Representing qualitative capacities
as families of possibility measures

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Abstract

This paper studies the structure of qualitative capacities, that is, monotonic set-functions, when they range on a finite totally ordered scale equipped with an order-reversing map. These set-functions correspond to general representations of uncertainty, as well as importance levels of groups of criteria in multiple-criteria decision-making. We show that any capacity or fuzzy measure ranging on a qualitative scale can be viewed both as the lower bound of a set of possibility measures and the upper bound of a set of necessity measures (a situation somewhat similar to the one of quantitative capacities with respect to imprecise probability theory). We show that any capacity is characterized by a non-empty class of possibility measures having the structure of an upper semi-lattice. The lower bounds of this class are enough to reconstruct the capacity, and the number of them is characteristic of its complexity. An algorithm is provided to compute the minimal set of possibility measures dominating a given capacity. This algorithm relies on the representation of the capacity by means of its qualitative Möbius transform, and the use of selection functions of the corresponding focal sets. We provide the connection between Sugeno integrals and lower possibility measures. We introduce a sequence of axioms generalizing the maxitivity property of possibility measures, and related to the number of possibility measures needed for this reconstruction. In the Boolean case, capacities are closely related to non-regular modal logics and their neighbourhood semantics can be described in terms of qualitative Möbius transforms.

Key words: fuzzy measures, possibility theory, modal logic

1. Introduction

A fuzzy measure [38] (or a capacity [10]) is a set-function that is monotonic under inclusion. In this paper, the capacity is said to be *qualitative* (or *q-capacity*, for short) if its range is a finite totally ordered set. It means we do not presuppose addition is available in the capacity range,

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only minimum and maximum. In such a context the connection with probability measures is lost. Consequently a number of notions, meaningful in the numerical setting, are lost as well, such as the Möbius transform [34], the conjugate, supermodularity [10] and the like. Likewise, some numerical capacities (such as convex ones, belief functions) can be viewed as encoding a convex family of probability distributions [35, 40]. This connection disappears if we give up using addition in the range of the capacity.

Yet, it is tempting to check whether counterparts of many such quantitative notions can be defined for qualitative capacities, if we replace probability measures by possibility measures. Conjugateness can be recovered if the range of the capacity is equipped with an order-reversing map. Moreover, a qualitative counterpart of a Möbius transform has been introduced by Mesiar [31] and Grabisch [25] in 1997 and further studied by Grabisch [26]. The qualitative Möbius transform can be viewed as the possibilistic counterpart to a basic probability assignment, whereby a capacity is defined with respect to the latter by a qualitative counterpart of the belief function definition, extending as well the definition of possibility measures. In fact the process of generation of belief functions, introduced by Dempster [9], was applied very early to possibility measures by Dubois and Prade [16, 17] so as to generate upper and lower possibilities and necessities. It was noticed that upper possibilities and lower necessities are still possibility and necessity measures respectively, but upper necessities and lower possibilities are not. However, as we shall see, the formal analogy between belief functions and qualitative capacities via the qualitative Möbius transform can be misleading at the interpretive level.

This situation leads to natural questions, namely, whether a qualitative capacity can be expressed in terms of a family of possibility measures, and if a qualitative Möbius transform can encode such a family. Previous recent works [11, 32] started addressing this issue, taking up a pioneering work by Banon [4]. In this paper we show that in the finite (qualitative) setting, special subsets of possibility measures play a role similar to convex sets of probability measures. We prove that any capacity can be defined in terms of a finite set of possibility measures, either as a lower possibility or an upper necessity function. This result should not come as a surprise. Indeed, it has been shown that possibility measures can be refined by probability measures using a lexicographic refinement of the basic axiom of possibility measures, and that capacities on a finite set can be refined by belief functions [13, 14]. Based on this fundamental result we can generalize the maxitivity and minitivity axiom of possibility theory so as to define families of qualitative capacities of increasing complexity. Finally, this property enables qualitative capacities to be seen as necessity modalities in a non-regular class of modal logics, extending the links between possibility theory and modal logic to a potentially larger range of uncertainty theories.

The structure of the paper is as follows[1]. The next section 2 provides basic definitions pertaining to capacities, recalls and discusses the similarity between belief functions and qualitative capacities, indicating the limitation of this analogy. Section 3 provides the main contribution of this paper, namely it shows the formal analogy between qualitative capacities and imprecise probabilities, proving any capacity comes down to any of two families of possibility distributions, and can be described by finite sets thereof, either as a lower possibility or an upper necessity function. This section also extends these results to Sugeno integrals. Section 4 provides an algorithm that computes the set of minimal elements among possibility measures that dominate a capacity from its qualitative Möbius transforms. Section 5 axiomatically defines subfamilies of qualitative capacities of increasing

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[1]This paper is based on and extends two previous conference papers [11, 21].

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complexity generalizing the maxitive and the minitivity axioms of possibility theory. Finally, in Section 6 we lay bare a connection between capacities and neighbourhood semantics in non-regular modal logics, which suggests potential applications to reasoning from conficting information coming from several sources.

2. Qualitative capacities and Möbius transforms

Consider a finite set S and a finite totally ordered scale $L = \{\lambda_0 = 0 < \lambda_1 < \cdots < \lambda_\ell = 1\}$ with top 1 and bottom 0. Moreover we assume that L is equipped with an order-reversing map, i.e., a strictly decreasing mapping $\nu : L \to L$ with $\nu(1) = 0$ and $\nu(0) = 1$. Note that ν is unique, and such that $\nu(\lambda_i) = \lambda_{\ell-i}$.

Definition 1. *A capacity (or fuzzy measure) is a mapping $\gamma : 2^S \to L$ such that $\gamma(\emptyset) = 0$; $\gamma(S) = 1$; and if $A \subseteq B$ then $\gamma(A) \leq \gamma(B)$. The conjugate of γ is the capacity γ^c defined as $\gamma^c(A) = \nu(\gamma(A^c)), \forall A \subseteq S$, where A^c is the complement of set A.*

The value $\gamma(A)$ can be interpreted as the degree of confidence in a proposition represented by the set A of possible states of the world, or, if S is a set of criteria, the degree of importance of the group of criteria A [27]. In this paper we basically use the first interpretation, unless specified otherwise. We here speak of *qualitative capacity* (or q-capacity, for short) to mean that we only rely on an ordinal structure, not an additive underlying structure.

Remark 1. *In fact, even if the scale is encoded by means of numbers in $[0, 1]$, we do not assume these figures represent orders of magnitude, so that their addition or substraction make no sense. Of course, we could construct a q-capacity using a probability measure P on S, and considering $\{P(A) : A \subseteq S\}$ as the totally ordered set L by renaming the numbers using symbols λ_i. It is always possible to do so using any numerical capacity on S. However, since the symbols λ_i only encode a ranking, we then are unable to distinguish between capacities that yield the same ordering of events (in particular the probability measure P can no longer be distinguished from the many non-additive numerical capacities that yield the same ordering of events as P). So* qualitative *here presupposes that the "distance" between two consecutive λ_i's can be arbitrary. And in our view, a numerical set-function is a special case of a qualitative one with additional structure in its range.*

Important special cases of capacity are possibility and necessity measures.

Definition 2. *A possibility measure is a capacity that satisfies the characteristic axiom of* maxitivity*:*

$$\gamma(A \cup B) = \max(\gamma(A), \gamma(B)).$$

A possibility measure is usually denoted by Π. In possibility theory [18, 20], the available information is represented by means of a possibility distribution [42]. This is a function, usually denoted by π, from the universe of discourse S to the scale L, such that $\pi(s) = 1$ for some element $s \in S$. The associated possibility measure is defined by

$$\Pi(A) = \max_{s \in A} \pi(s)$$

and in the finite case any possibility measure induces a unique possibility distribution $\pi(s) = \Pi(\{s\})$. The function π is supposed to rank-order potential values of (some aspect of) the state of the world

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- according to their plausibility. The value $\pi(s)$ is understood as the possibility that s be the actual state of the world. Precise information corresponds to the situation where $\exists s^*, \pi(s^*) = 1$, and $\forall s \neq s^*, \pi(s) = 0$, while complete ignorance is represented by the *vacuous possibility distribution* denoted by $\pi^?$, such that $\forall s \in S, \pi^?(s) = 1$. A possibility distribution π is said to be *more specific* (in the wide sense) than another possibility distribution ρ if $\forall s \in S, \pi(s) \leq \rho(s)$. It defines a partial order between possibility distributions, reflecting their relative informativeness.

Definition 3. *A necessity measure is a capacity that verifies the characteristic axiom of* minitivity*:*

$$\gamma(A \cap B) = \min(\gamma(A), \gamma(B)).$$

A necessity measure is usually denoted by N. Function N is characterized by an "impossibility" distribution $\iota : S \to L$ as $\iota(s) = N(S \setminus \{s\})$, since $N(A) = \min_{s \notin A} \iota(s)$. Note that the conjugate $N(A) = \Pi^c(A) = \nu(\Pi(A^c))$ of a possibility measure is a necessity measure such that $\iota(s) = \nu(\pi(s))$, which explains the interpretation of $\iota(s)$ as a degree of impossibility. It is also called a degree of potential surprize by Shackle [36], or a grade of disbelief by Spohn [37] (who use numerical representations thereof).

   Possibility distributions seem to play in the qualitative setting a role similar to probability distributions in the numerical setting, replacing sum by maximum. In this scope, it is well-known that, in the finite case, the probability distribution of a probability measure is generalized by the Möbius transform of the capacity. In this section we recall qualitative Möbius transforms that were first proposed by Grabisch [25] and Mesiar [31], and discuss the analogy it suggests between belief functions and qualitative capacities.

2.1. Qualitative Möbius transforms

Definition 4. *The inner (qualitative) Möbius transform of a capacity γ is a mapping $\gamma_\# : 2^S \to L$ defined by*

$$\gamma_\#(E) = \begin{cases} \gamma(E) & \text{if } \gamma(E) > \max_{B \subsetneq E} \gamma(B) \\ 0 & \text{otherwise.} \end{cases}$$

A set E such that $\gamma_\#(E) > 0$ is called a focal set.

In the above definition, due to the monotonicity property of γ, the condition $\gamma(E) > \max_{B \subsetneq E} \gamma(B)$ can be replaced by $\gamma(E) > \max_{x \in E} \gamma(E \setminus \{x\})$. It is easy to check that

- $\gamma_\#(\emptyset) = 0$; $\max_{A \subseteq S} \gamma_\#(A) = 1$;

- If $A \subset B$, and $\gamma_\#(B) > 0$, then $\gamma_\#(A) < \gamma_\#(B)$.

Let $\mathcal{F}(\gamma) = \{E, \gamma_\#(E) > 0\}$ be the family of focal sets associated to γ. The last property says that the inner qualitative Möbius transform of γ is strictly monotonic with respect to inclusion of focal sets.

Example 1. *Let $S = \{s_1, s_2, s_3\}$ and $\gamma(\{s_1\}) = \gamma(\{s_1, s_3\}) = 0.3$, $\gamma(\{s_1, s_2\}) = 0.7$, $\gamma(\{s_2, s_3\}) = \gamma(\{s_1, s_2, s_3\}) = 1$ (so L is chosen arbitrarily as $\{0, 0.3, 0.7, 1\}$), and $\gamma(A) = 0$ otherwise. Then $\mathcal{F}(\gamma) = \{\{s_1\}, \{s_1, s_2\}, \{s_2, s_3\}\}$, with $\gamma_\#(\{s_1\}) = 0.3$, $\gamma_\#(\{s_1, s_2\}) = 0.7$ and $\gamma_\#(\{s_2, s_3\}) = 1$.* $\square$

It is clear that the inner qualitative Möbius transform of a possibility measure coincides with its possibility distribution: $\Pi_\#(A) = \pi(s)$ if $A = \{s\}$ and 0 otherwise. Hence the set of focal sets $\mathcal{F}(\Pi)$ is made of singletons only. This property makes it clear that $\gamma_\#$ generalizes the notion of possibility distribution to the power set of S.

Just like in the numerical setting, the focal sets associated to a necessity measure N form a collection of nested sets $E_1 \subset E_2 \subset \cdots \subset E_k$ such that $N(A) = \max_{E_i \subseteq A} N_\#(E_i)$. If the necessity measure is based on a possibility distribution π such that $N(A) = \min_{s \notin A} \nu(\pi(s))$, where ν is the order-reversing map on L, then $\mathcal{F}(N) = \{E_i = \{s : \pi(s) \geq \lambda_i\} : \lambda_i \in L \setminus \{0\}\}$, and $N_\#(E_i) = \nu(\lambda_{i-1})$. A set of the form $\{s : \pi(s) \geq \lambda_i > 0\}$ is called a *cut* of π. So the focal sets of N are the cuts of the possibility distribution of its conjugate.

More generally, the inner (qualitative) Möbius transform contains the minimal information needed to reconstruct the capacity γ since, by construction [25, 13]:

$$\gamma(A) = \max_{E \subseteq A} \gamma_\#(E). \tag{1}$$

The reader can check that if one of the values $\gamma_\#(E)$ is changed, the corresponding capacity will be different, namely the values $\gamma(A)$ such that $\gamma(A) = \gamma_\#(E)$.

Remark 2. *Another qualitative counterpart of Möbius transforms, named* outer qualitative Möbius transform, *can be used to represent a capacity γ. It is a mapping $\gamma^\# : 2^S \to L$ defined by*

$$\gamma^\#(A) = \begin{cases} \gamma(A) & \text{if } \gamma(A) < \min_{A \subset F} \gamma(F) \\ 1 & \text{otherwise.} \end{cases} \tag{2}$$

In particular, $\gamma^\#(S) = 1$. Due to the monotonicity of γ, the condition $\gamma(A) < \min_{A \subset F} \gamma(F)$ can be equivalently replaced by $\gamma(A) < \min_{s \notin A} \gamma(A \cup \{s\})$.

The original capacity is then retrieved as [14]:

$$\gamma(A) = \min_{A \subseteq F} \gamma^\#(F). \tag{3}$$

Note that $\gamma(A)$ is recovered from $\gamma^\#$ via weights assigned to supersets of set A, which reminds of outer measures, while $\gamma_\#$ can be called an inner qualitative mass function.

It is worth noticing that the inner qualitative Möbius transform $\gamma_\#^c$ of γ^c is related to the outer qualitative Möbius transform $\gamma^\#$ [13, 14]:

$$\gamma^\#(E) = \nu(\gamma_\#^c(E^c)) \tag{4}$$

since $\gamma(A) < \min_{A \subset F} \gamma(F)$ also writes $\gamma^c(A^c) > \max_{F^c \subset A^c} \gamma^c(F^c)$. So if we use capacities and their conjugates, using the inner and the outer qualitative Möbius transforms conjointly is redundant. In this paper, we only use inner ones.

