Uncertain logical gates in possibilistic networks. An application to human geography

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Abstract. Possibilistic networks offer a qualitative approach for modeling epistemic uncertainty. Their practical implementation requires the specification of conditional possibility tables, as in the case of Bayesian networks for probabilities. This paper presents the possibilistic counterparts of the noisy probibilistic connectors (and, or, max, min, ...). Their interest is illustrated on an example taken from a human geography modeling problem. The difference of behaviors in some cases of some possibilistic connectors, with their respect to their probabilistic analogs, is discussed in details.

Introduction

Bayesian networks [6] can be built in two ways: statistical and subjective. In the first case, a supposedly large dataset involving a number of variables is available, and the Bayesian network is obtained by some machine learning procedure. Obtained probability tables have a frequentist flavor, and the simplest network possible is searched for. On the contrary, Bayesian networks can be specified using expert knowledge. In this case, the structure of a network relating the variables is first given, often relying on causal connections between variables and conditional independence relations the expert is aware of. Then probability tables must be filled by the expert. It consists, for each variable in the network, of conditional probabilities for this variable conditioned on each configuration of its parent variables. Note that even if causal relations as perceived by the expert are instrumental in building a simple and interpretable network, the joint probability distribution obtained by combining the probability tables no longer account for causality. Another difficulty arises for causality-based Bayes networks: if variables are not binary and/or the number of parents variable is more than two, the task of elicitation of numerical probability tables becomes tedious, if not impossible to fulfill. Indeed the number of probability values to be supplied increases exponentially with the number of parent variables.

To alleviate the elicitation task, the notion of noisy connective has been introduced, based on the assumption of independent causal influences that can be combined. As a result, one small conditional probability table is elicited per parent variables, and the probability table of each variable given its parents is obtained by combining the small ones by a so-called noisy connective [3, 5]. It includes a so-called leakage factor summarizing the causal effect of variables not explicitly present in the network.

While the notion of noisy connectives solves the combinatorial problem of collecting many probability values to a large extent, it remains the issue that people cannot always provide precise probability assessments. Let alone the fact that the probability scale is too fine-grained for human perception of belief or frequencies, some conditional probability values may be ill-known or plainly unknown to the experts. The usual Bayesian recommendation in the latter case is to use uniform distributions, but it is well-known that they do not properly model ignorance. Alternatively, one may use imprecise probability networks (called credal nets) [7], qualitative Bayesian networks [8] or possibilistic networks [2]. While the two first options extend probabilistic nets to ill-known parameters (with an interval-based approach for the first and and ordinal approach for the second approach), possibilistic networks represent a more drastic departure from probabilistic nets. In their qualitative version, possibilistic nets can be defined on a finite chain of possibility values and do not refer to numerical values. This feature may make the collection of expert information on conditional tables easier than requiring precise numbers following the laws of probability.

In this paper we try to introduce possibilistic counterparts of noisy connectives of probabilistic networks. As possibilistic uncertainty is merely epistemic and due to a lack of information, we shall speak of uncertain connectives. After recalling probabilistic nets with noisy gates, we present the corresponding approach for possibilistic nets, and present various uncertain gates, especially the AND, OR, MAX and MIN functions. Then the approach, including algorithmic issues, is illustrated on a belief network stemming from an application to geography.

Probabilistic nets with independent causal influences

Consider a set of independent variables $X_1, \dots X_n$ that influence the value of a variable Y. In the ideal case, there is a deterministic function f such that $Y = f(X_1, X_2, \dots X_n)$. In order to account for uncertainty, one may assume the existence of intermediary variables $Z_1, \dots Z_n$, with domains identical to the ones of $X_1, \dots X_n$, such that Z_i expresses the fact that X_i will have a causal influence on Y, and to what extent (Z_i has the same domain as Y). It is assumed that the relation between X_i and Z_i is probabilistic and that X_i is independent from other variables given Z_i. Besides, we consider the deterministic function as affected by the auxiliary variables Z_i only. In other words, we get a probabilistic network such that

$$P(Y, Z_1, \dots Z_n, X_1, X_2, \dots X_n) = P(Y, Z_1, \dots Z_n) \cdot \prod_{i=1}^{n} P(Z_i|X_i) \qquad (1)$$

