}_*(\gamma^c)} \oint_{N}^{\uparrow}(f)$. Conversely we consider the possibility distribution defined by $\pi(i) = \gamma^c(\overline{A_{i+1}}) = \gamma^c(\{1, \cdots, i\})$. For all A we have $\Pi(A) = \gamma^c(\{1, \cdots, i_A\})$ where $i_A = \vee_{i\in A} i$; so $\Pi(A) \geq \gamma^c(A)$ i.e; $\pi \in \mathcal{C}_*(\gamma^c)$. Moreover $\Pi(\overline{A_{i+1}}) = \Pi(1, \cdots, i) = \gamma^c(1, \cdots, i) = \gamma^c(\overline{A_{i+1}})$. So we have $\oint_{\gamma}^{\uparrow}(f) = \oint_{N}^{\uparrow}(f)$ where N is the fuzzy measure associated to the distribution defined above. Hence $\oint_{\gamma}^{\uparrow}(f) \leq \bigvee_{\pi\in\mathcal{C}_*(\gamma^c)} \oint_{N}^{\uparrow}(f)$.

Consider $\pi \in \mathcal{C}_*(\gamma)$. We have $\Pi(A) \rightarrow \vee_{i\in A}(1 - f_i) \leq \gamma(A) \rightarrow \vee_{i\in A}(1 - f_i)$ which entails $\oint_{\gamma}^{*}(f) \leq \oint_{\Pi}^{*}(f)$. So we have $\oint_{\gamma}^{*}(f) \leq \bigwedge_{\pi\in\mathcal{C}_*(\gamma)} \oint_{\Pi}^{*}(f)$. Conversely we consider π^f the marginal of γ. Hence $\oint_{\gamma}^{*}(f) = \oint_{\Pi_f}^{*}(f) \geq$

$$\bigwedge_{\pi \in \mathcal{C}_*(\gamma)} \oint_{\Pi}^{*} (f). \qquad \blacksquare$$

5. Qualitative aggregation schemes on negative scales

In this part, the evaluation scale for each criterion is decreasing, i.e., 0 is a better score than 1, but the scale of the global evaluation is increasing. In such a context, criteria i are impediments that justify the downgrading an object and the value f_i is interpreted as a penalty level, degree of defect, intensity of rejection, according to dimension i. So f_i is all the greater as the penalty is higher with respect to the criterion i, but the global score is all the lower as the local evaluations are higher. The counterpart of the two elementary aggregations on positive scales can be handled by first reversing the negative scales and then aggregating the results as previously, or on the contrary aggregating the negative scores and reversing the global result. We obtain the following elementary qualitative schemes, where the two methods coincide:

- The demanding evaluation takes the $\wedge_{i=1}^{n}(1-f_i) = 1-\vee_{i=1}^{n} f_i$. In order to obtain a good evaluation an object needs to have a strong local rejection levels on all criteria.
- The loose global evaluation is $\vee_{i=1}^{n}(1-f_i) = 1-\wedge_{i=1}^{n} f_i$. It is enough to have a very small penalty only on a criterion to obtain a good global evaluation.

5.1. Elementary desintegrals

These two aggregation schemes can be generalised defining permissiveness or tolerance levels t_i, on the criterion i, $i = 1, \cdots, n$. On negative scales, t_i is all the greater as the criterion i is less important. A fully tolerant criterion has $t_i = 1$ (tolerating high rejection levels) and a fully intolerant criterion has $t_i = 0$. Similarly to the importance weights, these permissiveness levels can alter each local evaluation f_i in different manners. More precisely, t_i can act as a saturation threshold that blocks the global score under or above a certain value dependent on the tolerance level of criterion i. Alternatively, t_i can be considered as a threshold under which the decision-maker is perfectly satisfied, the local rating being altered or not if above the threshold. Let us present all these cases in details.

Saturation levels. The result of applying tolerance t_i to the negative rating f_i results in a positive rating that cannot be below t_i. Moreover the local rating scale is reversed, which leads to a local positive rating $(1 - f_i) \vee t_i \in [t_i, 1]$ or $(1 - t_i) \wedge (1 - f_i) \in [0, 1 - t_i]$. An intolerant criterion can affect the global evaluation.