Example 1 (continued) *In the previous example, it can be checked that $\gamma^\#(\{s_2\}) = \gamma^\#(\{s_3\}) = 0$, $\gamma^\#(\{s_1, s_2\}) = 0.7$, $\gamma^\#(\{s_1, s_3\}) = 0.3$ (and $\gamma^\#(E) = 1$ otherwise).* □

2.2. May qualitative capacities be interpreted as qualitative belief functions ?

The similarity between capacities and belief functions is striking on Equation (1). A belief function [34] is a set function $2^S \to [0, 1]$ defined by

$$Bel(A) = \sum_{E : E \subseteq A} m(E), \forall A \subseteq S \tag{5}$$

where m is a basic probability assignment, i.e. a probability distribution over $2^S \setminus \{\emptyset\}$. The degree $Bel(A)$ is a degree of belief or certainty in the sense that $Bel(A) = 1$ means A is certain, while $Bel(A) = 0$ means A is at best unknown. The conjugate set function

$$Pl(A) = 1 - Bel(A^c) = \sum_{E : E \cap A \neq \emptyset} m(E), \forall A \subseteq S \tag{6}$$

expresses the idea of possibility or plausibility, as $Pl(A) = 0$ means that A is impossible. The mass assignment m can be reconstructed from Bel by inverting the system of equations defining the latter: this is the Möbius transform [34]. In Eq. (1), max replaces the sum in the expression of Bel.

Hence function $\gamma_\#$ stands as the qualitative counterpart to a basic probability assignment in the theory of evidence, obtained via a kind of Möbius transform [26]. By analogy with the numerical case, $\gamma_\#$ can be called a *basic possibility assignment* [32] associated with γ since $\gamma_\#$ is a possibility distribution on the power set of S. The subsets E that receive a positive support play the same role as the focal sets in Dempster-Shafer theory: they are the primitive items of knowledge. The parallel between possibility and probability is confirmed: the focal sets of a qualitative possibility measure in the sense of the inner Möbius mass function are singletons, just as the focal sets of a probability measure for the standard Möbius transform are singletons.

However there are differences between numerical and qualitative Möbius transforms. While an inner qualitative Möbius transform can be viewed as a possibility distribution on the power set of S, the converse is not true. Indeed, a possibility distribution π_m on a space 2^S is not requested to satisfy the monotonicity constraint of a qualitative Möbius transform. It means that several basic possibility assignments may yield the same capacity $\gamma(A) = \max_{E \subseteq A} \pi_m(E)$, in contrast with the setting of evidence theory, where there is a one-to-one correspondence between belief functions and basic probability assignments.

Remark 3. *If π_m is a possibility distribution over 2^S, monotonicity can be restored by deleting the terms $\pi_m(F)$ whenever $\exists E, E \subset F, \pi_m(E) > \pi_m(F)$. Clearly, if for some $F \subseteq A$, there is $E_1 \subset F, \pi_m(E_1) > \pi_m(F)$, then $\gamma(A) = \max_{E \subseteq A, E \neq F} \pi_m(E)$. So we define a unique qualitative Möbius mass from π_m as follows: $\gamma_\#(E) = \pi_m(E) > 0$ if and only if $\forall F \subset E, \pi_m(E) > \pi_m(F)$, otherwise $\gamma_\#(E) = 0$. The inner qualitative Möbius transform is the canonical (monotonic) possibility assignment associated with the capacity in the sense that all other possibility assignments contain redundant information with respect to reconstructing the capacity and only $\gamma_\#$ can be recovered in a non-ambiguous way from γ.*

It is also possible to introduce the counterpart of the *contour function* [34] [2] of a belief function (it is $\mu : S \to [0, 1]$ defined by $\mu(s) = \sum_{E \ni s} m(E)$). Namely,

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[2]also called *one-point coverage function* in the literature of random sets.

Definition 5. *The* qualitative contour function *of a capacity γ is the possibility distribution*

$$\pi_\gamma(s) = \max_{E \ni s} \gamma_\#(E).$$

Note that it is a possibility distribution since $\pi_\gamma(s) = 1$ for some $s \in S$. Finally, the qualitative counterpart of a plausibility function, in conformity with Eq. (6) is:

$$\Pi_\gamma(A) = \max_{A \cap E \neq \emptyset} \gamma_\#(E).$$

It is easy to see that Π_γ is just a possibility measure such that $\Pi_\gamma(A) \geq \gamma(A), \forall A$. In contrast, in the numerical setting, replacing max by a sum, and using a mass function, one gets a plausibility measure, i.e. a special case of upper probability function more general than a possibility measure and a probability measure. It can be proved that the possibility distribution associated with Π_γ is precisely the qualitative contour function:

Proposition 1. $\forall A, \Pi_\gamma(A) = \max_{s \in A} \pi_\gamma(s)$

  Proof: To see it just note that $\{E : A \cap E \neq \emptyset\} = \{E, \exists s \in A \cap E\}$. So $\Pi_\gamma(A) = \max_{s \in A} \max_{E \ni s} \gamma_\#(E)$. ∎

  In the numerical setting, we do not have that $Pl(A) = \max_{s \in A} \mu(s)$ (nor have we $Pl(A) = \sum_{s \in A} \mu(s)$). The meaning of the qualitative contour function as a special upper possibilistic approximation of γ will be laid bare at the end of Section 4.

Example 1 (continued) For the capacity in Example 1, the reader can check that $\pi_\gamma(s_1) = \max(\gamma_\#(\{s_1\}), \gamma_\#(\{s_1, s_2\})) = 0.7$, $\pi_\gamma(s_2) = \max(\gamma_\#(\{s_2, s_3\}), \gamma_\#(\{s_1, s_2\})) = 1$, and $\pi_\gamma(s_3) = \gamma_\#(\{s_2, s_3\}) = 1$. □

  In the construction of Dempster [9], a belief function is actually a special case of a lower probability bound induced by a probability space (Ω, P) and a multi-valued mapping, $\Gamma : \Omega \to 2^S$ as $Bel(A) = P(\{\omega \in \Omega : \Gamma(\omega) \subseteq A\})$. Considering the selection functions f of Γ (i.e., $\omega \in \Omega, f(\omega) \in \Gamma(\omega)$) and the probability functions P_f induced by (Ω, P) on S as $P_f(A) = P(f^{-1}(A))$, it holds (in the finite case) that $Bel(A) = \min_{f \in \Gamma} P_f(A)$ [8].

  In contrast, in the qualitative case, *any* capacity can be generated by this process, replacing probability by possibility. The possibilistic counterpart of Dempster process has been studied at length by Dubois and Prade [17] and more recently by De Baets and Tsiporkova [39]. Given a possibility distribution π on Ω, we can thus define an upper possibility function $\overline{\overline{\Pi}} = \max_{f \in \Gamma} \Pi_f(A)$ and a lower possibility function $\underline{\Pi} = \min_{f \in \Gamma} \Pi_f(A)$ where $\Pi_f(A) = \Pi(f^{-1}(A))$ and Π is the possibility measure associated to a possibility distribution $\pi : \Omega \to [0, 1]$.

  Since any capacity can be expressed by a monotonic possibility assignment, it can always be viewed as a lower possibility, as we shall see in the next section. Indeed if we choose $\Omega = \mathcal{F}$, the set of focal sets of γ, and let $\pi = \gamma_\#$, then $\gamma(A) = \min_{f \in \Gamma} \Pi_f(A) = \underline{\Pi}$, while $\overline{\overline{\Pi}} = \Pi_\gamma$. As a consequence, $\underline{\Pi}$ and $\overline{\overline{\Pi}}$ are *not* conjugate set-functions, contrary to the case of belief functions. Especially, we cannot use the conjugate function $\overline{\overline{\Pi}}^c$, that is, the necessity measure induced by the contour function of γ as a lower bound for the latter. Indeed

$$N_\gamma(A) = \nu(\Pi_\gamma(A^c)) = \min_{s \notin A} \nu(\max_{E \ni s} \gamma_\#(E)) = \min_{E \cap A^c \neq \emptyset} \nu(\gamma_\#(E)),$$

which cannot be compared with $\gamma(A)$.

Summarising, the situation of qualitative capacities with respect to possibility measures is different from the one of numerical capacities with respect to probabilities in several respects:

- Any qualitative capacity is a lower possibility, while not any numerical capacity is a lower probability. Some are upper probabilities only (for instance the vacuous possibility function $\Pi^?$) and some are neither upper nor lower probability bounds. There is no probability function acting as the counterpart of the vacuous possibility function. The interpretation of a belief function as measuring belief or certainty,[3] as opposed to plausibility, does not carry over to qualitative capacities, even if the definition of a capacity via $\gamma_\#$ bears a formal similarity with belief functions, and capacities can be generated as lower bounds of possibility functions.

- In particular, since the set-function $\gamma_\#$ can also be viewed as a possibility distribution over the power set 2^S, expression (1) is also a generalization of the definition of the degree of possibility of a set in terms of a possibility distribution on S as pointed out earlier.

- The one-to-one correspondence between random sets (using Möbius masses m) and belief functions does not carry over to qualitative capacities: the set of possibility distributions on 2^S can be partitioned into equivalence classes, one per capacity γ, each containing the inner qualitative Möbius transform of some capacity, and all possibility distributions in the class generating the same γ. It is clear that $\gamma_\#$ is the least element (eventwisely) of the equivalence class attached to γ.

- The qualitative counterpart to a plausibility function is a possibility function that dominates the capacity, and its possibility distribution is the qualitative contour function of the capacity.

Due to this state of facts, there is a big mathematical and semantic difference between belief functions and qualitative capacities.

Remark 4. *Wong et al. [41] have shown that, in order to be representable by a numerical belief function, a qualitative capacity must satisfy the axiom*

BEL*: $\forall A, B, C$ disjoint, if $\gamma(A \cup B) > \gamma(A)$ then $\gamma(A \cup B \cup C) > \gamma(A \cup C)$.*

This is a weakening of the well-known De Finetti axiom of comparative probability:

If $\forall A \cap (B \cup C) = \emptyset$ then, $\gamma(B) \geq \gamma(C)$ if and only if $\gamma(A \cup B) > \gamma(A \cup C)$.

which is known to be insufficient to characterize numerical probability on finite sets [24]. Curiously, even if a capacity γ fails to satisfy axiom **BEL***, it almost satisfies it in the sense that the following property always holds [14]:*

$\forall A, B, C$ disjoint, if $\gamma(A \cup B) > \gamma(A)$ then $\gamma(A \cup B \cup C) \geq \gamma(A \cup C)$.

However, using contraposition, the following dual axiom, another specialisation of the latter property (distinct from BEL):

PL*: $\forall A, B, C$ disjoint, if $\gamma(A \cup B \cup C) > \gamma(A \cup C)$ then $\gamma(A \cup B) > \gamma(A)$*

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[3]$Bel(A) = 1$ means that A is totally certain.

makes γ representable by a plausibility function. It means that the weak ordering of events induced by any capacity can be refined by a weak ordering representable by a belief function or by a plausibility function [14], which confirms that the formal analogy between capacities and belief functions proper is misleading.

3. Capacities as lower possibility and upper necessity functions

It is well-known that a belief function can be equivalently represented by a non-empty convex set of probabilities, namely $\mathcal{P}(Bel) = \{P, P(A) \geq Bel(A), \forall A \subseteq S\}$. Then it holds that $Bel(A) = \min_{P \in \mathcal{P}(Bel)} P(A)$. In other words a belief function is one example of coherent lower probability in the sense of Walley [40] (exact capacity after Schmeidler [35]). The set $\mathcal{P}(Bel)$ is called the credal set representing Bel [4]. If the range of the set-function is not equipped with addition and product, this construction is impossible. The natural question is then whether a similar construction may make sense with qualitative possibility measures in place of probability measures, using min and max instead of product and sum. From previous preliminary results [17, 39, 4] we know the answer is positive. In this paper, we investigate the problem in depth, without resorting to Dempster setting.

3.1. Capacities as lower possibilities

There is always at least one possibility measure that dominates any capacity: the vacuous possibility measure, based on the distribution $\pi^?$ expressing ignorance, since for any capacity γ, $\forall A \neq \emptyset \subset S, \Pi^?(A) = 1 \geq \gamma(A)$, and $\Pi^?(\emptyset) = \gamma(\emptyset) = 0$.

Let $\mathcal{R}(\gamma) = \{\pi : \Pi(A) \geq \gamma(A), \forall A \subseteq S\}$ be the non-empty set of possibility distributions whose corresponding set-functions Π dominate γ. In analogy to the corresponding notion in game theory [35], we call $\mathcal{R}(\gamma)$ the *possibilistic core* of the capacity γ. Clearly, $\mathcal{R}(\gamma)$, equipped with the partial order of specificity (introduced at the beginning of Section 2), is an upper semi-lattice, that is if $\pi_1, \pi_2 \in \mathcal{R}(\gamma)$, then $\max(\pi_1, \pi_2) \in \mathcal{R}(\gamma)$. Of course, the maximal element of $\mathcal{R}(\gamma)$ is the vacuous possibility distribution $\pi^?$, but it has several minimal elements. We denote by $\mathcal{R}_*(\gamma)$ the set of minimal elements in $\mathcal{R}(\gamma)$.

Any capacity γ can be reconstructed using possibility measures induced by permutations. Let us present this result in details. Let σ be a permutation of the $n = |S|$ elements in S. The ith element of the permutation is denoted by $s_{\sigma(i)}$. Moreover let $S_\sigma^i = \{s_{\sigma(i)}, \ldots, s_{\sigma(n)}\}$. Define the possibility distribution π_σ^γ as follows:

$$\forall i = 1, \ldots, n, \pi_\sigma^\gamma(s_{\sigma(i)}) = \gamma(S_\sigma^i) \tag{7}$$

There are at most $n!$ (number of permutations) such possibility distributions, that are called the *marginals* of γ.

Example 1 (continued) If we take the permutation $\sigma = (1, 2, 3)$, we get $\pi_\sigma^\gamma(s_3) = \gamma(\{s_3\}) = 0$, $\pi_\sigma^\gamma(s_2) = \gamma(\{s_2, s_3\}) = 1$, $\pi_\sigma^\gamma(s_1) = \gamma(S) = 1$.
If we take the permutation $\tau = (3, 2, 1)$, we get $\pi_\tau^\gamma(s_1) = \gamma(\{s_1\}) = 0.3$, $\pi_\tau^\gamma(s_2) = \gamma(\{s_1, s_2\}) = 0.7$, $\pi_\tau^\gamma(s_3) = \gamma(S) = 1$. □

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[4]In the case of a general numerical capacity g, the probability set set $\mathcal{P}(g)$ may be empty.

This is similar to the case of a numerical capacity g from which a set of probability distributions p_σ^g of the form $p_\sigma^g(s_{\sigma(i)}) = g(S_\sigma^i) - g(S_\sigma^{i+1})$ can be extracted. If g is a belief function,[5] the convex hull of these probabilities coincide with the (non-empty) credal set $\mathcal{P}(g)$.

Similarly, it can be checked that the marginals of γ lie in $\mathcal{R}(\gamma)$ and enable γ to be reconstructed (already in [4]):

Lemma 1. $\Pi_\sigma^\gamma(A) \geq \gamma(A), \forall A \subseteq S$.