Where $P(Y, Z_1, \dots Z_n) = 1$ if $Y = f(Z_1, Z_2, \dots Z_n)$ and 0 otherwise. This is called a noisy function. In particular, note that the dependence tables between Y and $X_1, \dots X_n$ can now be obtained by combining simple conditional probability distributions pertaining to single factors:

$$P(y|x_1,\dots x_n) = \sum_{z_1,\dots,z_n:y=f(z_1,\dots,z_n)} \prod_{i=1}^{n} P(z_i|x_i). \qquad (2)$$

This is the assumption of *independence of causal influence* (ICI [3]). In the case of Boolean variables, it is assumed that $P(z_i = 0|x_i = 0) = 1$ (no cause, no effect), while $P(z_i = 0|x_i = 1)$ can be positive (the effect may or not appear when the cause is present).

Canonical ICI models are obtained by means of specific choice of the function f. For instance, if all variables are Boolean, f will be a logical connective. Then, we speak of noisy OR ($f = \vee$), noisy AND ($f = \wedge$; if the range of the Z_i's and Y is a totally ordered set, usual gates are the noisy MAX ($f = \max$), or MIN ($f = \min$).

Additionally the approach is further refined by allowing to summarize the potential effect of external variables not taken into account: this is the leaky model. Now, Y also depends on a leak variable Z_ℓ not explicitly related to specific identified causes, i.e., $Y = f(Z_1, Z_2, \dots Z_n, Z_\ell)$. The domain of Z_ℓ is supposed to be the range of f, i.e. the domain of Y and this variable is independent of the other ones. Hence the leakage model writes:

$$P(Y, Z_1, \dots Z_n, Z_\ell, X_1, X_2, \dots X_n) = P(Y, Z_1, \dots Z_n) \cdot P(Z_\ell) \cdot \prod_{i=1}^{n} P(Z_i|X_i)$$

so that

$$P(y|x_1,\dots x_n) = \sum_{z_1,\dots,z_n,z_\ell:y=f(z_1,\dots,z_n,z_\ell)} P(z_\ell) \cdot \prod_{i=1}^{n} P(z_i|x_i). \qquad (3)$$

For instance in the case of Boolean variables $P(y = 1|x_1 = 0, \dots x_n = 0)$ may be positive due to such external causes.

The thrust of this paper is to see whether the same kind of ICI approach can be used to elicitate possibilistic networks.

Canonical possibilistic networks

Possibility theory [4, 9] is based on maxitive set functions associated to possibility distributions. Formally, given a universe of discourse U, and a possibility distribution π from U to $[0, 1]$, pertains to a variable X ranging on U. It represents the available (incomplete) information about the more or less possible values of X, assumed to be single-valued. Thus, $\pi(u) = 0$ means that $X = u$ is impossible. The consistency of information is expressed by the normalization of $\pi : \exists u \in U, \pi(u) = 1$, namely, at least one value is fully possible for X. Distinct values u and u' may be simultaneously possible at degree 1. A state of complete ignorance is represented by the distribution $\pi_?(u) = 1, \forall u \in U$. A possibility measure of an event $A \subseteq U$ is defined by

$$\mathit{\Pi}(A) = \sup_{u \in A} \pi(u)$$

Possibility measures are maxitive, i.e.,

$$\forall A, \forall B, \Pi(A \cup B) = \max(\Pi(A), \Pi(B)).$$

The underlying assumption is that the agent focuses on most plausible values, neglecting other ones. A dual measure of necessity $N(A) = 1 - \Pi(U \setminus A)$ expresses the certainty of event A as the impossibility of non-A.

A possibilistic network [1, 2] has the same structure as a Bayesian network: The joint possibility for n variables linked by an acyclic directed graph is defined by

$$\pi(x_1, \dots, x_n) = *_{i=1,\dots,n} \pi(x_i \mid pa(X_i))$$

where x_i is an instantiation of the variable X_i, and $pa(X_i)$ an instantiation of the parent variables of x_i. The operation $*$ is the minimum (in the qualitative case) or the product (in the numerical case).