- The corresponding demanding aggregation scheme is

  $$SLMIN_t^{neg}(f) = \wedge_{i=1}^n (1-f_i) \vee t_i = \wedge_{i=1}^n f_i \to_D t_i = \wedge_{i=1}^n (1-t_i) \to_D (1-f_i).$$

  An intolerant criterion can alone downgrade the global evaluation and a non intolerant criterion cannot bring the global evaluation under the level t_i.

- The loose aggregation is of the form

  $$SLMAX_t^{neg}(f) = \vee_{i=1}^n (1 - t_i) \wedge (1 - f_i) = \vee_{i=1}^n (1 - t_i) \star_D (1 - f_i).$$

  An intolerant criterion can alone brings a good global evaluation and a tolerant one cannot alone brings to a global evaluation above $(1 - t_i)$.

Note that $SLMIN_t^{neg}(f) = SLMIN_{1-t}(1-f)$ and likewise, $SLMAX_t^{neg}(f) = SLMAX_{1-t}(1 - f)$ so we have:

$$SLMAX_t^{neg}(f) = 1 - SLMIN_t^{neg}(1 - f) \tag{20}$$

Softening thresholds. Here t_i is viewed as a tolerance threshold such that it is enough not to reach $(f_i \leq t_i)$ (i.e. the defect rating should remain smaller than this threshold) for the impediment associated to the criterion to be totally avoided. So if f_i is lower than t_i then f_i becomes 0 otherwise we keep the local evaluation f_i. Next, the local evaluation is reversed before the agregation is performed so the local rating f_i is replaced by $(1 - t_i) \to_G (1 - f_i)$.

- The corresponding demanding aggregation has the form

  $$STMIN_t^{neg}(f) = \wedge_{i=1}^n (1 - t_i) \to_G (1 - f_i).$$

  A completely intolerant criterion is fully satisfied only if $f_i = 0$. A less intolerant criterion is satisfied only if $f_i \leq t_i$ even if f_i is high.

- The loose counterpart is

$$STMAX_t^{neg}(f) = \vee_{i=1}^{n}(1 - t_i) \star_G (1 - f_i).$$

  A tolerant criterion is taken into account only if f_i is very low and an intolerant criterion is involved in the global evaluation even if f_i is high since the condition is $f_i \neq 1$.

Note that $STMIN_t^{neg}(f) = STMIN_{1-t}(1 - f)$ and $STMAX_t^{neg}(f) = STMAX_{1-t}(1 - f)$ so we have

$$STMAX_t^{neg}(f) = 1 - STMIN_t^{neg}(1 - f). \tag{21}$$

Drastic thresholds. Here if $f_i > t_i$ then the local rating is considered bad and the result is set to t_i on the opposite scale. If $f_i \leq t_i$ then the local rating is fine and the result is on the opposite scale is 1. It corresponds again to using Gödel implication and now computing $f_i \rightarrow_G t_i = (1 - t_i) \rightarrow_{GC} (1 - f_i)$.

- The demanding aggregation is

$$DTMIN_t^{neg}(f) = \wedge_{i=1}^{n}(1 - t_i) \rightarrow_{GC} (1 - f_i).$$

  A completely intolerant negative criterion, if fulfilled, can alone can downgrade the global evaluation to 0. A less intolerant criterion can just downgrade the result to t_i.

- The loose counterpart is

$$DTMAX_t^{neg} = \vee_{i=1}^{n}(1 - t_i) \star_{GC} (1 - f_i).$$

  A complete intolerant criterion can bring the global evaluation to 1. A less intolerant criterion cannot bring the global evaluation above $1 - t_i$.

Note that $DTMIN_t^{neg}(f) = DTMIN_{1-t}(1 - f)$ and $DTMAX_t^{neg}(f) = DTMAX_{1-t}(1 - f)$ so we have

$$DTMAX_t^{neg}(f) = 1 - DTMIN_t^{neg}(1 - f). \tag{22}$$

These elementary aggregations can actually be derived from the ones defined in Section 3 for merging positive ratings. Namely, it is routine to check that the aggregation AG_t^{neg} that merges negative ratings with weight distribution t can be defined as

$$AG_t^{neg}(f) = AG_{1-t}(1 - f).$$

for $AG \in \{SLMIN, SLMAX, STMIN, STMAX, DTMIN, DTMAX\}$.

5.2. Qualitative desintegrals

We now presents the generalisation of the previous aggregation schemes that merge negative ratings into a global positive one.