Proof: Consider a given permutation σ and an event A. Let $i_A = \max\{i, A \subseteq S_\sigma^i\}$. Then $S_\sigma^{i_A}$ is the smallest set in the sequence $\{S_\sigma^i\}_i$ that contains A and $\Pi_\sigma^\gamma(A) = \Pi_\sigma^\gamma(S_\sigma^{i_A}) = \pi_\sigma^\gamma(s_{\sigma(i)})$ since A contains $s_{\sigma(i_A)}$ and other elements in A are of the form $s_{\sigma(j)}, j \geq i_A$ by construction. Hence, by construction, $\Pi_\sigma^\gamma(A) = \gamma(S_\sigma^{i_A}) \geq \gamma(A)$. ∎

Proposition 2. $\forall A \subseteq S, \gamma(A) = \min_\sigma \Pi_\sigma^\gamma(A)$.

Proof: From the Lemma, $\gamma(A) \leq \min_\sigma \Pi_\sigma^\gamma(A)$. Conversely, $\forall A \subseteq S$, there is a permutation σ_A such that $A = S_{\sigma_A}^i$. By construction $\gamma(A) = \Pi_{\sigma_A}^\gamma(A) \geq \min_\sigma \Pi_\sigma^\gamma(A)$. ∎

As consequences, we have the following results:

Proposition 3. *For all $A \subseteq S$, $\gamma(A) = \min_{\pi \in \mathcal{R}(\gamma)} \Pi(A)$.*

Proof: For all $\pi \in \mathcal{R}(\gamma)$, $\Pi \geq \gamma$ so $\gamma(A) \leq \min_{\pi \in \mathcal{R}(\gamma)} \Pi(A)$. Moreover there exists σ such that $\gamma(A) = \Pi_\sigma^\gamma(A) \geq \min_{\pi \in \mathcal{R}(\gamma)} \Pi(A)$, and, by Lemma 1, we know that $\pi_\sigma^\gamma \in \mathcal{R}(\gamma)$. ∎

Proposition 4. $\forall \pi \in \mathcal{R}(\gamma), \pi(s) \geq \pi_\sigma^\gamma(s), \forall s \in S$ *for some permutation σ of S.*

Proof: Just consider a permutation σ induced by π, that is $\sigma(i) \geq \sigma(j) \iff \pi(s_i) \leq \pi(s_j)$. For this permutation, $\Pi(S_\sigma^i) = \pi(s_i) \geq \gamma(S_\sigma^i) = \pi_\sigma^\gamma(s_i), \forall i = 1, \ldots, n$. ∎

This result says that the set of marginals π_σ^γ includes the set $\mathcal{R}_*(\gamma)$ of least elements of $\mathcal{R}(\gamma)$, i.e., the most specific possibility distributions dominating γ. In other terms, $\mathcal{R}(\gamma) = \{\pi : \exists \sigma, \pi \geq \pi_\sigma^\gamma\}$. Not all the $n!$ possibility distributions π_σ^γ are least elements of $\mathcal{R}(\gamma)$. As a trivial example, if $\gamma = \Pi$, this least element is unique and is precisely π. But other permutations yield other less specific possibility distributions. A more efficient method for generating $\mathcal{R}_*(\gamma)$ is provided in Section 4.

By construction $\mathcal{R}_*(\gamma)$ is a finite set of possibility distributions none of which is more specific that another, that is, if $\pi, \rho \in \mathcal{R}_*(\gamma), \exists s_1 \neq s_2 \in S, \pi(s_1) > \rho(s_1)$ and $\pi(s_2) < \rho(s_2)$. We thus can prove the following basic result:

Proposition 5. $\forall A \subseteq S, \gamma(A) = \min_{\pi \in \mathcal{R}_*(\gamma)} \Pi(A)$.

Proof: As $\forall \pi \in \mathcal{R}_*(\gamma), \Pi \geq \gamma$, it holds that $\gamma(A) \leq \min_{\pi \in \mathcal{R}_*(\gamma)} \Pi(A)$. Now suppose $\exists A, \gamma(A) < \min_{\pi \in \mathcal{R}_*(\gamma)} \Pi(A)$. Then $\forall \pi \in \mathcal{R}_*(\gamma), \Pi(A) > \gamma(A)$. However there is $\pi' \in \mathcal{R}(\gamma) \setminus \mathcal{R}_*(\gamma)$ such that $\Pi'(A) = \gamma(A) < \Pi(A)$. Hence π' is either more specific than some $\pi \in \mathcal{R}_*(\gamma)$ (which is impossible since $\mathcal{R}_*(\gamma)$ contains the most specific elements in $\mathcal{R}(\gamma)$) or incomparable with all $\pi \in \mathcal{R}_*(\gamma)$, which would mean inside it, which is contrary to the assumption. ∎

Example 1 (continued) For instance, the three least specific possibility distributions that dominate γ in Example 1 are given in Table 1. They are not comparable and one can check that

Table 1: Least elements of the possibilistic core $(\mathcal{R}_*(\gamma))$ for Example 1

S	s_1	s_2	s_3
π_1	0.3	1	0
π_2	0.7	0	1
π_3	0.3	0.7	1

$$\gamma(A) = \min(\Pi_1(A), \Pi_2(A), \Pi_3(A)). \qquad\qquad \square$$

These findings also show that *any* capacity can be represented by a finite set of possibility measures. Conversely, for any set $\mathcal{T}$ of possibility distributions, the set-function $\gamma(A) = \min_{\pi \in \mathcal{T}} \Pi(A)$ is a capacity, and it is easy to see that $\mathcal{T} \subseteq \mathcal{R}(\gamma)$. If $\mathcal{T}$ only contains possibility distributions that are not comparable with respect to specificity, $\mathcal{T} = \mathcal{R}_*(\gamma)$, the most specific elements of $\mathcal{R}(\gamma)$.

Remark 5. *Note that the set function $\max_{\pi \in \mathcal{T}} \Pi(A)$ is not only a capacity, but also a possibility measure with possibility distribution $\pi^{\max}(s) = \max_{\pi \in \mathcal{T}} \pi(s)$ [19].*

It is interesting to consider the closure of $\mathcal{R}_*(\gamma)$ under the qualitative counterpart of a convex combination [15]:

Definition 6. *If $\pi_1, \ldots, \pi_k$ are possibility distributions and $\forall \alpha_1, \ldots, \alpha_k \in L$, such that $\max_{i=1}^{k} \alpha_i = 1$, the qualitative mixture of the π_i's is $\max_{i=1}^{k} \min(\alpha_i, \pi_i)$. The q-convex closure $\mathcal{C}(\mathcal{T})$ of the set $\mathcal{T}$ of possibility distributions is the set of qualitative mixtures of elements of $\mathcal{T}$.*

Note that the qualitative mixture of possibility distributions is a possibility distribution (it is normalized) and $\max_{i=1}^{k} \min(\alpha_i, \Pi_i)$ is a possibility measure with distribution $\max_{i=1}^{k} \min(\alpha_i, \pi_i)$ [19]. We then define the *q-convex core* of the capacity γ by $\mathcal{C}(\mathcal{R}_*(\gamma))$, which possesses a maximal element, i.e., the possibility measure with distribution $\max_{\pi \in \mathcal{R}_*(\gamma)} \pi$. Clearly $\mathcal{C}(\mathcal{R}_*(\gamma))$ is in general a proper q-convex subset of $\mathcal{R}(\gamma)$, that is analogous to a numerical credal set [6]. The minimal set $\mathcal{R}_*(\gamma)$ plays the same role as extreme probabilities in credal sets (the vertices of a convex polyhedron), from which the latter can be reconstructed via convex closure.

3.2. Capacities as upper necessities

We can dually describe capacity functions as upper necessities by means of a family of necessity functions that stem from the lower possibility description of their conjugates. Clearly, possibility measures that dominate γ^c are conjugates of necessity measures dominated by γ. In other words γ is also an upper necessity measure in the sense that

Proposition 6.
$$\gamma(A) = \max_{\pi \in \mathcal{R}_*(\gamma^c)} N(A).$$

---

[5]More generally, γ may be a convex capacity: $g(A \cup B) + g(A \cap B) \geq g(A) + g(B)$.
[6]In the numerical setting, core and credal set coincide

Proof: $\gamma^c(A) = \min_{\pi \in \mathcal{R}_*(\gamma^c)} \Pi(A)$. Hence, $\gamma(A)$ can also be expressed as:
$\gamma(A) = \nu(\min_{\pi \in \mathcal{R}_*(\gamma^c)} \Pi(A^c)) = \max_{\pi \in \mathcal{R}_*(\gamma^c)} \nu(\Pi(A^c))$. ■

However, this result can be directly obtained from the expression (1) of γ. Indeed, if E is a focal set of γ, define the necessity measure N_E by $\forall A \neq S, N_E(A) = \gamma_\#(E)$ if $E \subseteq A$ and 0 otherwise (its focal sets are E, S, with qualitative Möbius weights $N_{E\#}(A) = \gamma_\#(E)$ and $N_{E\#}(S) = 1$). It is clear that $\gamma(A) = \max_{E \in \mathcal{F}(\gamma)} N_E(A)$. This is not the minimal form of course. To get the minimal form one may consider all maximal chains of nested focal subsets $\mathcal{C}_i = \{E_i^1 \subset \cdots \subset E_i^{k_i}\} \subseteq \mathcal{F}(\gamma), i = 1, \ldots, n$: each such chain $\mathcal{C}_i$ defines a necessity measure N_i whose nested focal sets are as follows:

- if $\gamma_\#(E_i^{k_i}) = 1$ then $\mathcal{F}(N_i) = \mathcal{C}_i$ and $N_{i\#}(E_i^k) = \gamma_\#(E_i^k)$

- otherwise $\mathcal{F}(N_i) = \mathcal{C}_i \cup \{S\}$ and $N_{i\#}(E_i^k) = \gamma_\#(E_i^k), N_{i\#}(S) = 1$.

It is easy to check that if there are n maximal chains of subsets in $\mathcal{F}(\gamma)$,

$$\gamma(A) = \max_{i=1}^n \max_{E \in \mathcal{C}_i} \gamma_\#(E) = \max_{i=1}^n N_i(A) \tag{8}$$

and $N_i(E) = \gamma_\#(E), \forall E \in \mathcal{C}_i$.

Example 1 (continued) The set of focal sets of γ in Example 1 contains two maximal chains: $(\{s_1\}, \{s_1, s_2\})$ and $\{s_2, s_3\}$ with respective weights $(0.3, 0.7)$, and 1. Hence we need two necessity measures to represent γ: N_1 with focal sets defined by $N_{1\#}(\{s_1\}) = 0.3$, $N_{1\#}(\{s_1, s_2\}) = 0.7$, $N_{1\#}(S) = 1$; and a Boolean necessity measure N_2 with one focal set defined by $N_{2\#}(\{s_2, s_3\}) = 1$. It can be checked that $\gamma(A) = \max(N_1(A), N_2(A))$. □

By construction, the cuts of possibility distributions π_i underlying N_i and different from S are focal sets of γ. These possibility distributions are not comparable with respect to specificity: if they were, they would not correspond to maximal nested chains of focal sets. In other words:

Proposition 7. *The set of possibility distributions whose cuts are built from the maximal nested chains in $\mathcal{F}(\gamma)$ is the set $\mathcal{R}_*(\gamma^c)$ of possibility distributions dominating the conjugate of γ.*

Proof: The π_i's induced by maximal chains $\mathcal{C}_i$ dominate γ^c since the corresponding N_i is dominated by γ. These distributions are not comparable and the Π_i's generate γ^c due to Eq. (8). ■

The conjugate capacity is thus such that $\gamma^c(A) = \min_{i=1}^n \Pi_i(A)$ where Π_i are the conjugates of the N_i induced by the maximal chains in $\mathcal{F}(\gamma)$. Let $\mathbb{C}(\pi)$ be the set of cuts of π. Of course, $\mathbb{C}(\pi_i) = \mathcal{F}(N_i)$. We can show that the focal sets of γ are precisely the cuts of all π_i's in $\mathcal{R}_*(\gamma^c)$ (up to set S).

Proposition 8. $\mathcal{F}(\gamma) \subseteq \bigcup_{\pi \in \mathcal{R}_*(\gamma^c)} \mathbb{C}(\pi) \subseteq \mathcal{F}(\gamma) \cup \{S\}$.

Proof: If E is focal for γ then it appears in one maximal chain $\mathcal{C}_i$ hence it is a cut of $\pi_i \in \mathcal{R}_*(\gamma^c)$. Conversely, if $E \neq S$ is a cut of $\pi \in \mathcal{R}_*(\gamma^c)$, it is a focal set of some N induced by a max-chain $\mathcal{C}$ of focal sets of γ. If $E = S$ is a cut of some $\pi \in \mathcal{R}_*(\gamma^c)$, π corresponds to a maximal chain whose largest set G may differ from S (when $\gamma_\#(G) < 1$). ■

So it is interesting to notice that there is almost an identity between the focal sets of a capacity and the cuts of the most specific possibility distributions that generate its conjugate.

Example 1 (continued) In Example 1, $\gamma(A) = \max(N_1(A), N_2(A))$ where $N_1(A)$ is induced by $\pi_1(s) = 1$ if $s = s_1$, $\nu(0.3) = 0.7$ if $s \in \{s_1, s_2\} \setminus \{s_1\}$, i.e., $s = s_2$ and $\nu(0.7)$ otherwise, that is $s = s_3$. We check that $\mathcal{F}(N_1) = \{\{s_1\}, \{s_1, s_2\}, S\}$ are the cuts of π_1. And N_2 is induced by $\pi_2(s) = 1$ if $s \in \{s_2, s_3\}$ and 0 otherwise. $\quad\square$

One of the representations of γ by means of $\mathcal{R}_*(\gamma)$ or $\mathcal{R}_*(\gamma^c)$ may contain less elements than the other. For instance, if γ is a necessity measure based on possibility distribution π, then $\mathcal{R}_*(\gamma^c) = \{\pi\}$ while $\mathcal{R}_*(\gamma)$ contains several possibility distributions (whose focal sets are singletons or S). Note that $\Pi(A) \geq N(A) = \Pi^c(A)$, so that it looks more natural to reach N from below and Π from above. For the capacity in Example 1, we see that it needs three possibility measures to represent it, but only two necessity measures are needed. The results of this section generalize to several possibility measures (the least ones that dominate γ) the obvious remark that the cuts of a possibility distribution inducing a possibility measure are the focal sets of the corresponding necessity measure.

3.3. Sugeno integrals as upper or lower possibilistic expectations

The fact that a convex capacity (or a belief function) coincides with a lower probability can be carried out to integrals. Namely, the Choquet integral of a real-valued function with respect to a numerical convex capacity is equal to the lower expectation with respect to probabilities in the credal set induced by this convex capacity [10]. However, this is not true for any capacity and any convex probability set. In this section, we examine the situation with qualitative capacities for which a counterpart of Choquet integral exists, namely Sugeno integral (see [29] for an overview of this operator).

Let $f : S \to L$ be a function that may serve as a utility function if S is a set of attributes. Sugeno integral is often defined as follows [38]:

$$\mathcal{S}_\gamma(f) = \max_{\lambda \in L} \min(\lambda, \gamma(F_\lambda)) \tag{9}$$

where $F_\lambda = \{s : f(s) \geq \lambda\}$ is the set of attributes having best ratings for some object, above threshold λ, and $\gamma(A)$ is the degree of importance of feature set A.