Deterministic models $Y = f(X_1, \dots, X_n)$ are defined as in the probabilistic case:

$$\pi(y|x_1, \dots, x_n) = \begin{cases} 1 \text{ si } y = f(x_1, \dots x_n); \\ 0 \text{ sinon} \end{cases} \tag{4}$$

Let us define possibilistic models with *independent causal influences* (ICI). We use a deterministic function $Y = f(Z_1, \dots, Z_n)$ with n intermediary causal variables Z_i, as for probabilistic models. Now, $\pi(y|x_1, \dots, x_n)$ is of the form:

$$\pi(y|z_1, \dots, z_n) * \pi(z_1, \dots, z_n|x_1, \dots, x_n),$$

where $\pi(y|z_1, \dots, z_n)$ obeys the equation (4). Again, each variable Z_i only depends (in an uncertain way) on the variable X_i. Thus, we have $\pi(z_1, \dots, z_n|x_1, \dots, x_n) = *_{i=1,\dots,n} \pi(z_i|x_i)$. This leads to the equality, whose similarity with (2) is striking:

$$\pi(y|x_1, \dots, x_n) = \max_{z_1,\dots,z_n : y = f(z_1,\dots,z_n)} *_{i=1,\dots,n} \pi(z_i|x_i). \tag{5}$$

Note that when $* = \min$, it comes down to applying the extension principle [**?**] to the function f assuming fuzzy-valued inputs $F_1, \dots F_n$, where the membership function of F_i is defined by $\mu_{F_i}(z_i) = \pi(z_i|x_i)$.

In case we suppose that y depends also in an uncertain way on other causes summarized by a leak variable Z_ℓ then the counterpart of (3) reads:

$$\pi(y|x_1, \dots x_k) = \max_{z_1,\dots,z_n,z_\ell : y = f(z_1,\dots,z_n,z_\ell)} *_{i=1,\dots,n} \pi(z_i|x_i) * \pi(z_\ell) \tag{6}$$

In the following we provide a detailed analysis of possibilistic counterparts of noisy gates.

Uncertain OR and AND gates

The variables are assumed to be Boolean (i.e., $Y = y$ or $\neg y$, etc.). The uncertain OR (counterpart of the probabilistic "noisy OR") assumes that $X_i = x_i$ for at least one variable X_i represents a sufficient cause for getting $Y = y$, and $Z_i = z_i$ indicates that $X_i = x_i$ has caused $Y = y$. This gives $f(Z_1, \dots, Z_n) = \vee_{i=1}^n Z_i$. The uncertainty indicates that the causes may fail to produce their effects. $Z_i = \neg z_i$ indicates that $X_i = x_i$ did not cause $Y = y$ due to the presence of some inhibitor that prevents the effect from taking place. We assume it is more possible that $X_i = x_i$ causes $Y = y$ than the opposite (otherwise one could not say that $X_i = x_i$ is sufficient for causing $Y = y$). Then we must define $\pi(z_i|x_i) = 1, \pi(\neg z_i|x_i) = \kappa_i < 1$. Besides $\pi(z_i|\neg x_i) = 0$ since whenX_i is absent, it does not cause y. Hence the causal elementary possibility table:

$\pi(Z_i\|X_i)$	x_i	$\neg x_i$
z_i	1	0
$\neg z_i$	κ_i	1

Note that in the case of a probabilistic network, $\pi(z_i|x_i)$ is replaced by $1 - \kappa_i$ in the above table. We can then obtain the table of the conditional possibility distribution $\pi(Y|X_1, \dots, X_n)$ by means of equation (5).

$$\begin{aligned}\pi(y|X_1, \dots, X_n) &= \max_{z_1,\dots,z_n : z_1 \vee \dots \vee z_n = 1} *_{i=1}^n \pi(z_i|X_i)\\ &= \max_{i=1}^n \pi(z_i|X_i) * (*_{j\neq i} \max(\pi(z_j|X_j)\pi(\neg z_j|X_j));\\ \pi(\neg y|X_1, \dots, X_n) &= \max_{z_1,\dots,z_n : z_1 \vee \dots \vee z_n = 0} *_{i=1}^n \pi(z_i|X_i)\\ &= \pi(\neg z_1|X_1) * \dots * \pi(\neg z_n|X_n).\end{aligned}$$