Here the tolerance level is assigned to sets of criteria by means of an anti-capacity (or anti-fuzzy measure) which is a set function $\nu : 2^C \to L$ such that $\nu(\emptyset) = 1$, $\nu(C) = 0$, and if $A \subseteq B$ then $\nu(B) \leq \nu(A)$. The conjugate ν^c of an anti-capacity ν is an anti-capacity defined by $\nu^c(A) = 1 - \nu(A^c)$.

The reason why we are using an anti-capacity is that more impediments lead to downgrading the overall positive score. if B is a set of Boolean impediments and $\nu(B)$ is the overal score induced, then $\nu(B)$ should decrease when B becomes larger.

A special case of anti-capacity is the guaranteed possibility measure [10] defined by $\Delta(A) = \wedge_{i\in A} t_i$, where t is a (guaranteed) possibility distribution such that $\wedge_i t_i = 0$. In a multiple criteria perspective, t_i is the tolerance level of negative criterion i.

Qualitative desintegrals can be defined from the corresponding variants of Sugeno integral, by reversing the direction of the local value scales (f becomes $1 - f$), and by considering a capacity induced by the anti-capacity ν, as follows:

$$\oint_{\nu}^{\downarrow}(f) = \oint_{1-\nu^c}^{\uparrow}(1-f) \text{ and } \oint_{\nu}^{*\downarrow}(f) = \oint_{1-\nu^c}^{*}(1-f) \tag{23}$$

where $(\to, \star) \in \{(\to_D, \star_D), (\to_G, \star_G), (\to_{GC}, \star_{GC})\}$.

We obtain the following desintegrals:

Definition 2.

$$\oint_{\nu}^{\downarrow}(f) = \bigwedge_{A\subseteq C}(1-\nu(A)) \to \vee_{i\in A}(1-f_i) = \bigwedge_{A\subseteq C}\nu^c(\bar{A}) \to \vee_{i\in A}(1-f_i)$$

$$\oint_{\nu}^{\star\downarrow}(f) = \bigvee_{A\subseteq C}(1-\nu^c(A)) \star \wedge_{i\in A}(1-f_i) = \bigvee_{A\subseteq C}\nu(\bar{A}) \star \wedge_{i\in A}(1-f_i).$$

where $(\to, \star) \in \{(\to_D, \star_D), (\to_G, \star_G)(\to_{GC}, \star_{GC})\}$.

The drastic implication-based desintegral can also be expressed using Gödel implication as $\oint_{\nu}^{\downarrow GC}(f) = \bigwedge_{A\subseteq C}(\wedge_{i\in A} f_i) \to_G \vee_{i\in A}\nu(A)$. This kind of aggregation is already proposed by Dvořák and Holčapek [**?**], in connection with the modelling of fuzzy quantifiers. However, the underlying algebraic structure they consider is the one of MV-algebras (they use Łukasiewicz implication) hence closer to numerical representations.

In the following, we shall see that the saturation desintegrals, $\oint_{\nu}^{\downarrow_D}$, $\oint_{\nu}^{*_{\downarrow}D}$, generalise $SLMIN_t^{neg}$ and $SLMAX_t^{neg}$; the soft desintegrals, $\oint_{\nu}^{\downarrow G}$, $\oint_{\nu}^{*_{\downarrow}G}$, generalise $STMIN_t^{neg}$ and $STMAX_t^{neg}$; and the drastic desintegrals, $\oint_{\nu}^{\downarrow GC}$, $\oint_{\nu}^{*_{\downarrow}GC}$, generalise $DTMIN_t^{neg}$ and $DTMAX_t^{neg}$.

First, using the relation 24 and the property 2 of the integrals we obtain the following expected duality relation:

Proposition 10.
$\oint_{\nu}^{\star_{\downarrow}}(f) = 1 - \oint_{\nu^c}^{\downarrow}(1-f)$ *where* $(\rightarrow, \star) \in \{(\rightarrow_D, \star_D), (\rightarrow_G, \star_G), (\rightarrow_{GC}, \star_{GC})\}$.