When γ is a possibility measure Π, Sugeno integral simplifies into the possibility integral [23]:

$$\mathcal{S}_\Pi(f) = \max_{s \in S} \min(\pi(s), f(s)) \tag{10}$$

which is the prioritized max operator.

An equivalent expression of Sugeno integral is [29]:

$$\mathcal{S}_\gamma(f) = \max_{A \subseteq S} \min(\gamma(A), \min_{s \in A} f(s))$$

In this disjunctive form, the set-function γ can be replaced without loss of information by the inner qualitative Möbius transform $\gamma_\#$ defined earlier.

$$\mathcal{S}_\gamma(f) = \max_{A \in \mathcal{F}(\gamma)} \min(\gamma_\#(A), f_A) \tag{11}$$

where $f_A = \min_{s \in A} f(s)$. This form makes it clear that it simplifies into the possibility integral (10) if $\gamma = \Pi$. The above expression of Sugeno integral has the standard maxmin form of a possibility

integral, viewing $\gamma_\#$ as a possibility distribution over 2^S. The similarity of Sugeno integral in the form (11) with the discrete form of Choquet integral using Möbius transform [27] is patent. And indeed, it has been proved elsewhere [13, 14] that it is possible to refine the ordering induced by Sugeno integral with respect to a qualitative capacity on the set of functions f by means of a Choquet integral with respect to a belief function that refines the ordering induced by the capacity.

Yet another equivalent expression of Sugeno integral is [29]:

$$\mathcal{S}_\gamma(f) = \min_{A \subseteq S} \max(\gamma(A^c), f^A) \tag{12}$$

where $f^A = \max_{s \in A} f(s)$. Likewise when γ is a necessity measure $N = \Pi^c$, based on possibility distribution π, Sugeno integral simplifies into a necessity integral [23], as clear in the form (12):

$$\mathcal{S}_N(f) = \min_{s \in S} \max(\nu(\pi(s)), f(s)) \tag{13}$$

which is the prioritized min operator. Besides it is easy to check that the conjugacy property extends to Sugeno integral as

$$\mathcal{S}_\gamma(f) = \nu(\mathcal{S}_{\gamma^c}(\nu(f))) \tag{14}$$

As a consequence of results in this section, it can be proved that Sugeno integral is a lower prioritized maximum and an upper prioritized minimum:

Proposition 9. $\mathcal{S}_\gamma(f) = \inf_{\pi \in \mathcal{R}_*(\gamma)} \mathcal{S}_\Pi(f) = \sup_{\pi \in \mathcal{R}_*(\gamma^c)} \mathcal{S}_N(f)$.

Proof: Viewing γ as a lower possibility (Proposition 5) and using the fact that $\max_x \min_y f(x, y)$ is never greater than $\min_x \max_y f(x, y)$, it comes:

$$\mathcal{S}_\gamma(f) = \max_{A \subseteq S} \min(\min_{\pi \in \mathcal{R}_*(\gamma)} \Pi(A), f_A) = \max_{A \subseteq S} \min_{\pi \in \mathcal{R}_*(\gamma)} \min(\Pi(A), f_A)$$
$$\leq \min_{\pi \in \mathcal{R}_*(\gamma)} \max_{A \subseteq S} \min(\Pi(A), f_A).$$

Hence we have $\mathcal{S}_\gamma(f) \leq \inf_{\pi \in \mathcal{R}_*(\gamma)} \mathcal{S}_\Pi(f)$.

Conversely, let π_f be the marginal of γ obtained from the nested sequence of sets F_λ induced by function f, then it is clear that $\Pi_f(F_\lambda) = \gamma(F_\lambda)$, and thus $\mathcal{S}_\gamma(f) = \mathcal{S}_{\Pi_f}(f)$. As $\exists \pi \in \mathcal{R}_*(\gamma), \pi_f \geq \pi$, by definition, $\mathcal{S}_{\Pi_f}(f) \geq \mathcal{S}_\Pi(f) \geq \inf_{\pi \in \mathcal{R}_*(\gamma)} \mathcal{S}_\Pi(f)$. Finally, using conjugacy property (14), the other property showing that Sugeno integral is an upper prioritized minimum obviously follows. ∎

Remark 6. *The obtained equality is not surprizing because Sugeno integral with respect to a function coincides with the possibility integral with respect to a specific marginal possibility distribution computed from the permutation σ_f such that $f(s_{\sigma_f(1)}) \leq f(s_{\sigma_f(2)}) \leq \cdots \leq f(s_{\sigma_f(|S|)})$, just as the discrete Choquet integral is a weighted arithmetic mean with respect to the probability distribution computed from the capacity with the same permutation.*

The significance of the above result stems from the fact that if the set of minimal possibility (resp. necessity) measures dominating (resp. dominated by) a qualitative capacity is very small, the computational complexity of the capacity is low. Namely, representing a capacity is theoretically exponential in the number of elements $|S|$ of S, while representing a possibility or a necessity measure is linear (we just need to know the $|S|$ possibility values of singletons). If $\mathcal{R}_*(\gamma)$ contains

k distributions, only $k|S|$ values are needed to represent γ. A direct consequence is to cut down the complexity the Sugeno integral computation with respect to such capacities accordingly (from exponential to linear), which may be instrumental in optimisation problems using Sugeno integral as a criterion.

4. Computing the minimal dominating possibility measures in the qualitative convex core

In this section we consider the problem of determining the minimal set of n possibility distributions π_i that are sufficient to generate a given qualitative capacity γ in the form $\gamma = \min_{i=1}^n \Pi_i$. The aim of this section is to show that the qualitative Möbius transform is instrumental in finding these least elements.

4.1. Selection functions

We need the notion of a selection function from $\mathcal{F}(\gamma)$, the set of focal sets of γ. A selection function $sel : \mathcal{F}(\gamma) \to S$ assigns to each focal set A an element $s = sel(A) \in A$. We denote by $\Sigma(\mathcal{F}(\gamma))$ the set of selection functions with domain $\mathcal{F}(\gamma)$. Given a capacity γ for any selection function on $\mathcal{F}(\gamma)$, one can define a possibility distribution π_{sel}^γ by letting $\max \emptyset = 0$ and

$$\pi_{sel}^\gamma(s) = \max_{E:sel(E)=s} \gamma_\#(E), \forall s \in S. \tag{15}$$

Note that it parallels a similar construction in the theory of belief functions, where the probability assignments $p_{sel}^{Bel}(s) = \sum_{A:sel(A)=s} m(A), \forall s \in S$ turn out to be the vertices of the credal set induced by Bel. In fact, similar results for π_{sel}^γ as for the $n!$ possibility distributions π_σ^γ can be obtained. Note that if $\gamma = \Pi$, then there is only one possible selection function (since focal sets are singletons) and $\pi_{sel}^\Pi = \pi$ (as recalled above in Section 2.1).

Proposition 10. *For any selection function sel in $\Sigma(\mathcal{F}(\gamma))$, $\Pi_{sel}^\gamma(A) \geq \gamma(A), \forall A \subseteq S$.*

Proof:
$$\Pi_{sel}^\gamma(A) = \max_{s \in A} \pi_{sel}^\gamma(s) = \max_{s \in A} \max_{E:sel(E)=s} \gamma_\#(E)$$
$$\geq \max_{s \in A} \max_{E \subseteq A:sel(E)=s} \gamma_\#(E) = \gamma(A)$$
since for a focal set $E \subseteq A, sel(E) \in A$ too, then $\{E \in \mathcal{F}(\gamma) \subseteq A : \exists s \in A : sel(E) = s\} = \{E \in \mathcal{F}(\gamma) : E \subseteq A\}$. $\blacksquare$

In fact, not all possibility distributions π_{sel}^γ are of the form π_σ^γ for a permutation σ.

Example 2 (Counterexample). *Suppose $S = \{s_1, s_2, s_3\}$, $\gamma_\#(\{s_1, s_3\}) = 1$, $\gamma_\#(\{s_2, s_3\}) = \lambda < 1$. Then consider $sel(\{s_1, s_3\}) = s_3$ and $sel(\{s_2, s_3\}) = s_2$, so that $\pi_{sel}^\gamma(s_3) = 1 > \pi_{sel}^\gamma(s_2) = \lambda > \pi_{sel}^\gamma(s_1) = 0$. Using the corresponding permutation, note that $\pi_\sigma^\gamma(s_3) = 1$ but $\pi_\sigma^\gamma(s_2) = \gamma(\{s_1, s_2\}) = 0$. However if we choose another selection function sel' such that $sel'(\{s_1, s_3\}) = sel'(\{s_2, s_3\}) = s_3$, $\pi_{sel'}^\gamma$ corresponds to the above π_σ^γ.* $\square$

Nevertheless we do have that the set $\{\pi_{sel}^\gamma : sel \in \Sigma(\mathcal{F}(\gamma))\}$ also reconstructs γ.

Proposition 11. $\forall A \subseteq S$, $\gamma(A) = \min_{sel \in \Sigma(\mathcal{F}(\gamma))} \Pi_{sel}^\gamma(A)$.

Proof: $\Pi^{\gamma}_{sel}(A) = \max_{s \in A} \max_{E:sel(E)=s} \gamma_{\#}(E)$. To get the equality, let $E \subseteq A$ be such that $\gamma(A) = \gamma_{\#}(E)$. Choose the selection function as follows: assign $\gamma_{\#}(E)$ to an element $s_E \in E$; then if $\gamma_{\#}(F) > \gamma_{\#}(E)$, assign $\gamma_{\#}(F)$ to some element not in A. This is possible, because if $\gamma_{\#}(F) > \gamma_{\#}(E)$, then $F \not\subseteq E$, and since $\gamma_{\#}(F) > \gamma(A), F \not\subseteq A$ either. For such a selection function, $\Pi^{\gamma}_{sel}(A) = \pi^{\gamma}_{sel}(s_E)$ holds since the only elements in A to which a possibility weight is assigned are $s_E \in E$, and possibly other $s_C \in C \cap A \neq \emptyset$, such that $\gamma_{\#}(C) < \gamma_{\#}(E)$. So $\gamma(A) \geq \min_{sel \in \Sigma(\mathcal{F}(\gamma))} \Pi^{\gamma}_{sel}(A)$ and due to Prop. 8, the converse is true as well. ∎

4.2. An algorithm based on useful selection functions

Due to Proposition 11, the set of minimal elements (maximally specific) of $\mathcal{R}(\gamma)$ is also included in $\{\pi^{\gamma}_{sel} : sel \in \Sigma(\mathcal{F}(\gamma))\}$. But not all selection functions are of interest since we can have redundant possibility distributions.

Example 3. *We consider a capacity γ such that the set of focal sets has a common intersection I. We denote E the focal set such that $\gamma_{\#}(E) = 1$ and F a focal set such that $\gamma_{\#}(F) < 1$. We can define a possibility such that $\pi^{\gamma}_{sel}(s^*) = 1$ for some $s^* \in I \subseteq E$ and $\pi^{\gamma}_{sel}(s) = \gamma_{\#}(F)$ for some $s \in F \setminus I$. But we can define a more specific possibility distribution dominating γ: $\pi^{\gamma}_{sel}(s^*) = 1$ for some $s^* \in I \subseteq E$ and 0 otherwise.* □

Definition 7. *A selection function sel is said to be* useful *for a capacity γ if it is a minimal element in $\{\pi^{\gamma}_{sel} : sel \in \Sigma(\mathcal{F}(\gamma))\}$.*

Let us consider the following algorithm to calculate selection functions *sel* and the associated possibility distribution π^{γ}_{sel}.

Algorithm **MSUP: Maximal specific upper possibility generation**
Input: n Ranking focal sets E_k $k = 1, \ldots, n$ in decreasing order of $\gamma_{\#}(E_k)$
        i.e., $\gamma_{\#}(E_1) \geq \cdots \geq \gamma_{\#}(E_n)$
$\mathcal{F} \leftarrow \{E_1, \cdots, E_n\}$
for all $s \in S$ let $\pi(s) = 0$
repeat
> $E \leftarrow$ the first element of $\mathcal{F}$
> Define $sel(E) = s$ for some $s \in E$
> Let $\pi(s) = \gamma_{\#}(E)$
> Delete E from $\mathcal{F}$
> $\mathcal{T} \leftarrow \{G \in \mathcal{F}$ such that $s \in G\}$
> **repeat**
> > $G \leftarrow$ the first element of $\mathcal{T}$
> > Define $sel(G) = s$
> > Delete G from $\mathcal{T}$
> > Delete G from $\mathcal{F}$.
> **until** $\mathcal{T} = \emptyset$;

until $\mathcal{F} = \emptyset$;
Result: A possibility distribution π^{γ}_{sel}.

In fact the above algorithm applies the trick in the proof of Proposition 11: the value $\gamma(A) = \gamma_{\#}(E_j)$ is retrieved by a possibility distribution where the weights $\gamma_{\#}(E_k), k < j$ are assigned

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outside A (which is one option offered by the above procedure). A selection function sel thus generated satisfies the following property: If $sel(E) = s$ for some $E \in \mathcal{F}(\gamma)$ and $\pi_{sel}(s) = \gamma_\#(E)$, then $\forall F \in \mathcal{F}(\gamma)$, such that $s \in F$:

- if $\gamma_\#(E) \geq \gamma_\#(F)$ then $sel(F) = s$

- if $\gamma_\#(E) < \gamma_\#(F)$ then $sel(F) \neq s$.

Conversely, a possibility distribution π_{sel} satisfying the above properties is a possible solution for the algorithm MSUP.

Let $\Sigma_*(\mathcal{F}(\gamma))$ be the set of selection functions generated by repeated applications of algorithm MSUP, and $MSUP(\gamma)$ be the corresponding set of possibility distributions.

Note that while not all selection functions π_{sel} yield a permutation σ such that $\pi_{sel} = \pi_\sigma$, this is true for useful selection functions. This is due to the ordering of elements generated by the procedure from great to small masses $\gamma_\#(E_j)$: define σ from the sequence of elements s_j obtained by one application of the algorithm constructing sel. For π_{sel}, it is clear that $\pi_{sel}(s_{\sigma(i)}) \leq \pi_{sel}(s_{\sigma(j)}), \forall i > j$. Hence $\pi_\sigma(i) = \gamma(S_i^\sigma) = \Pi_{sel}(S_i^\sigma) = \pi_{sel}(i)$. Moreover, if two possibility distributions are generated by the algorithm, they correspond to different permutations (since each time, different elements receive positive masses). So we conclude that $MSUP(\gamma)$ is a subset of $\{\pi_\sigma : \sigma \text{ permutation of } S\}$.