Let us denote by $\mathbf{x}$ a configuration of $(X_1, \dots, X_n)$, and let $I_+(\mathbf{x}) = \{i : X_i = x_i\}$ and $I_-(\mathbf{x}) = \{i : X_i = \neg x_i\}$. Then we get:

- $\pi(\neg y|\mathbf{x}) = *_{i=1,\dots,n}\pi(\neg z_i|X_i = \mathbf{x}_i) = *_{i\in I_+(\mathbf{x})}\kappa_i$
- $\pi(y|\mathbf{x}) = 1$ when $\mathbf{x} \neq (\neg x_1, \dots, \neg x_n)$
- $\pi(\neg y \mid \neg x_1, ..., \neg x_n) = 1, \pi(y \mid \neg x_1, ..., \neg x_n) = 0$: $\neg y$ (no effect) can be obtained for sure only if all the causes are absent.

For n = 2, it gives the conditional tables:

$\pi(y\|X_1X_2)$	x_1	$\neg x_1$
x_2	1	1
$\neg x_2$	1	0

$\pi(\neg y\|X_1X_2)$	x_1	$\neg x_1$
x_2	$\kappa_1 * \kappa_2$	κ_2
$\neg x_2$	κ_1	1

More generally, if there are n causes, we have to provide the values of n parameters κ_i.

For the uncertain OR with leak, we now assume that $f(Z_1, \dots Z_n) = \vee_{i=1}^n Z_i \vee Z_\ell$, where Z_ℓ is an unknown external cause. We assign $\pi(z_\ell) = \kappa_\ell < 1$ considering that z_ℓ is not a usual cause. We thus obtain

- $\pi(\neg y|\mathbf{x}) = *_{i=1,\dots,n}\pi(\neg z_i|X_i = \mathbf{x}_i) * \pi(\neg z_\ell) = *_{i\in I_+(\mathbf{x})}\kappa_i$
- $\pi(y|\mathbf{x}) = 1$, si $\mathbf{x} \neq (\neg x_1, \dots, \neg x_n)$
- $\pi(\neg y|\neg x_1, \dots, \neg x_n) = 1$
- $\pi(y|\neg x_1, \dots, \neg x_n) = \kappa_\ell$ (even if the causes x_i are absent, there is still a possibility for having $Y = y$, namely if the external cause is present).

Indeed we get (letting $\neg x = \neg x_1, \ldots, \neg x_n$,

$$\pi(y|(\neg x_1, \ldots, \neg x_n)) = \max(\pi(y|(\neg x, z_\ell)) * \pi(z_\ell), \pi(y|(\neg x, \neg z_\ell)) * \pi(\neg z_\ell)))$$
$$= \max(1 * \kappa_\ell, 0 * 1) = \kappa_\ell$$

For n = 2, the conditional table becomes:

$\pi(y\|X_1X_2)$	x_1	$\neg x_1$	$\pi(\neg y\|X_1X_2)$	x_1	$\neg x_1$
x_2	1	1	x_2	$\kappa_1 * \kappa_2$	κ_2
$\neg x_2$	1	κ_ℓ	$\neg x_2$	κ_1	1

The only 0 entry has been replaced by the leakage coefficient. For n causes, we have now to provide the values of $n + 1$ parameters κ_i.

The uncertain AND (counterpart of the probabilistic "noisy AND") uses the same local conditional tables but it assumes that $X_i = x_i$ represents a *necessary* cause for $Y = y$. We again build the conditional possibility table $\pi(Y|(X_1, \ldots, X_n))$ by means of equation (5) with $f(Z_1, \ldots, Z_n) = \wedge_{i=1}^n Z_i$. Thus, we find

- $\pi(\neg y|x_1, \ldots, x_n) = \max_{z_1,\ldots,z_n:\neg y = z_1 \wedge \cdots \wedge z_n} *_{i=1}^n \pi(z_i|x_i) = \max_{i=1}^n \pi(\neg z_i|x_i) = \max_{i=1}^n \kappa_i$.
- $\pi(y|(x_1, \ldots, x_n)) = 1$.
- $\pi(\neg y|\mathbf{x}) = 1, \pi(y|\mathbf{x}) = 0$ si $\mathbf{x} \neq (x_1, \ldots x_k)$ (if at least one of the causes is absent, the effect is necessarily absent).