When we consider the desintegral counterpart of Sugeno integral, we obviously observe that $\oint_{\nu}^{\downarrow D}(f) = \oint_{1-\nu^c}^{\uparrow D}(1-f) = \oint_{1-\nu^c}^{\star D}(1-f) = \oint_{\nu}^{\star_{\downarrow}D}(f)$ This equality is not true for the other desintegrals: $\oint_{\nu}^{\downarrow G} \neq \oint_{\nu}^{\star_{\downarrow}G}$ and $\oint_{\nu}^{\downarrow GC} \neq \oint_{\nu}^{\star_{\downarrow}GC}$. The failure of the corresponding equalities is not surprising since the following extreme cases can be observed:

desintegrals	**attain** 0	**attain** 1
$\oint_{\nu}^{\downarrow D}(f)$	$\exists A, \nu(A) = 0$ and $\forall i \in A, f_i = 1$	$\exists A \nu^c(A) = 0$ and $\forall i \in A, f_i = 0$
$\oint_{\nu}^{\downarrow G}(f)$	$\exists A\, \nu(A) < 1$ and $\forall i \in A, f_i = 1$	$\forall A, \exists i \in A, f_i \leq \nu(A)$
$\oint_{\nu}^{\star_{\downarrow}G}(f)$	$\forall A, \exists i \in A, f_i \geq \nu(\overline{A})$	$\exists A, \nu(\overline{A}) > 0$ and $\forall i \in A, f_i = 0$
$\oint_{\nu}^{\downarrow GC}(f)$	$\exists A,\ \nu(A) = 0$ and $\forall i \in A,\ f_i > 0$	$\forall A, \exists i \in A, f_i \leq \nu(A)$
$\oint_{\nu}^{\star_{\downarrow}GC}(f)$	$\forall A, \exists i \in A, f_i \geq \nu(\bar{A})$	$\exists A,\ \nu^c(A) = 0$ and $\forall i \in A,\ f_i < 1$

Like for integrals, the expression of the desintegrals can be reduced to expression of linear size:

Proposition 11. $\oint_{\nu}^{\downarrow D}(f) = \bigvee_{i=1}^{n}(1-f_i) \wedge \nu(A_{i+1}) = \bigwedge_{i=1}^{n}(1-f_i) \vee \nu(A_i)$.

$\oint_{\nu}^{\downarrow}(f) = \bigwedge_{i=1}^{n} \nu^c(\overline{A_i}) \to (1-f_i) \quad \oint_{\nu}^{\star\downarrow}(f) = \bigwedge_{i=1}^{n} \nu(A_{i+1}) \star (1-f_i)$;
where $(\to, \star) \in \{(\to_G, \star_G), (\to_{GC}, \star_{GC})\}$.

Proof: : Let us define $g = 1-f$. We have $g_1 = 1-f_n \leq g_2 = 1-f_{n-1} \leq \cdots \leq g_i = 1-f_{n-i+1} \leq \cdots \leq g_n = 1-f_1$ and $A_i^g = \{j | g_j \geq g_i\} = \overline{A_{n-i+2}}$. So we have the following equalities:

- $\oint_{1-\nu^c}^{\uparrow D}(1-f) = \bigvee_{i=1}^{n}(1-f_{n-i+1}) \wedge (1-\nu^c)(\overline{A_{n-i+2}}) = \bigvee_{i=1}^{n}(1-f_{n-i+1}) \wedge \nu(A_{n-i+2})$. We denote $j = n-i+1$, $\oint_{\nu}^{\downarrow D}(f) = \bigvee_{j=1}^{n}(1-f_j) \wedge \nu(A_{j+1})$.
  $\oint_{1-\nu^c}^{\uparrow D}(1-f) = \bigwedge_{i=1}^{n}(1-f_{n-i+1}) \vee (1-\nu^c)(\overline{A_{n-i+1}}) = \bigwedge_{i=1}^{n}(1-f_{n-i+1}) \wedge \nu(A_{n-i+1})$. We denote $j = n-i+1$, $\oint_{\nu}^{\downarrow D}(f) = \bigwedge_{j=1}^{n}(1-f_j) \vee \nu(A_j)$.

- For any $(\to, \star) \in \{(\to_G, \star_G), (\to_{GC}, \star_{GC})\}$ we have
  $\oint_{1-\nu^c}^{\uparrow}(1-f) = \bigwedge_{i=1}^{n}(1-\nu)(A_{n-i+1}) \to (1-f_{n-i+1}) = \bigwedge_{j=1}^{n} \nu^c(\overline{A_j}) \to (1-f_j)$.
  $\oint_{1-\nu^c}^{\star\uparrow}(1-f) = \bigvee_{i=1}^{n}(1-\nu^c)(\overline{A_{n-i+2}}) \star (1-f_{n-i+1}) = \bigvee_{j=1}^{n} \nu(A_{j+1}) \star (1-f_j)$.