Proposition 12. *If $\pi \neq \rho \in MSUP(\gamma)$, then neither $\pi > \rho$ nor $\pi > \rho$ hold.*

Proof: Suppose $\pi > \rho$ generated by algorithm MSUP, i.e., $\pi \geq \rho$ and let s be such that $\pi(s) > \rho(s)$. Let sel_π and sel_ρ be the selection functions associated with π and ρ, respectively. Let E be such that $sel_\pi(E) = s$ and $\pi(s) = \gamma_\#(E) > \rho(s)$. By construction, $sel_\rho(E) = s' \neq s$. We have $\rho(s') \geq \gamma_\#(E) > \rho(s)$, so $\pi(s') \geq \rho(s') > 0$ and $\exists G$, such that $sel_\pi(G) = s'$ and $\pi(s') = \gamma_\#(G)$. But since by construction, $s' \in E \cap G$, it follows that $\gamma_\#(G) \geq \gamma_\#(E)$ and the algorithm MSUP would enforce $sel_\pi(E) = s'$, which contradicts the assumption that $sel_\pi(E) = s$. So $\pi \geq \rho$ with $\pi(s) > \rho(s)$ is in conflict with the assumption that π comes from the algorithm. ∎

So the possibility distributions obtained with the algorithm MSUP are not pairwise comparable (they are based on useful selections).

Example 4. *Let $S = \{s_1, s_2, s_3, s_4\}$ and the focal sets $\mathcal{F}(\gamma) = \{\{s_1, s_2\}, \{s_1, s_3\}, \{s_2, s_3\}, \{s_2, s_4\}\}$ where $\gamma_\#(\{s_1, s_2\}) = 1 > \gamma_\#(\{s_1, s_3\}) = .8 > \gamma_\#(\{s_2, s_3\}) = .4 > \gamma_\#(\{s_2, s_4\}) = .2$. We show all possible applications of the MSUP algorithm, starting with $\mathcal{F} = \mathcal{F}(\gamma)$.*

- $sel(\{s_1, s_2\}) = s_1, \pi(s_1) = \gamma_\#(\{s_1, s_2\}) = 1, \mathcal{T} = \{\{s_1, s_3\}\}$, and $\mathcal{F} = \{\{s_2, s_3\}, \{s_2, s_4\}\}$.

  - $sel(\{s_2, s_3\}) = s_2, \pi(s_2) = \gamma_\#(\{s_2, s_3\}) = .4, \mathcal{T} = \{\{s_2, s_4\}\}$, and $\mathcal{F} = \emptyset$. So, $\pi(s_1) = 1, \pi(s_2) = .4, \pi(s_3) = \pi(s_4) = 0$. The permutation σ can be $(1, 2, 3, 4)$.
  - $sel(\{s_2, s_3\}) = s_3, \pi(s_3) = \gamma_\#(\{s_2, s_3\}) = .4, \mathcal{T} = \emptyset$ and $\mathcal{F} = \{\{s_2, s_4\}\}$.
    * $sel(\{s_2, s_4\}) = s_2, \pi(s_2) = \gamma_\#(\{s_2, s_4\}) = .2, \mathcal{F} = \emptyset$. So, $\pi(s_1) = 1, \pi(s_2) = .2, \pi(s_3) = .4, \pi(s_4) = 0$. The permutation σ is $(1, 3, 2, 4)$.
    * $sel(\{s_2, s_4\}) = s_4, \pi(s_4) = \gamma_\#(\{s_2, s_4\}) = .2, \mathcal{F} = \emptyset$. So $\pi(s_1) = 1, \pi(s_2) = 0, \pi(s_3) = .4, \pi(s_4) = .2$ The permutation σ is $(1, 3, 4, 2)$.

- $sel(\{s_1, s_2\}) = s_2, \pi(s_2) = \gamma_\#(\{s_1, s_2\}) = 1, \mathcal{T} = \{\{s_2, s_3\}, \{s_2, s_4\}\}$ and $\mathcal{F} = \{\{s_1, s_3\}\}$.

- $sel(\{s_1, s_3\}) = s_1$, $\pi(s_1) = \gamma_\#(\{s_1, s_3\}) = .8$, $\mathcal{F} = \emptyset$.
  We have $\pi(s_1) = .8, \pi(s_2) = 1, \pi(s_3) = \pi(s_4) = 0$. The permutation σ can be $(2, 1, 3, 4)$.
- $sel(\{s_1, s_3\}) = s_3$, $\pi(s_3) = \gamma_\#(\{s_1, s_3\}) = .8$, $\mathcal{F} = \emptyset$.
  We have $\pi(s_1) = 0, \pi(s_2) = 1, \pi(s_3) = .8, \pi(s_4) = 0$. The permutation σ can be $(2, 3, 4, 1)$.

$\square$

Moreover the family of possibility distributions which can be obtained with the algorithm MSUP is necessary and sufficient to reconstruct the capacity. In order to prove this main result the following one is needed.

Proposition 13. *For any permutation σ, there exists a selection function sel corresponding to another permutation τ such that $\pi_\sigma \geq \pi_\tau = \pi_{sel}$.*

Proof: Let σ be a permutation of S, and we denote the corresponding elements as s_i for simplicity. Let $\pi(s_i) = \gamma(S_i)$ where $S_i = \{s_i, \ldots, s_n\}$. Let us show that we can derive a useful selection function from σ. For each s_i we shall decide whether not to make it a selection from $E \in \mathcal{F}(\gamma)$. Let U be the set of elements not yet decided, and $\mathcal{F}$ the current set of focal sets.

- Let $U = S$ and $\mathcal{F} = \mathcal{F}(\gamma)$. If $\pi_\sigma(s_j) = 0$ for $j = i, \ldots, n$, the elements s_j must be discarded from S and will not be selected from focal sets. Set $U =: \{s_1, \ldots s_{i-1}\}$.

- For $j = i - 1$ down to 1

  - Select $E \in \mathcal{F}$ such that $s_j \in E$, $E \subseteq S_j$, $\gamma_\#(E)$ is maximal, and let $sel(E) = s_j$. If there is none, let $\pi_{sel}(s_j) = 0$. Delete s_j from U.
  - Let $sel(F) = s_j$ for all F such that $s_j \in F$ and $F \subseteq S_j$ and $\gamma_\#(F) \leq \gamma_\#(E)$, and delete such F from $\mathcal{F}$.
  - Set $j =: j - 1$.

It is clear that the obtained selection function sel is useful. Moreover, it is also obvious that $\pi_{sel}(s_i) \leq \pi(s_i)$ since $\pi_{sel}(s_i) = \gamma_\#(E_i)$ for some $E_i \subseteq S_i$. Now if we use the permutation τ such that $\pi_{sel}(s_\tau(i)) \leq \pi_{sel}(s_\tau(j))$ whenever $i > j$, then we have that $\pi_{sel} = \pi_\tau$ (this is because $\pi_{sel}(s_\tau(i)) < \pi_{sel}(s_\tau(j))$ if and only if $\gamma_\#(E_i) < \gamma_\#(E_j)$). $\blacksquare$

Example 5. Generation of a useful selection from a permutation.
Consider $S = \{s_1, \cdots, s_6\}$. Let γ be such that
$\gamma_\#(\{s_1, s_2\}) = 1 > \gamma_\#(\{s_2, s_3\}) = .8 > \gamma_\#(\{s_5, s_6\}) = .7 > \gamma_\#(\{s_3, s_4, s_5\}) = .6 > \gamma_\#(\{s_1, s_6\}) = .5$.
  Consider the permutation $\sigma = (s_6, s_5, s_4, s_3, s_2, s_1)$. It is clear that $\pi_\sigma(s_i) = 1, \forall i < 6$ and $\pi_\sigma(s_6) = 0$. This possibility distribution is a very loose upper bound of γ. Let us find a more informative permutation via a selection.
We have $U = S$ and $\mathcal{F} = \{\{s_1, s_2\}, \{s_2, s_3\}, \{s_5, s_6\}, \{s_3, s_4, s_5\}, \{s_1, s_6\}\}$.

1. *First we can delete s_1 from U and $\pi_{sel}(s_1) = 0$; $U = \{s_2, \cdots, s_6\}$.*
2. *Then we select $E = \{s_1, s_2\}$ and $sel(\{s_1, s_2\}) = s_2$. We also let $sel(\{s_2, s_3\}) = s_2$.*
   Now $U = \{s_3, s_4, s_5, s_6\}$ and $\mathcal{F} = \{\{s_5, s_6\}, \{s_3, s_4, s_5\}, \{s_1, s_6\}\}$.
3. *We can select no focal set in $S_3 = \{s_1, s_2, s_3\}$, nor in $S_4 = \{s_4, s_3, s_2, s_1\}$. So $\pi_{sel}(s_3) = \pi_{sel}(s_4) = 0$.*
   Now $U = \{s_5, s_6\}$ and $\mathcal{F} = \{\{s_5, s_6\}, \{s_3, s_4, s_5\}, \{s_1, s_6\}\}$.

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4. *Then we select $E = \{s_3, s_4, s_5\} \subset S_5 = \{s_1, \cdots, s_5\}$ and $sel(\{s_3, s_4, s_5\}) = s_5$.*
   Now $U = \{s_6\}$ and $\mathcal{F} = \{\{s_5, s_6\}, \{s_1, s_6\}\}$.
5. *Finally select $E = \{s_5, s_6\}$ and let $sel(\{s_5, s_6\}) = sel(\{s_1, s_6\}) = s_6$.*

By construction, $\pi_{sel}(\{s_2\}) = 1, \pi_{sel}(\{s_5\}) = .6, \pi_{sel}(\{s_6\}) = .7$ and 0 otherwise.
It corresponds to several permutations, for instance $\{s_3, s_4, s_1, s_5, s_6, s_2\}$, where indeed
$\Pi_{sel}(\{s_2, s_6, s_5, s_1, s_4, s_3\}) = \pi_{sel}(\{s_2\}) = 1$, $\Pi_{sel}(\{s_6, s_5, s_1, s_4, s_3\}) = \pi_{sel}(\{s_6\}) = .7$,
$\Pi_{sel}(\{s_5, s_1, s_4, s_3\}) = \pi_{sel}(\{s_5\}) = .6$, $\Pi_{sel}(\{s_1, s_4, s_3\}) = \Pi_{sel}(\{s_4, s_3\}) = \Pi_{sel}(\{s_3\}) = 0.$ $\qquad\square$

Now we are in a position to show that useful selection functions provide a minimal set of possibility distributions that are enough to represent the possibilistic core of a capacity γ.

Proposition 14. $MSUP(\gamma) = \mathcal{R}_*(\gamma)$, *the set of maximally specific possibility distributions such that $\Pi(A) \geq \gamma(A)$.*

Proof: Suppose $\pi \in \mathcal{R}(\gamma)$. From Proposition 4, there is a permutation σ of S such that $\pi \geq \pi_\sigma$. Moreover there is a useful selection function sel such that $\pi_\sigma \geq \pi_{sel}$. Since possibility distributions constructed from a useful selection function are not comparable, they form the minimal elements of $\pi \in \mathcal{R}(\gamma)$. $\qquad\blacksquare$

So we have $\gamma(A) = \min_{\pi_{sel}^\gamma \in MSUP(\gamma)} \Pi_{sel}^\gamma(A)$. We let the reader apply Algorithm MSUP to Example 1, and realise that it yields possibility distributions in Table 1.

Remark 7. *Extreme cases:*

- *Note that if $\gamma_\#(E) > 0, \forall E \neq \emptyset \subseteq S$, $\mathcal{R}_*(\gamma)$ contains all π_σ for all permutations, since at each round of the algorithm MSUP, it is possible to choose any remaining element and assign the weight $\gamma_\#(E) > 0$ to it for some focal set E.*

- *If γ is a necessity measure N then its focal sets form a collection of nested sets $E_1 \subset E_2 \subset \cdots \subset E_k$ such that $N(A) = \max_{E_i \subseteq A} N_\#(E_i)$. The useful selection functions are then such that if $sel(E_i) = s$, $sel(E_k) \neq s$ for $k > i$ and $sel(E_k) = s$ if $s \in E_k, k < i$. In particular one may choose $sel(E_i) = s \in E_1, \forall E_i$. But a necessity measure is based on a simple possibility distribution π such that $N(A) = \min_{s \notin A} \nu(\pi(s))$, where ν is the order-reversing map on L. So it is not worth approximating N from above by a family of possibility distributions.*

Finally, the above considerations enable us to lay bare the connection between $\mathcal{R}_*(\gamma)$ and the contour function of a qualitative capacity already characterised in Proposition 1, via the set $\Sigma_*(\mathcal{F}(\gamma))$ of useful selections.

Proposition 15. $\pi_\gamma(s) = \max_{\pi \in \mathcal{R}_*(\gamma)} \pi(s)$.

Proof: Indeed, $\max_{\pi \in \mathcal{R}_*(\gamma)} \pi(s) = \max_{sel \in \Sigma_*(\mathcal{F}(\gamma))} \pi_{sel}(s) = \max_{sel \in \Sigma_*(\mathcal{F}(\gamma)), sel(E)=s} \gamma_\#(E)$ and $\{E : sel(E) = s, \forall sel \in \Sigma(\mathcal{F}(\gamma))\} = \{E : s \in E\}$. Note that $\forall E, \gamma_\#(E) > 0$, there is a useful selection s. t. $sel(E) = s$ and $\pi_{sel}(s) = \gamma_\#(E)$. So, $\max_{sel \in \Sigma_*(\mathcal{F}(\gamma))} \pi_{sel}(s) = \max_{E \ni s} \gamma_\#(E)$. $\qquad\blacksquare$

So the contour function can be viewed as the most specific "consensual" upper possibilistic approximation of γ (viewing $\mathcal{R}_*(\gamma)$ as containing conflicting consonant approximations of γ).

5. Generalized minitivity and maxitivity axioms

It was pointed out at the end of Section 3 that the representation complexity of a qualitative capacity is measured by the number of elements in $\mathcal{R}_*(\gamma)$ or in $\mathcal{R}_*(\gamma^c)$. This notion can be formalised by properties that generalize maxitivity and minitivity in this section.

5.1. The n-adjunction property

For each capacity γ, there is a least integer n along with n necessity measures such that $\gamma(A) = \max_{i=1}^{n} N_i(A)$. We now show that this property can be described by means of an extension of the minitivity axiom of necessity measures of the form:

n-**adjunction**: $\forall A_1, A_2, \ldots, A_{n+1} \subseteq S, \min_{i=1}^{n+1} \gamma(A_i) \leq \max_{1 \leq i < j \leq n+1} \gamma(A_i \cap A_j)$

Note that the property trivially holds if some of the $n+1$ sets are not distinct. When $n = 1$, this is the usual adjunction property $\min(\gamma(A), \gamma(B)) \leq \gamma(A \cap B)$. It is then equivalent to the minitivity axiom of necessity measures: $N(A \cap B) = \min(N(A), N(B))$ since γ is inclusion-monotonic: 1-adjunctive capacities are necessity measures. Note that it fits the idea of modelling accepted belief by means of a capacity γ and a threshold $\lambda \in L$ in the sense that the truth of A is accepted whenever $\gamma(A) \geq \lambda$. Then the property $\min(\gamma(A), \gamma(B)) \leq \gamma(A \cap B)$ is equivalent to saying that whatever the acceptance threshold λ, if the truth of A is accepted and the one of B as well, then the truth of the conjunction $A \cap B$ should be accepted as well. The above remark shows that the only capacities that model this situation are necessity measures.