For n = 2, equation (5) yields the conditional tables:

$\pi(y\|X_1X_2)$	x_1	$\neg x_1$	$\pi(\neg y\|X_1X_2)$	x_1	$\neg x_1$
x_2	1	0	x_2	$\max(\kappa_1, \kappa_2)$	1
$\neg x_2$	0	0	$\neg x_2$	1	1

More generally, if there are n causes, we have to assess n values for the parameters κ_i. The case of the uncertain AND with leak corresponds to the possibility $\pi(z_L) = \kappa_L < 1$ that an external factor $Z_L = z_L$ causes $Y = y$ independently from the values of the X_i. Namely $f(Z_1, \ldots, Z_n, Z_L) = (\wedge_{i=1}^n Z_i) \vee Z_L$. For n = 2, equation (5) then gives the conditional tables:

$\pi(y\|X_1X_2)$	x_1	$\neg x_1$	$\pi(\neg y\|X_1X_2)$	x_1	$\neg x_1$
x_2	1	κ_L	x_2	$\max(\kappa_1, \kappa_2)$	1
$\neg x_2$	κ_L	κ_L	$\neg x_2$	1	1

Comparison with probabilistic gates

It is interesting to compare the possibilistic and probabilistic tables . Consider those of the noisy OR [], where $\kappa_i = P(\neg z_i|x_i)$.

$p(y\|X_1X_2)$	x_1	$\neg x_1$	$p(\neg y\|X_1X_2)$	x_1	$\neg x_1$
x_2	$1 - \kappa_1\kappa_2$	$1 - \kappa_2$	x_2	$\kappa_1\kappa_2$	κ_2
$\neg x_2$	$1 - \kappa_1$	0	$\neg x_2$	κ_1	1

There is an important difference between the behaviors of uncertain and noisy OR if $* = \min$. In the possibilistic tables, we see that $N(y|11) = \max(N(y|10), N(y|01))$ while $P(y|11) > \max(P(y|10), P(y|01))$, so that the presence of two causes does not reinforce the certainty of the effect wrt the presence of the most influential cause. Hence qualitative possibility networks will be less expressive than probabilistic nets. If $* =$ product, $N(y|11) = 1 - \kappa_1\kappa_2 > \max(N(y|10), N(y|01))$ as with the probability case.

Another major difference will occur in the case when the effects of causes are not frequent, namely suppose $P(\neg z_i|x_i) = \kappa_i > 0.5, i = 1, 2$. Then it may occur that $P(y|x_1x_2) = 1 - \kappa_1\kappa_2 > 0.5$, that is the presence of the two causes makes the effect frequent. Then a possibilistic rendering of this case must be such that $\pi(\neg z_i|x_i) = 1 > \pi(z_i|x_i) = \lambda_i$ (say). However, there is no way of observing this reversal effect, since $\pi(y|x_1x_2) = \max(\lambda_1 * \lambda_2, \lambda_1, \lambda_2) = \max(\lambda_1, \lambda_2) < 1$. Hence $\pi(y|x_1x_2) = 1$ and $N(y|x_1x_2) > 0$. In other words, using the uncertain OR, two causes that are individually insufficient to make an effect plausible are still insufficient to make it plausible if joined together. Note that this fact reminds of the property of closure under conjunction for necessity measures in possibility theory ($N(y_1) > 0$ and $N(y_2) > 0$ imply $N(y_1 \wedge y_2) > 0$) which fail to hold in probability theory.

One way to address this problem is to define the global conditional possibility tables $\pi(Y|X_1X_2)$ enforcing $\pi(y|x_1x_2) > \pi(\neg y|x_1x_2)$ even if $\pi(y|x_1) < \pi(\neg y|x_1)$ and $\pi(y|x_2) < \pi(\neg y|x_2)$, which is perfectly compatible in possibility theory. However one cannot build the global table from the marginal ones using an uncertain OR.