■

Moreover we have the following expression for the qualitative desintegrals.

Proposition 12. $\oint_{\nu}^{\downarrow}(f) = \bigwedge_{a \in L} \nu^c(\{f < a\}) \to (1-a)$ *and*
$\oint_{\nu}^{\star\downarrow}(f) = \bigvee_{a \in L} \nu(\{f > a\}) \star (1-a)$,
where $(\to, \star) \in \{(\to_D, \star_D), (\to_G, \star_G), (\to_{GC}, \star_{GC})\}$.

Proof: $\oint_{\nu}^{\downarrow}(f) = \oint_{1-\nu^c}^{\uparrow}(1-f) = \bigwedge_{a\in L} \nu^c(\{1-f>a\}) \rightarrow a$
$= \bigwedge_{a\in L} \nu^c(\{f<1-a\}) \rightarrow a = \bigwedge_{a\in L} \nu^c(\{f<a\}) \rightarrow (1-a)$.
$\oint_{\nu}^{\star\downarrow}(f) = 1 - \oint_{\nu^c}^{\downarrow}(1-f) = 1 - \bigwedge_{a\in L} \nu(\{1-f<a\}) \rightarrow (1-a)$
$= \bigvee_{a\in L}(1-\nu(\{f>1-a\}) \rightarrow (1-a)) = \bigvee_{a\in L}(1-\nu(\{f>a\}) \rightarrow a) = \bigvee_{a\in L} \nu(\{f>a\}) \star (1-a)$. ■

We can also compare the desintegrals.

Proposition 13. $\oint_{\nu}^{\star\downarrow} \geq \oint_{\nu}^{\downarrow}$ *with* $(\rightarrow,\star) \in \{(\rightarrow_G,\star_G),(\rightarrow_{GC},\star_{GC})\}$.

Proof: $\oint_{\nu}^{\star\downarrow}(f) = \oint_{1-\nu^c}^{\star}(1-f) \geq \oint_{1-\nu^c}^{\uparrow}(1-f) = \oint_{\nu}^{\downarrow}(f)$. ■

On this basis we can establish the connection between the desintegrals and the elementary weigthed aggregation schemes on negative scales.

Proposition 14.

- $\oint_{\Delta}^{\downarrow D}(f) = SLMIN_{\delta}^{neg}(f)$, $\oint_{\Delta}^{\star\downarrow G}(f) = STMIN_{\delta}^{neg}(f)$ *and* $\oint_{\Delta}^{\star\downarrow GC}(f) = DTMIN_{\delta}^{neg}(f)$.
- $\oint_{\nabla}^{\downarrow D}(f) = SLMAX_{\delta}^{neg}(f)$, $\oint_{\nabla}^{\downarrow G}(f) = STMAX_{\delta}^{neg}(f)$ *and* $\oint_{\nabla}^{\downarrow GC}(f) = DTMAX_{\delta}^{neg}(f)$.

Proof: We use the relation between the integrals and the desintegrals; and the following remarks: $1-\Delta^c$ is a necessity measure, $1-\nabla^c$ is a possibility measure. Hence we apply the results proved for the integrals. ■

5.3. Desintegrals as upper or lower possibility desintegrals

Just as capacities have possibilistic cores, an anti-capacity ν has a possibilistic support $\mathcal{S}(\nu)$, defined by $\mathcal{S}(\nu) = \{\delta : \Delta(A) \leq \nu(A), \forall A \subseteq C\}$. For each ν, $\mathcal{S}(\nu)$ is a lower semi-lattice which is not empty since there is always at least one guaranteed possibility under any anti-measure based on the following tolerance function t expressing complete intolerance: $\forall A \neq \emptyset \subset C, t(A) = 0$ and $t(\emptyset) = \nu(\emptyset) = 1$. A result dual of Proposition 1 can then be established.