A weaker requirement than minitivity is 2-adjunction that reads.

$$\forall A, B, C, \min(\gamma(A), \gamma(B), \gamma(C)) \leq \max(\gamma(A \cap B), \gamma(B \cap C), \gamma(A \cap C)).$$

It can be shown that the property is equivalent to the existence of two necessity measures such that $\forall A, \gamma(A) = \max(N_1(A), N_2(A))$ (see [21] for a detailed proof of this case). For instance, the capacity in Example 1 is 2-adjunctive.

In the general case, it holds that

Proposition 16. $\min_{i=1}^{n+1} \gamma(A_i) \leq \max_{i \neq j} \gamma(A_i \cap A_j), \forall A_1, A_2, \ldots, A_{n+1} \subseteq S$ *if and only if there exist n necessity measures such that* $\forall A, \gamma(A) = \max_{j=1}^{n} N_j(A)$.

Proof: $\Leftarrow$: Suppose $\forall A, \gamma(A) = \max_{j=1}^{n} N_j(A)$. As a consequence:

$$\min_{i=1}^{n+1} \gamma(A_i) = \min_{i=1}^{n+1} \max_{j=1}^{n} N_j(A_i) = \min_{i=1}^{n+1} N_{j_i}(A_i)$$

where $N_{j_i}(A_i) \geq N_k(A_i), \forall k \neq j_i, k = 1, \ldots, n, i = 1, \ldots, n+1$. It is clear that at least two among indices $j_i, i = 1, n+1$ are equal, since there are only n distinct values of j. Suppose they are $j_1 = 1 = j_2$ without loss of generality, that is, $\min_{i=1}^{n+1} \gamma(A_i) = \min(N_1(A_1), N_1(A_2), \min_{i=3}^{n+1} N_{j_i}(A_i))$.

Now $\gamma(A_1 \cap A_2) = \max_{i=1}^{n} N_i(A_1 \cap A_2) = \max_{i=1}^{n} \min(N_i(A_1), N_i(A_2))$. However by assumption $N_1(A_1) \geq N_k(A_1), k = 2, \ldots, n$ and $N_1(A_2) \geq N_k(A_2), k = 2, \ldots, n$ so

$$\min(N_1(A_1), N_1(A_2)) \geq \min(N_k(A_1), N_k(A_2)), k = 2, \ldots, n.$$

As a consequence,

$$\gamma(A_1 \cap A_2) = \min(N_1(A_1), N_1(A_2)) = \min(\gamma(A_1), \gamma(A_2)) \geq \min_{i=1}^{n+1} \gamma(A_i).$$

$\Rightarrow$: For the converse, suppose that non trivially, $\gamma(A) = \max_{i=1}^{n+1} N_i(A)$. Then one may find a family of $n + 1$ distinct sets A_i such that $\gamma(A_i) = N_i(A_i), i = 1, \ldots, n + 1$ and also choose them such that

$$\min_{i=1}^{n+1} \gamma(A_i) > \max_{1 \leq i < j \leq n+1} \gamma(A_i \cap A_j).$$

Indeed, choose the $n+1$ distinct sets A_i with $\gamma(A_i) = N_i(A_i)$ and $\gamma(A) = 0, \forall A \subset A_i, i = 1, \ldots, n+1$. These are the least elements of the family: $\{D : \gamma(D) > 0\}$ that is formed by a union of $n+1$ filters exactly [7] (they are the cores of the possibility distributions inducing $N_i, i = 1, n+1$). It is then clear that none of the $A_i's$ are included into one another, so that $\forall i < j, A_i \cap A_j \subset A_i$ and $A_i \cap A_j \subset A_j$ (strict inclusion) hence $\gamma(A_i \cap A_j) = 0$ by construction; so, $\max_{1 \leq i < j \leq n+1} \gamma(A_i \cap A_j) = 0$. ■

Due to Proposition 7, if a capacity is n-adjunctive, it means that there are exactly n chains of nested focal sets in $\mathcal{F}(\gamma)$.

Remark 8. *In the numerical setting, the n-superadditivity of a capacity is implied by but does not imply its $n + 1$-superadditivity. The above concept of n-adjunction seem to play a similar role: we can generalize necessity functions by steps since n-adjunction implies but is not implied by $n + 1$-adjunction. However, note that the n-superadditivity of a capacity does not determine the number of extreme points in its corresponding credal set, while the n-adjunction property of a capacity ensures that the number of least elements in its possibilistic core is upper-bounded by n.*

Of course the above results can be adapted, replacing necessity measures by possibility measures and weakening the notion of maxitivity. We can consider the following axiom, dual to n-adjunction:

n-**disjunctive dominance**: $\forall A_1, A_2, \ldots, A_{n+1} \subseteq S, \max_{i=1}^{n+1} \gamma(A_i) \geq \min_{1 \leq i < j \leq n+1} \gamma(A_i \cup A_j).$

and prove the counterpart to the last previous proposition:

Proposition 17. $\max_{i=1}^{n+1} \gamma(A_i) \geq \min_{i \neq j} \gamma(A_i \cup A_j), \forall A_1, A_2, \ldots, A_{n+1} \subseteq S$ *if and only if there exist n possibility measures such that* $\gamma(A) = \min_{i=1}^{n} \Pi_i(A)$.

It is clear that if a capacity γ satisfies the n-adjunction property, its conjugate γ^c satisfies the n-disjunctive dominance property. Moreover any capacity γ is $|MSUP(\gamma)|$-disjunctive-dominating and γ^c is $|MSUP(\gamma)|$-adjunctive (where $|\cdot|$ denotes cardinality).

5.2. Connection between n-adjunction and n-maxitive capacities in the sense of Grabisch-Mesiar

A capacity γ is k-maxitive if and only if $\gamma_\#(A) = 0$ if $|A| > k$ and $\exists A$ such that $|A| = k$ and $\gamma_\#(A) \neq 0$. This notion has been introduced by Mesiar [31] and Grabisch [25]. This is clearly another way of cutting down the complexity of the representation of a capacity, that contrasts with the idea of limiting the number of minimal possibility measures dominating it. This section explores the connections between the notions of k-maxitivity and n-adjunction. As we shall see, this issue involves the relations between the focal sets of a capacity and the focal sets of its conjugate. We first consider the case of Boolean capacities.

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[7]A filter of the Boolean algebra 2^S is a family $\mathcal{F}$ of sets such that for all $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$ and if $A \in F, A \subseteq B$ then $B \in \mathcal{F}$.

k-maxitive Boolean capacities. A Boolean capacity β is such that $\beta(A) \in \{0,1\}$. It is clear that its focal sets are not nested (they form an antichain for inclusion)[8]. We first highlight the condition for which the conjugate of a Boolean capacity takes value 1.

Lemma 2. *Suppose* $\mathcal{F}(\beta) = \{E_1, \ldots, E_k\}$ *for a Boolean capacity* β. *Then* $\beta^c(A) = 1$ *if only if* $\forall E \in \mathcal{F}(\beta), E \cap A \neq \emptyset$.

Proof: It can the be checked that: $\beta^c(A) = 1 \iff \beta(A^c) = 0 \iff \forall E \in \mathcal{F}(\beta), E \nsubseteq A^c$.
Hence: $\beta^c(A) = 1 \iff \forall E \in \mathcal{F}(\beta), E \cap A \neq \emptyset$. ∎

In the case of a Boolean necessity measure N, there is a unique focal set E such that $N(A) = 1$ if $E \subseteq A$ and 0 otherwise. And $\Pi(A) = 1$ if $E \cap A \neq \emptyset$ and 0 otherwise. The above Lemma extends this definition to capacities adding a quantification over focal sets, since we have:

$$\beta(A) = 1 \text{ if } \exists E \in \mathcal{F}(\beta), E \subseteq A \text{ and } 0 \text{ otherwise;}$$
$$\beta^c(A) = 1 \text{ if } \forall E \in \mathcal{F}(\beta), E \cap A \neq \emptyset \text{ and } 0 \text{ otherwise.}$$

We can then compute the focal sets of β^c from those of β by picking one element in each focal set of β. Given a non-empty family of sets $\mathcal{F} = \{E_1, \ldots E_k\}$ a so-called *dual family* $\mathbb{D}(\mathcal{F})$ can be defined by

$$\mathbb{D}(\mathcal{F}) = \min_{\subseteq}\{\{s_1, \ldots, s_k\}, s_i \in E_i, i = 1, \ldots, k\},$$

where $\min_{\subseteq}$ picks the smallest subsets for inclusion.

Proposition 18. *For a Boolean capacity* β, *the set of focal sets of* β^c *is* $\mathcal{F}(\beta^c) = \mathbb{D}(\mathcal{F}(\beta))$.

Proof: Note that $\beta^c(A) = 1 \iff \forall E \in \mathcal{F}(\beta), \exists s_E \in E \cap A \iff \exists F = \{s_E : E \in \mathcal{F}(\beta)\}, F \subseteq A$, where for each focal set E of β, s_E is picked in E. So, $\beta^c(A) = 1$ if only if A contains a set the form $\{s_1, \ldots, s_k\}, s_i \in E_i, i = 1, \ldots, k$. It is obvious that $\mathcal{F}(\beta^c) = \min_{\subseteq}\{A : \beta^c(A) = 1\}$. ∎

As an obvious example, while a necessity measure has a single focal set E, the focal sets of the conjugate possibility measure $\Pi(A) = 1 - N(A^c)$ are obviously the singletons $\{s\}$ such that $s \in E$. Note that when there are several focal sets, the elements s_E, s_F picked in overlapping focal sets E and F need not be distinct.

Example 6. *Let* $\mathcal{F}(\beta) = \{E_1, E_2\}$ *with* $E_1 = \{s_0, s_1, s_3\}, E_2 = \{s_0, s_2, s_4\}$, *then the focal sets of the conjugate are the least elements among the family*

$$\{\{s_0\}\} \cup \{\{s_0, s_i\}, i = 1, \ldots, 4\} \cup \{\{s_1, s_2\}, \{s_1, s_4\}, \{s_3, s_2\}, \{s_3, s_4\}\},$$

that is $\mathcal{F}(\beta^c) = \{\{s_0\}, \{s_1, s_2\}, \{s_1, s_4\}, \{s_3, s_2\}, \{s_3, s_4\}\}$. □

Proposition 19. $\mathbb{D}(\mathbb{D}(\mathcal{F}(\beta))) = \mathcal{F}(\beta)$.

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[8]Note that qualitative Boolean capacities strikingly differ quantitative 0-1 capacities, if 0 and 1 are viewed as usual numbers and we apply the usual Möebius transform to the latter. Indeed, for instance Boolean belief functions are necessity measures with a single focal set having mass 1.

Proof: It is obvious because $(\beta^c)^c = \beta$. A direct proof is far less obvious. ∎

Example 6 (continued) We can write $\mathcal{F}(\beta^c) = \{\{s\} : s \in E_1 \cap E_2\} \cup \{\{s', s''\} : s' \in E_1 \setminus E_2, s'' \in E_2 \setminus E_1\}$. To build dual focal sets from the latter family, each such focal set must contain $E_1 \cap E_2 (= \{s_0\})$. Then suppose we pick $s_1 \in E_1 \setminus E_2 = \{s_1, s_2\}$. Clearly, this choice eliminates all focal sets $\{s_1, s\}, s \in E_2 \setminus E_1$ from further consideration, that is, $\{s_1, s_4\}$. It thus prevents us from picking the next element in $E_2 \setminus E_1$. So the next elements to be picked lie in E_1, here s_3. In fact, the focal sets left $\{s, s_2\}, s \neq s_1$ can be deprived of s_2 since there is a focal set of the form $\{s_1, s_2\}$ that forbids s_2 from further consideration. So this process does reconstruct the focal set E_1. We could similarly build E_2 and this would end the process. Any other choices of elements would provide sets containing E_1 or E_2. □

The next result lays bare the connection between k-adjunction of capacities and the notion of k-maxitive capacities, i.e. whose focal sets have at most k elements. From Prop. 18, it is clear that

Corollary 1. *A Boolean capacity is k-adjunctive if and only if its conjugate is k-maxitive.*

Proof: A Boolean capacity β is k-adjunctive if and only if it has k focal sets, since each of them stands for a Boolean necessity measure and they are not nested. Now, the above results on computing the focal sets of β^c clearly implies that the focal sets of its conjugate will have not more than k elements, as they are formed by picking one element in each focal subset of β. However, since the focal sets of a Boolean capacity forms an antichain, it is possible to pick k elements, one per focal set that is contained in the focal set and not in the other ones. So k is reached. The converse is obvious due to Prop. 19. ∎

Example 7. *We consider $S = \{s_1, s_2, s_3\}$ and necessities N_1, N_2 associated to the distribution π_1 and π_2 respectively with $\pi_1(s_1) = 1$, $\pi_1(s_2) = 1$, $\pi_1(s_3) = 0$ and $\pi_2(s_1) = 0$, $\pi_2(s_2) = 1$, $\pi_2(s_3) = 1$. Focal sets are $\{s_1, s_2\}$ and $\{s_3, s_2\}$. Let β be the 2-adjunctive capacity defined by $\beta(A) = \max(N_1(A), N_2(A))$. Hence $\mathcal{F}(\beta^c) = \{\{s_2\}, \{s_1, s_3\}\}$ and β^c is 2-maxitive.* □

The reader may notice that the procedure for building focal sets of β^c from those of the Boolean β is related to the algorithm MSUP for deriving possibility distributions dominating β from its focal sets, in Section 4. In the light of Propositions 7 and 8, this is not surprising since according to these propositions, each of the focal sets of β^c can be viewed as a minimal specific possibility distribution that dominates β.

k-maxitive general capacities.

Proposition 20. *For a general capacity γ, it holds that,*

- *$\forall \lambda \in L \setminus \{0\}$, $\gamma^c(A) \geq \nu(\lambda)$ if and only if $\forall E$, if $\gamma_\#(E) > \lambda$, then $E \cap A \neq \emptyset$.*

- *$\gamma^c(A) = \nu(\lambda) \neq 0, 1$ if and only if $\forall E$, if $\gamma_\#(E) > \lambda$ then $E \cap A \neq \emptyset$ and $\exists E, \gamma_\#(E) = \lambda$ such that $E \cap A = \emptyset$.*

Proof: For the first statement, $\gamma^c(A) \geq \nu(\lambda)$ if and only if $\gamma(A^c) \leq \lambda$ if and only if $\forall E$, if $\gamma_\#(E) > \lambda$, then $E \not\subseteq A^c$, that is $E \cap A \neq \emptyset$. For the second one, the equality $\gamma^c(A) = \nu(\lambda)$ is attained if on top of the condition in Prop. 20, there is a focal set $E \subseteq A^c$ such that $\gamma_\#(E) = \lambda$. ∎

In particular, $\gamma^c(A) = 1$ if and only if $E \cap A \neq \emptyset, \forall E \in \mathcal{F}(\gamma)$ and $\gamma^c(A) > 0$, if and only if A intersect all focal sets of γ with $\gamma_\#(E) = 1$, which generalizes the case of valued possibility measures.