Uncertain MAX and MIN gates

The uncertain MAX is a multiple-valued extension of the uncertain OR, where the output variable (hence the variables Z_i) is valued on a finite, totally ordered, gravity or intensity scale $L = \{0 < 1 < \cdots < m\}$. We assume that $Y = \max(Z_1, \ldots, Z_n)$. $Z_i = z_i \in L$ represents the fact that X_i alone has increased the value of Y at level z_i.The conditional possibility distributions $\pi(y|x_i)$ are supposed to be given. We can then compute the conditional tables, as

$$\begin{aligned}\pi(y|x_1, \ldots, x_n) &= \max_{z_1,\ldots,z_n : y = \max(z_1,\ldots,z_n)} *_{i=1}^{n} \pi(z_i|x_i) \\ &= \max_{i=1}^{n} \quad \pi(Z_i = y|x_i) * (*_{j\neq i} \Pi(Z_j \leq y|x_j))\end{aligned}$$

In a causal setting, we assume that $y = 0$ is a normal state, and $y > 0$ is more or less abnormal, $y = m$ being fully abnormal. Suppose that the domain of X_i is L as well. It is natural to assume that:

- if $X_i = j$ then $Z_i = j$, which means $\Pi(Z_i = j|X_i = j) = 1$.
- $\Pi(Z_i > j|X_i = j) = 0$ (a cause having a weak intensity cannot induce an effect with strong gravity);
- $0 < \Pi(Z_i < j|X_i = j) < 1$ (a cause having strong intensity may sometimes only induce an effect with weak gravity, or may even have no effect at all);
- An effect with gravity weaker than the intensity of a cause is all the less plausible as the effect is weak. This leads to suppose the following inequalities:
  $0 < \pi(Z_i = 0|X_i = j) < \pi(Z_i = 1|X_i = j) < \cdots < \pi(Z_i = j|X_i = j) = 1.$

This leads to state the left-hand side table below (for 3 levels of strength 0, 1, 2).

$\pi(Z_i\|X_i)$	$X_i = 2$	$X_i = 1$	$X_i = 0$
$Z_i = 2$	1	0	0
$Z_i = 1$	κ_i^{12}	1	0
$Z_i = 0$	κ_i^{02}	κ_i^{01}	1

$\pi(Z_i\|X_i)$	$X_i = 2$	$X_i = 0$
$Z_i = 2$	1	0
$Z_i = 1$	κ_i^{12}	0
$Z_i = 0$	κ_i^{02}	1

where $0 < \kappa_i^{02} < \kappa_i^{12} < 1$, $0 < \kappa_i^{01} < 1$. In case we have m levels of strength, we have to assess $\frac{m(m+1)}{2}$ coefficients. On the right-hand side is the corresponding table when the variables X_i are Boolean (then the middle column is dropped).

The global conditional possibility tables are thus obtained by applying equation (5), using the values of $\pi(Z_i|X_i)$, as given in the above table.

$$\pi(Y = j|\mathbf{x}) = \max_{i=1}^{n} \pi(Z_i = j|x_i) * (*_{\ell \neq i} \Pi(Z_\ell \leq j|x_\ell)).$$

For n = 2, $m = 2$, when the X_i's are three-valued and Boolean, respectively, the following conditional tables are obtained (in the Boolean case, only 4 lines remain):

$\mathbf{x}$	$\pi(2\|\mathbf{x})$	$\pi(1\|\mathbf{x})$	$\pi(0\|\mathbf{x})$
$(2,2)$	1	$\max(\kappa_1^{12}, \kappa_2^{12})$	$\kappa_1^{02} * \kappa_2^{02}$
$(2,1)$	1	1	$\kappa_1^{02} * \kappa_2^{01}$
$(2,0)$	1	κ_1^{12}	κ_1^{02}
$(1,2)$	1	1	$\kappa_1^{01} * \kappa_2^{02}$
$(1,1)$	0	1	$\kappa_1^{01} * \kappa_2^{01}$
$(1,0)$	0	1	κ_1^{01}
$(0,2)$	1	κ_2^{12}	κ_2^{02}
$(0,1)$	0	1	κ_2^{01}
$(0,0)$	0	0	1