Proposition 15.

$$\nu(A) = \bigvee_{\delta \in \mathcal{S}(\nu)} \Delta(A) = \bigwedge_{\delta \in \mathcal{S}(\nu^c)} \nabla(A).$$

Proof:
According to the definition of $\mathcal{S}(\nu)$, $\bigvee_{\delta \in \mathcal{S}(\nu)} \Delta(A) \leq \nu(A)$ for all A.

Let us prove the converse. With the notations used for the capacity, we define the tolerance level $t_\nu^\sigma(i) = \nu(C_\sigma^i)$ for all i in C. Hence the associated guaranteed possibility Δ_ν^σ belongs to $\mathcal{S}(\nu)$.

Let A be a set of criteria and $\mathcal{C}_\sigma^{i_\sigma}$ be the smallest set in the sequence $\{\mathcal{C}_\sigma^i\}_i$ such that $A \subseteq \mathcal{C}_\sigma^i$. By construction we have $i_\sigma \in A$. The inclusion $A \subseteq \mathcal{C}_{i_\sigma}$ entails $\nu(A) \geq \nu(\mathcal{C}_\sigma^{i_\sigma}) = t_\nu^\sigma(i_\sigma)$. Moreover $\Delta_\nu^\sigma(A) = \wedge_{j \in A} t_\nu^\sigma(j) \leq t_\nu^\sigma(i_\sigma) \leq \nu(A)$.

When we consider a set of criteria A, there exists a permutation σ_0 such that $A = C_{\sigma_0}^i$. Hence, $\nu(A) = \nu(C_{\sigma_0}^i) = t_\nu^{\sigma_0}(i)$.

Moreover we have $\Delta_\nu^{\sigma_0}(A) = t_\nu^{\sigma_0}(i) \wedge \cdots \wedge t_\nu^{\sigma_0}(n) = \nu(C_{\sigma_0}^i) \wedge \cdots \wedge \nu(C_{\sigma_0}^n) = \nu(C_{\sigma_0}^i) = t_\nu^{\sigma_0}(i)$; so $\nu(A) = \Delta_\nu^{\sigma_0}(A)$ and $\nu(A) \leq \bigvee_{\delta \in \mathcal{S}(\nu)} \Delta(A)$.

Applying the first result to ν^c, and the relations linking Δ to ∇ yields the second expression. ■

One can restrict the scope of the minimum and that of the maximum to the maximal elements of $\mathcal{S}(\nu)$ using the following proposition.

Proposition 16. *$S(\nu) = \{\delta, \exists\sigma, \delta \leq t_\sigma^\nu\}$ where $t_\nu^\sigma(i) = \nu(C_\sigma^i)$ for all i in C.*

Proof: According to the poof of the previous proposition there exists σ_0 such that $\nu(A) = \Delta_{\sigma_0}^\nu(A)$ so $\nu(A) \leq \bigvee_\sigma \Delta_\sigma^\nu(A)$. Moreover for all permutation σ, t_ν^σ is in $S(\nu)$ i.e $\Delta_\nu^\sigma(A) \leq \nu(A)$ so $\bigvee_\sigma \Delta_\sigma^\nu(A) \leq \nu(A)$. Hence $\nu(A) = \bigvee_\sigma \Delta_\sigma^\nu(A)$ and forall $\delta \in S(\nu)$, $\exists\sigma$ such that $\delta \leq t_\nu^\sigma$. ■

Let $\mathcal{S}_*(\nu)$ be the set of maximal elements in $S(\nu)$. We have

$$\nu(A) = \bigvee_{\delta \in \mathcal{S}_*(\nu)} \Delta(A) = \bigwedge_{\delta \in \mathcal{S}_*(\nu^c)} \nabla(A). \tag{24}$$

Hence the following result can be proved:

Proposition 17. $\oint_\nu^{\downarrow}(f) = \bigvee_{\delta \in \mathcal{S}_*(\nu)} \oint_\Delta^{\downarrow}(f)$ *and* $\oint_\nu^{*\downarrow}(f) = \bigwedge_{\delta \in \mathcal{S}_*(\nu^c)} \oint_\nabla^{*\downarrow}(f)$
where $(\rightarrow, \star) \in \{(\rightarrow_G, \star_G), (\rightarrow_{GC}, \star_{GC})\}$.