To find the focal sets of γ^c in the general case one may use the decomposition of γ as a weighted combination of Boolean capacities (see Appendix). Recall that $L = \{\lambda_0 = 0 < \lambda_1 < \cdots < \lambda_\ell = 1\}$. Then $\nu(\lambda_i) = \lambda_{\ell-i}$, and let γ_λ be the λ-cut of γ, a Boolean capacity such that $\gamma_\lambda(A) = 1$ if and only if $\gamma(A) \geq \lambda$. Then, Lemma 2 also writes: $\gamma^c_{\lambda_i}(A) = 1$ if and only if $\forall E \in \mathcal{F}(\gamma_{\lambda_{\ell-i+1}}), E \cap A \neq \emptyset$. So applying results for Boolean capacities, E is a focal set of $\gamma^c_{\lambda_i}$ if and only if it is a minimal element of the family $\{E = \{s_F : \gamma_\#(F) > \lambda_{\ell-i}\}\} = \{E = \{s_F : F \in \mathcal{F}(\gamma_{\lambda_{\ell-i+1}})\}\}$. In other words,

$$\mathcal{F}(\gamma^c_{\lambda_i}) = \mathbb{D}(\mathcal{F}(\gamma_{\lambda_{\ell-i+1}})).$$

We can then compute $\gamma^c_\#(E)$ as per Corollary 3 in Appendix: $\gamma^c_\#(E) = \max_{\lambda:E \in \mathcal{F}(\gamma^c_\lambda)} \lambda$.

Example 1 (continued) The focal sets of γ are such that $\gamma_\#(\{s_1\}) = 0.3$, $\gamma_\#(\{s_1, s_2\}) = 0.7$ and $\gamma_\#(\{s_2, s_3\}) = 1$. Let us compute the focal sets of its conjugate.

- $\gamma^c_\#(E) = \lambda_3 = 1$, if and only if $E \in \mathbb{D}(\mathcal{F}(\gamma))$ (it must intersect all focal sets of γ.) So $\gamma^c_\#(\{s_1, s_2\}) = \gamma^c_\#(\{s_1, s_3\}) = 1$.

- $\gamma^c_\#(E) = \lambda_2 = 0.7$, if and only if $E \in \mathbb{D}(\mathcal{F}(\gamma_{0.7}))$ ($\ell = 3$ so $0.7 = \lambda_{\ell-2+1}$) and must not contain s_1. So $\gamma^c_\#(\{s_2\}) = 0.7$.

- $\gamma^c_\#(E) = \lambda_1 = 0.3$, if and only if $E \in \mathbb{D}(\mathcal{F}(\gamma_1))$ (focal sets of γ with weight 1) and must not hit $\{s_1, s_2\}$. So $\gamma^c_\#(\{s_3\}) = 0.3$.

γ^c can be easily computed as $\gamma^c(\{s_1\}) = 0, \gamma^c(\{s_2\}) = 0.7, \gamma^c(\{s_3\}) = 0.3, \gamma^c(\{s_1, s_3\}) = 1$, $\gamma(\{s_1, s_2\}) = 1$, $\gamma^c(\{s_2, s_3\}) = 0.7$, using its focal sets given above. □

These results show how the inner qualitative Möbius transform of a capacity can be computed from the one of its conjugate. Note that alternatively, as shown in Section 3.2, we can recover the focal sets of γ^c from the cuts of the possibility distributions dominating γ obtained by repeated applications of algorithm MSUP.

Regarding k-maxitive functions, it is easy to infer that also in the general case,

Corollary 2. *If a capacity has k weighted focal sets, its conjugate will be k-maxitive.*

Proof: Indeed, the largest focal sets of γ^c (they have weight equal to 1) are obtained by picking one element in each focal set of γ. ∎

The following figure summarizes the results obtained so far when switching between representations of γ and its conjugate γ^c using focal sets and possibility distributions.

6. Modal logic and qualitative capacities

In this section we show the connection between qualitative capacities and weak epistemic modal formalisms relying on non-regular modal logics [6]. Consider a propositional language $\mathcal{L}$ with variables $V = \{a, b, c, \dots\}$ with standard connectives $\wedge, \vee, \neg, \rightarrow$. $\top$ and $\bot$ denote tautology and contradiction respectively. In other words, formulas p in $\mathcal{L}$ are as usual generated as follows:

- If $a \in V$ then $a \in \mathcal{L}$;

- if $p, q \in \mathcal{L}$ then $\neg p \in \mathcal{L}$, $p \wedge q \in \mathcal{L}$;

- $p \vee q \in \mathcal{L}$ is short for $\neg(\neg p \wedge \neg q)$.

In this section the universe S is the set of interpretations of $\mathcal{L}$.

Figure 1: Passages between representations

6.1. The logic of Boolean possibility theory

Consider a higher level propositional language $\mathcal{L}_\square$ defined by

- If $p \in \mathcal{L}$ then $\square p \in \mathcal{L}_\square$;

- if $\phi, \psi \in \mathcal{L}_\square$ then $\neg\phi \in \mathcal{L}_\square$, $\phi \wedge \psi \in \mathcal{L}_\square$;

Note that the language $\mathcal{L}$ is embedded inside $\mathcal{L}_\square$, since atomic variables of $\mathcal{L}_\square$ are of the form $\square p, p \in \mathcal{L}$. As usual $\Diamond p$ stands for $\neg\square\neg p$. It defines a very elementary fragment of a modal logic language proposed by Banerjee and Dubois under the name MEL [2, 3].

Given a necessity measure N on S, denote by $\models \square p$ the statement $N(A) \geq \lambda > 0$, where $A = [p]$ the set of models of p. $\square p$ corresponds to a Boolean necessity measure based on a possibility distribution that is the characteristic function of $E = \{s : \pi(s) > \nu(\lambda)\}$ the largest focal set of N such that $N_\#(E) \geq \lambda$. Then $\models \Diamond p$ stands for $\Pi(A) \geq \nu(\lambda)$ where Π is the conjugate of N. Under this view, the language of MEL can serve as an elementary epistemic logic without nested modalities and without non-modal formulas from $\mathcal{L}$. Indeed, the following KD axioms and inference rule are valid under this understanding [28]:

- All axioms of propositional logics for $\mathcal{L}_\square$-formulas.

- $(K) : \square(p \to q) \to (\square p \to \square q)$.

- $(N) :$ $\square p$ whenever p is a propositional tautology.

- $(D) :$ $\square p \to \Diamond p$.

- Modus Ponens: If $\phi, \phi \to \psi$ then ψ.

Axiom (C): $\square(p \wedge q) \equiv (\square p \wedge \square q)$, which is the adjunction axiom (the Boolean form of the minitivity axiom) is valid in this system. In fact, MEL is a (higher-order) propositional logic.

A model for this modal logic is a *nonempty* subset $E \subseteq S$ (of propositional models). The set E is understood as an epistemic state (a *meta-model*), i.e., the information possessed by an agent that only knows that the real world lies in E. It is not empty due to axiom D, which means the agent has a consistent epistemic state. The satisfaction of MEL-formulae is then defined recursively:

- $E \models \Box p$, if and only if $E \subseteq [p]$,

- $E \models \neg\phi$, if and only if $E \not\models \phi$,

- $E \models \phi \wedge \psi$, if and only if $E \models \phi$ and $E \models \psi$, where ϕ, ψ are any $\mathcal{L}_\Box$-formulae,

- So, $E \models \Diamond p$ if and only if $E \cap [p] \neq \emptyset$.

For any set $\Gamma \cup \{\phi\}$ of $\mathcal{L}_\Box$-formulae, ϕ is a semantic consequence of Γ, written $\Gamma \models \phi$, provided that, for every epistemic state E, $E \models \Gamma$ implies $E \models \phi$.

This Boolean possibilistic logic, equipped with modus ponens, (the $\mathcal{L}_\Box$-fragment of KD) is sound and complete w.r.t. this semantics [3]. In particular, it does not require the use of accessibility relations.

6.2. The logic of Boolean capacities

This construction can be extended to qualitative capacities using the same language. Interpret now $\models \Box p$ as standing for $\gamma([p]) \geq \lambda > 0$ for any qualitative capacity γ and an arbitrary such threshold λ, or equivalently $\gamma_\lambda([p]) = 1$, using cuts. The following axioms are then verified [12]:

- All axioms of propositional logics for $\mathcal{L}_\Box$-formulas.

$(RM) : \Box p \to \Box q$, whenever $\vdash p \to q$.

$(N) : \quad \Box\top$.

$(P) : \quad \Diamond\top$.

- Modus Ponens: If $\phi, \phi \to \psi$ then ψ.

This system seems to be the natural logical account of qualitative capacities: RM expresses the monotonicity of capacities, and the two other axioms stand for $\gamma(S) = 1, \gamma(\emptyset) = 0$ respectively.

This modal logic we call QC is a known monotonic (non-regular) modal logic. It is a special case of the *monotonic modal* logic EMN (Chellas [6]), a fragment where modalities only apply to propositions, not to modal formulas. This logic no longer satisfies axioms K, C nor D.

A QC-model for this modal logic is a Boolean capacity β and the satisfaction of QC-formulae is then defined recursively as in MEL, replacing E by β:

- $\beta \models \Box p$, if and only if $\beta([p]) = 1$;

- $E \models \neg\phi$, $E \models \phi \wedge \psi$ in the standard way, as above.

Semantic entailment is defined as in the previous subsection, and syntactic entailment is classical propositional entailment taking RM, N, P as axioms: $\Gamma \vdash_{QC} \phi$ if and only if $\Gamma \cup \{$all instances of $RM, N, P\} \vdash \phi$ (classically defined).

Proposition 21. *The logic QC is sound and complete with respect to the semantics in terms of qualitative capacities.*

Proof: It goes along the same lines as the soundness and completeness of the MEL logic [3]. As QC is a propositional logic, all we have to do is to show that there is a one-to-one correspondence between standard propositional models of QC (assignments $t : \mathcal{L}_{\square} \to \{0, 1\}$) and Boolean capacities. To see it, note that if we let $t_{\beta}(\square p) = 1$ if and only if $\beta([p]) = 1$, we do define a standard propositional valuation that obeys RM, N, P. Conversely given a standard propositional valuation t that obeys RM, N, P, we immediately see that the set-function β_t defined by $\beta_t([p]) = 1$ if and only if $t(\square p) = 1$ defines a qualitative capacity, since

- from RM, $\beta_t([p]) = 1$ and $p \models q$ implies $\beta_t([q]) = 1$,

- from N, $\beta_t([\top]) = \beta_t(S) = 1$

- from P, $\beta_t([\bot]) = t(\square \bot) = 1 - t(\Diamond \top) = 0$.

So QC is sound and complete with respect to qualitative capacities, just because as for any knowledge base in propositional logic, syntactic inference is equivalent to semantic entailment. ∎

The usual semantics of such kinds of logics in the general case, also called Scott-Montague semantics, is based on so-called neighbourhood frames [6]. A neighbourhood is a family of non-empty subsets $\mathcal{N}$ of 2^S and a neighbourhood frame is an application from S to 2^S. Here we do not nest modalities and stick to neighbourhoods.

The neighbourood semantics takes the following form:

$$\mathcal{N} \models \square p \text{ if and only if } [p] \in \mathcal{N};$$
$$\mathcal{N} \models \Diamond p \text{ if and only if } [\neg p] \notin \mathcal{N}.$$

The following properties are easy to see:

- Axiom RM holds if and only if whenever $A \in \mathcal{N}$, and $A \subseteq B$, $B \in \mathcal{N}$ as well (closure under inclusion).

- Axiom N means $S \in \mathcal{N}$ (non-triviality) and axiom P means $\emptyset \notin \mathcal{N}$ (consistency).

We shall call a nontrivial consistent neighbourhood that is closed under inclusion a *standard neighbourhood*. It is then easy to see that

Proposition 22. *The family of sets $\{A : \gamma_{\lambda}(A) = 1\}$ is a standard neighbourhood and for any standard neighbourhood $\mathcal{N}$, the Boolean set-function β, defined by $\beta(A) = 1$ if and only if $A \in \mathcal{N}$, is a qualitative capacity.*

Besides, since $\gamma = \max_{i=1}^n N_i$ for some number n of necessity measures, denoting by $\square_i p$ the statement $N_i([p]) \geq \lambda > 0$, it is clear that $\gamma([p]) \geq \lambda > 0$ stands for $\square p \equiv \vee_{i=1}^n \square_i p$, where $\square_i$ is the KD modality induced by a focal set E_i of N_i. By duality, $\Diamond p = \neg \square \neg p = \wedge_{i=1}^n \Diamond_i p$.

The semantics of the QC logic in terms of Boolean capacities β can be thus equivalently expressed in terms of non-empty subsets of S (i.e. focal sets of β) as follows:

$$\beta \models \square p \iff \exists E_i \in \mathcal{F}(\beta), E_i \models \square_i p.$$

Interestingly, we can put an upper bound on the number of KD modalities that define $\square$ by adding the n-adjunction axiom to the axioms of QC (see [12] for $n = 2$):

$$C_n : \quad (\wedge_{i=1}^{n+1} \square p_i) \to \vee_{i \neq j=1}^{n+1} \square (p_i \wedge p_j).$$

for some fixed positive integer n. In QC, it implies that if $p_i, i = 1, \ldots, n+1$ are mutually inconsistent, then $\vdash \neg \wedge_{i=1}^{n+1} \Box p_i$. This property claims that we cannot have $\gamma([p_i]) \geq \lambda > 0$ for all $i = 1, \ldots, n+1$. When $n = 1$, we recover axiom C (adjunction) and the MEL setting. The models of this restricted QC logic with axiom C_n are n-adjunctive Boolean capacities. The satisfaction relation in this restricted logic can be expressed using n-tuples of non-empty subsets of S as follows:

$$(E_1, \ldots, E_n) \models \Box p \text{ if } \exists i \in [1, n], E_i \models \Box_i p.$$

By construction, an n-adjunctive Boolean capacity has n focal sets $E_1, \ldots, E_n$ that can refer to a society of n agents. As in the case of a KD modality, where a non-empty set E of interpretations stands for the epistemic state E of a single agent, one can assume that an n-adjunctive Boolean capacity stands for the joint epistemic states $(E_1, \ldots, E_n)$ of several agents, whereby the truth of $\Box p$ means that p is true for at least one agent i. Note that such agents hold globally non-redundant epistemic states E_i, since these sets are focal sets of a Boolean capacity and thus form an antichain for inclusion: as there is no inclusion between the E_i's, there are beliefs each agent possesses that are held by none of the other agents.

This capacity-based semantics corresponds to neigborhoods of the form $\mathcal{N} = \{A : \exists i : 1 \leq i \leq n, E_i \subseteq A\}$. In particular,

- For the KD-logic MEL, it is obvious that $\mathcal{N} = \{A : N(A) \geq \lambda\} = \{A : A \supseteq E\}$ for some non-empty $E \subseteq S$ ($\mathcal{N}$ is a proper filter).

- For the QC logic, $\mathcal{N} = \{A : \gamma(A) \geq \lambda > 0\} \neq 2^S$ is closed under inclusion and not empty).

- For QC logic with C_n, $\mathcal{N} = \{A : \gamma(A) \geq \lambda > 0\}$ is the union of n proper filters of the form $\{A : N_i(A) \geq \lambda\} = \{A : A \supseteq E_i\}$.