Proof: $\oint_{\nu}^{\downarrow}(f) = \oint_{1-\nu^c}^{\uparrow}(1-f) = \bigvee_{\pi\in\mathcal{C}_*(1-\nu)} \oint_{N}^{\uparrow}(1-f) = \bigvee_{1-\pi\in\mathcal{S}_*(\nu)} \oint_{1-N^c}^{\downarrow}(f) = \bigvee_{\delta\in\mathcal{S}_*(\nu)} \oint_{\Delta}^{\downarrow}(f)$.

$\oint_{\nu}^{\star\downarrow}(f) = \oint_{1-\nu^c}^{\star}(1-f) = \bigwedge_{\pi\in\mathcal{C}_*(1-\nu^c)} \oint_{\Pi}^{\star}(1-f) = \bigwedge_{1-\pi\in\mathcal{S}_*(\nu^c)} \oint_{1-\Pi^c}^{\star\downarrow}(f) = \bigwedge_{\delta\in\mathcal{S}_*(\nu^c)} \oint_{\nabla}^{\star\downarrow}(f)$. ∎

5.4. Potential application

The motivation for desintegrals is decision-making based on bipolar evaluations. An alternative f is then a vector $(f_1^+, \ldots, f_n^+, f_1^-, \ldots, f_m^-)$ where the f_i^+ are ratings in a positive scale expressing the strength of the reasons for accepting f and f_i^- are ratings in a negative scale expressing the strength of the reasons for rejecting f. The overall evaluation of f is then expressed by means of qualitative integrals and desintegrals, for instance a pair $(\oint_{\gamma}^{\uparrow}(f^+), \oint_{\nu}^{\downarrow}(f^-))$. See [14] for a discussion and an example of such a bipolar evaluation process.

In order to compare two alternatives, one may either merge the positive evaluations obtained from a integral over positive criteria and a desintegral with respect to negative ones, or on the contrary handle them separately for making a final comparison of objects. Approaches like Cumulative Prospect Theory follow the first principle, but they are numerical. Approaches proposed by Grabisch [**?** 19] work with a single qualitative bipolar scale. However the merging of positive and negative values in a finite bipolar scale is problematic [19]. The other principle is more in the spirit of bivariate bipolar approaches to evaluation, leading to a partial preference order. Yet another principe for the comparison between two alternatives f and g proposed by Bonnefon and colleagues [**? ?**] is that a reason for rejecting g is viewed as a reason for prefering f. However, the approach in [**? ?**] is restricted to Boolean valuation scales (all-or-nothing positive or negative criteria) and importance levels bear on single criteria.

The framework presented in this paper opens the way to a generalization of such qualitative bipolar decision evaluation methods to criteria with more refined value scales and generalized weightings of groups of criteria. For instance, the following is an extension of a decision rule in [**?** **?**]:

$$f \succeq g \iff \oint_\gamma^{\uparrow}(f^+) \geq \oint_\gamma^{\uparrow}(g^+) \text{ and } \oint_\gamma^{\downarrow}(f^-) \geq \oint_\gamma^{\downarrow}(g^-).$$

An alternative decision rule can be (generalizing a proposal of Dubois and Fargier [**?**])

$$f \succeq g \iff \oint_\gamma^{\uparrow}(f^+) \oplus \oint_\gamma^{\downarrow}(g^-) \geq \oint_\gamma^{\uparrow}(g^+) \oplus \oint_\gamma^{\downarrow}(f^-).$$

for some suitably chosen operation $\oplus$ (for instance a maximum).

6. Conclusion

In this paper, we proposed new variants of Sugeno integral based on the Heyting algebra setting augmented with a Kleene involutive negation. These proposals were motivated by alternative ways of using weights of qualitative criteria in min- and max-based aggregations, that make intuitive sense as tolerance threshold. We have shown the strong similarity between the equivalent expressions of Sugeno integrals and the expressions of our residuation-based integrals. However, in the latter case, the implication-based and conjuntion-based expressions are not equivalent, contrary to the case of Sugeno integrals.

The next step is to find characteristic properties of residuation integrals, including the one proposed in [**?**] that uses Łukasiewicz implication.

We have also proposed counterparts of Sugeno integral and their variants, for dealing with negative local value scales, we call desintegrals where degrees measure the extent of a defect or penalty. We showed that desintegrals can be easily expressed in terms of qualitative integrals, which allow us to easily establish their equivalent forms. This work paves the way to decision methods where pros and cons (for or against an alternative) can be separately evaluated using positive and negative criteria.

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