Then we shall have again soundness and completeness of the QC logic with axiom C_n with respect to n-adjunctive Boolean capacities, since classical models of QC are accordingly constrained by axiom C_n: $t((\wedge_{i=1}^{n+1} \Box p_i) \to \vee_{i \neq j=1}^{n+1} \Box(p_i \wedge p_j)) = 1$ if and only if β_t is n-adjunctive.

Remark 9. *The QC logic framework comes close to Belnap logic [5] as pointed out in [12], namely we have several sources (here, with respective epistemic states $E_1, E_2, \ldots, E_n$), each of which asserting, denying each formula or being silent about it. A formula is then considered true if at least one source asserts it and none denies it, which does correspond to a disjunction of KD modalities. In Belnap logic, these formulas are only literals, the epistemic status of other formulas being computed via a truth-table. See also [7] for the case when the sets $(E_1, \ldots, E_n)$ are singletons and its connection to three-valued paraconsistent logics.*

7. Conclusion

We have studied the representation of capacities having values on a finite totally ordered scale by families of qualitative possibility distributions. It turns out that any capacity can be viewed either as a lower possibility measure or as an upper necessity measure with respect to two distinct families of possibility distributions. This remark has led to propose a generalisation of maxitivity and minitivity properties of possibility theory, thus offering a classification of qualitative capacities in terms of increasing levels of complexity and generality, based on the minimal number of possibility distributions needed to represent them. A connection between the size of focal sets of a capacity

and the number of focal sets of its conjugate has been laid bare. Finally, we have shown that the corresponding property of n-adjunction enables qualitative capacities and non-regular modal logics to be connected, which generalizes KD-style modal logic in the same sense as capacities generalize necessity measures. This suggests a potential application to reasoning with conflicting sources of information.

It is clear that our results also apply to numerical capacities with a finite range, or defined on a finite set, as they are a special case of our setting, which does not use addition. The extension of our results to qualitative capacities on an infinite set, ranging on an ordinal scale, would deserve a specific investigation. Similarly it could be interesting to generalise the presented results to capacities ranging on a lattice [33].

Numerous open alleys of research are opened by the above results. We can mention three of them:

- On the logical side, we may reconsider the study of non-regular modal logics in the light of capacity-based semantics. It looks like a general setting for reasoning about uncertainty due both to incomplete and conflicting information. The fact that it comes down to disjunctions of KD necessity operators is clearly pointing towards Belnap epistemic set-ups [12], and paraconsistent logics [7].

- One may also wish to evaluate the quantity of information (or uncertainty) contained in a qualitative capacity. This is done in the numerical case by Marichal and Roubens [30]. Some preliminary results in the qualitative setting appear in [22], based on an information ordering reminiscent of a qualitative counterpart of the specialisation of belief functions (inclusion of focal sets) introduced in [32].

- The multi-source interpretation of qualitative capacities could be instrumental for the study of advanced methods for information fusion, especially qualitative counterparts to Dempster rule of combination [32, 1].

Appendix: Decomposition of qualitative capacities in terms of Boolean capacities

Let γ_λ denote the Boolean capacity obtained as $\gamma_\lambda(A) = 1$ if $\gamma(A) \geq \lambda$, and 0 otherwise, and we call it its λ-cut. Obviously, the focal sets of γ are among those of its λ-cut: $\mathcal{F}(\gamma) \subseteq \cup_{\lambda \in L} \mathcal{F}(\gamma_\lambda)$. The following is the counterpart, for capacities, of the decomposition of a fuzzy set in terms of its cuts.

Proposition 23. *Each qualitative capacity can be decomposed as follows:* $\gamma = \max_{\lambda \in L} \min(\lambda, \gamma_\lambda)$.
Proof: For all $\lambda \in L$, $\min(\lambda, \gamma_\lambda(A)) = \lambda$ if $\gamma(A) \geq \lambda$ and 0 otherwise. So for all $\lambda \in L$, $\min(\lambda, \gamma_\lambda(A)) \leq \gamma(A)$ i.e, $\max_{\lambda \in L} \min(\lambda, \gamma_\lambda(A)) \leq \gamma(A)$.
Conversely, $\gamma(A) = \min(\gamma(A), \gamma_{\gamma(A)}(A)) \leq \max_{\lambda \in L} \min(\lambda, \gamma_\lambda(A))$. ∎

Moreover the focal sets of γ are the ones of its cuts.

Proposition 24. $\mathcal{F}(\gamma) = \cup_{\lambda \in L} \mathcal{F}(\gamma_\lambda)$.
Proof: Let A be a focal set of γ. $\gamma_{\gamma(A)}(A) = 1$ and $\forall B \subset A$ we have $\gamma_{\gamma(A)}(B) = 0$ since $\gamma(B) < \gamma(A)$. So A is a focal set of $\gamma_{\gamma(A)}$ and $A \in \cup_{\lambda \in L} \mathcal{F}(\gamma_\lambda)$.

Let $A \in \cup_{\lambda \in L} \mathcal{F}(\gamma_\lambda)$. Hence there exists λ such that $\gamma_\lambda(A) = 1$ and $\forall B \subset A \; \gamma_\lambda(B) = 0$. We have $\gamma(A) \geq \lambda$ and $\forall B \subset A, \; \gamma(B) < \lambda$ so $\forall B \subset A, \; \gamma(B) < \gamma(A)$ i.e, $A \in \mathcal{F}(\gamma)$. ∎

The decomposition process is valid as well for the inner qualitative Möbius transform:

Proposition 25. $\gamma_\# = \max_{\lambda \in L} \min(\lambda, \gamma_\#^\lambda)$.

Proof: According to Proposition 24, $\gamma_\#(A) = 0$ if and only if for all λ we have $\gamma_\#^\lambda(A) = 0$.

If $\gamma_\#(A) \neq 0$ then $\gamma_\#(A) = \min(\gamma_\#(A), \gamma_\#^{\gamma_\#(A)}(A)) \leq \max_{\lambda \in L} \min(\lambda, \gamma_\#^\lambda(A))$. Consider $\lambda \in L$, if $\lambda > \gamma_\#(A)$ then $\min(\lambda, \gamma_\#^\lambda(A)) = 0$ and if $\lambda < \gamma_\#(A)$ then $\min(\lambda, \gamma_\#^\lambda(A)) \leq \lambda < \gamma_\#(A)$. So we have $\max_{\lambda \in L} \min(\lambda, \gamma_\#^\lambda(A)) \leq \gamma_\#(A)$. ∎

Corollary 3. $\gamma_\#(E) = \max_{\lambda : E \in \mathcal{F}(\gamma_\lambda)} \lambda$.

References

[1] Z. Assaghir, A. Napoli, M. Kaytoue, D. Dubois, H. Prade. Numerical information fusion: Lattice of answers with supporting arguments. *Proc. 23rd IEEE Int. Conf. on Tools with Artificial Intelligence (ICTAI'11)*, Boca Raton, Nov. 7-9, IEEE Press, 621-628, 2011.

[2] M. Banerjee, D. Dubois. A simple modal logic for reasoning about revealed beliefs. In: C. Sossai, G. Chemello (eds.) *Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU'09)*, (Proc. 10th Europ. Conf. , Verona, July 1-3, 2009), Springer, LNAI 5590, 805-816, 2009.

[3] M. Banerjee, D. Dubois. A simple logic for reasoning about incomplete knowledge. *Int. J. of Approximate Reasoning*, 55, 639-653, 2013.

[4] G. Banon. Constructive decomposition of fuzzy measures in terms of possibility and necessity measures. *Proc. VIth Int. Fuzzy Systems Assoc. (IFSA) World Congress*, São Paulo, Brazil, vol. I, 217-220, 1995.

[5] N. D. Belnap. How a computer should think. In: G. Ryle (ed.) *Contemporary Aspects of Philosophy*, Oriel Press, Boston, 30-56, 1977.

[6] B. F. Chellas, *Modal Logic: An Introduction.* Cambridge University Press, Cambridge, 1980.

[7] D. Ciucci, D. Dubois. From paraconsistent three-valued logics to multiple-source epistemic logic. *Proc. 8th. Conf. of the Europ. Soc. for Fuzzy Logic and Technology (EUSFLAT'13)*, Milano, Sept. 11-13, Atlantis Press, 780-707, 2013.

[8] I. Couso, D. Dubois, L. Sanchez, *Random Sets and Random Fuzzy Sets as Ill-Perceived Random Variables*, SpringerBriefs in Computational Intelligence, Springer, 2014.

[9] A. P. Dempster. Upper and lower probabilities induced by a multivalued mapping. *Annals of Mathematical Statistics*, 38, 325-339, 1967.

[10] D. Denneberg. *Nonadditive Measure and Integral.* Kluwer Academic, Dordrecht, The Netherlands, 1994.

[11] D. Dubois. Fuzzy measures on finite scales as families of possibility measures. *Proc. 7th Europ. Society For Fuzzy Logic and Technology (EUSFLAT-LFA)*, Aix-Les-Bains, July 18-22, Atlantis Press, 822-829, 2011.

[12] D. Dubois. Reasoning about ignorance and contradiction: Many-valued logics versus epistemic logic. *Soft Computing*, 16 (11), 1817-1831, 2012.

[13] D. Dubois, H. Fargier. Making discrete Sugeno integrals more discriminant. *Int. J. of Approximate Reasoning*, 50, 880-898, 2009.

[14] D. Dubois, H. Fargier. Capacity refinements and their application to qualitative decision evaluation.In: C. Sossai, G. Chemello (eds.) *Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU'09)*, (Proc. 10th Europ. Conf. , Verona, July 1-3, 2009), Springer, LNAI 5590, 311-322, 2009.

[15] D. Dubois, J. C. Fodor, H. Prade, M. Roubens. Aggregation of decomposable measures with application to utility theory. *Theory and Decision*, 41, 1996, 59-95.

[16] D. Dubois, H. Prade. Upper and lower possibilities induced by a multivalued mapping. *Proc. IFAC Symp. on Fuzzy Information, Knowledge Representation and Decision Analysis* (Marseille, Jul. 19-21, 1983, Preprints p. 147-152), Pergamon Press, Oxford, 1984.

[17] D. Dubois, H. Prade. Evidence measures based on fuzzy information. *Automatica*, 21, 547-562, 1985.

[18] D. Dubois and H. Prade, *Possibility theory: An approach to computerized processing of uncertainty*, Plenum Press, 1988.

[19] D. Dubois, H. Prade. Aggregation of possibility measures, In: J. Kacprzyk, M. Fedrizzi (eds.) *Multiperson Decision Making using Fuzzy Sets and Possibility Theory*, Kluwer, Dordrecht, 55-63, 1990.

[20] D. Dubois, H. Prade. Possibility theory: Qualitative and quantitative aspects. In: P. Smets (ed.) *Handbook on Defeasible Reasoning and Uncertainty Management Systems - Vol. 1: Quantified Representation of Uncertainty and Imprecision*, Kluwer Academic Publ., Dordrecht, The Netherlands, 169-226, 1998.

[21] D. Dubois, H. Prade, A. Rico. Qualitative capacities as imprecise possibilities. In: L. van der Gaag (ed.) *Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU'13)*, (Proc. 12th Europ. Conf. on Utrecht, July 7-10), Lecture Notes in Computer Science, Springer, 169-180, 2013.

[22] D. Dubois, H. Prade, A. Rico. On the informational comparison of qualitative fuzzy measures. In: A. Laurent, O. Strauss, B. Bouchon-Meunier, R. R. Yager (Eds.), *Information Processing and Management of Uncertainty in Knowledge-Based Systems* (15th Int. Conf., IPMU 2014, Montpellier, France, July 15-19, 2014), Proceedings, Part I, Springer, 216–225, 2014.

[23] D. Dubois, H. Prade. R. Sabbadin. Decision theoretic foundations of qualitative possibility theory. *European Journal of Operational Research*, 128, 459-478, 2001.

[24] T. Fine. *Theories of Probability.* Academic Press, New-York, 1983.

[25] M. Grabisch. On the representation of k-decomposable measures. *Proc. 7th Int. Fuzzy Systems Assoc. (IFSA) World Congress*, Prague, June 25 - 29, vol. 1, 478-483, 1997.

[26] M. Grabisch. The Möbius transform on symmetric ordered structures and its application to capacities on finite sets. *Discrete Mathematics*, 287, 17-34, 2004.

[27] M. Grabisch, T. Murofushi, and M. Sugeno, eds., *Fuzzy Measures and Integrals - Theory and Applications*, Physica Verlag, 2000.

[28] J. Hintikka. *Knowledge and Belief: An Introduction to the Logic of the Two Notions.* Cornell University Press, 1962.

[29] J.-L. Marichal. On Sugeno integrals as an aggregation function. *Fuzzy Sets and Systems*, 114(3), 347-365, 2000.

[30] J.- L. Marichal, M. Roubens. Entropy of discrete fuzzy measures. *Int. J. of Uncertainty, Fuzziness and Knowledge-Based Systems* 8(6), 625-640, 2000.

[31] R. Mesiar, k-order pan-discrete fuzzy measures. *Proc. 7th Int. Fuzzy Systems Assoc. (IFSA) World Congress*, Prague, June 25 - 29, vol. 1, 488-490, 1997.

[32] H. Prade, A. Rico. Possibilistic evidence. In: Weiru Liu (ed.) *Symbolic and Quantitative Approaches to Reasoning with Uncertainty* (ECSQARU'11), (11th Europ. Conf., Belfast, June 29-July 1, 2011), Proceedings, LNAI 6717, Springer, 713-724, 2011.

[33] A. Rico, Sugeno integral in a finite Boolean algebra, Fuzzy Sets and Systems, 159(13), 1709-1718, 2008.

[34] G. Shafer. *A Mathematical Theory of Evidence.* Princeton University Press, Princeton, 1976.

[35] D. Schmeidler. Core of exact games. I. *J. Math. Analysis and Appl.*, 40, 214-225, 1972.

[36] G. L. S. Shackle. *Expectation in Economics.* Cambridge University Press, UK, 1949. 2nd edition, 1952.

[37] W. Spohn. *The Laws of Belief: Ranking Theory and Its Philosophical Applications.* Oxford Univ. Press, 2012.

[38] M. Sugeno. Fuzzy measures and fuzzy integrals: a survey. In: *Fuzzy Automata and Decision Processes,* (M.M. Gupta, G.N. Saridis, and B.R. Gaines, eds.), North-Holland, 89-102, 1977.

[39] E. Tsiporkova, B. De Baets. A general framework for upper and lower possibilities and necessities. *Int. J. of Uncertainty, Fuzziness and Knowledge-Based Systems*, 6(1), 1-34, 1998.

[40] P. Walley. *Statistical Reasoning with Imprecise Probabilities.* Chapman and Hall, 1991.

[41] S. M. K. Wong, Y. Y. Yao, P. Bollman, and H. C. Burger. Axiomatization of qualitative belief structure. *IEEE Trans. Systems Man and Cybern.*, 21, 726-734, 1993.

[42] L. A. Zadeh: Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets and Systems*, 1, 3-28, 1978.