$\mathbf{x}$	$\pi(2\|\mathbf{x})$	$\pi(1\|\mathbf{x})$	$\pi(0\|\mathbf{x})$
$(2,2)$	1	$\max(\kappa_1^{12}, \kappa_2^{12})$	$\kappa_1^{02} * \kappa_2^{02}$
$(2,0)$	1	κ_1^{12}	κ_1^{02}
$(0,2)$	1	κ_2^{12}	κ_2^{02}
$(0,0)$	0	0	1

More generally, If we have m levels of strength, and n causal variables, we need $\frac{nm(m+1)}{2}$ coefficients for defining the uncertain MAX. If we take into account the leak, we have to add $\frac{m(m+1)}{2}$ coefficients per variable, in order to replace the 0 by a leak coefficient in the conditional tables $\pi(Z_i|X_i)$ (assuming that an effect of strong gravity may take place even if the causes present have a weak intensity).

As for the uncertain MAX wrt uncertain OR, the uncertain MIN is a multiple-valued extension of the uncertain AND, where variables are valued on a the intensity scale $L = \{0 < 1 < \cdots < m\}$. We assume that $Y = \min(Z_1, \ldots, Z_n)$. We can then compute the conditional tables, as

$$\begin{aligned}\pi(y|x_1, \ldots, x_n) &= \max_{z_1,\ldots,z_n : y=\min(z_1,\ldots,z_n)} *_{i=1}^{n} \pi(z_i|x_i) \\ &= \max_{i=1}^{n} \quad \pi(Z_i = y|x_i) * (*_{j \neq i} \Pi(Z_j \geq y|x_j))\end{aligned}$$

The conditional possibility tables are thus obtained by applying equation (5), using the same values of $\pi(Z_i|X_i)$, as in the case of the uncertain MAX. For n = 2, $m = 2$, this gives the following conditional tables (for ternary and binary inputs, respectively):

$\mathbf{x}$	$\pi(2\|\mathbf{x})$	$\pi(1\|\mathbf{x})$	$\pi(0\|\mathbf{x})$
$(2,2)$	1	$\max(\kappa_1^{12},\kappa_2^{12})$	$\max(\kappa_1^{02},\kappa_2^{02})$
$(2,1)$	0	1	$\max(\kappa_1^{02},\kappa_2^{01})$
$(2,0)$	0	κ_1^{12}	1
$(1,2)$	0	1	$\max(\kappa_1^{01},\kappa_2^{02})$
$(1,1)$	0	1	$\max(\kappa_1^{01},\kappa_2^{01})$
$(1,0)$	0	0	1
$(0,2)$	0	κ_2^{12}	1
$(0,1)$	0	0	1
$(0,0)$	0	0	1

$\mathbf{x}$	$\pi(2\|\mathbf{x})$	$\pi(1\|\mathbf{x})$	$\pi(0\|\mathbf{x})$
$(2,2)$	1	$\max(\kappa_1^{12},\kappa_2^{12})$	$\max(\kappa_1^{02},\kappa_2^{02})$
$(2,0)$	0	κ_1^{12}	1
$(0,2)$	0	κ_2^{12}	1
$(0,0)$	0	0	1

Illustration

IN ENGLISH

Implementation

Here are some details about the practical implementation of the uncertain operators defined in the paper.

The way the uncertain MAX operator is implemented is shown in Algorithm 1. The parameter *prm* taken as input by this algorithm may be though of as representing a set of rules of the form

$$X_{i_1} = x_{i_1} \wedge \ldots \wedge X_{i_m} = x_{i_m} \Rightarrow Y \sim (\kappa(y_1), \ldots, \kappa(y_n)), \tag{7}$$

where the X_{i_j} on the left-hand side are parent variables of Y in the possibilistic graphical model, the x_{i_j} are one of their modalities, and $(\kappa_1(y_1), \ldots, \kappa_1(y_n))$ is a normalized possibility distribution over the values of variable Y, i.e., for all $y \in Y$, $\kappa(y) \in [0,1]$, and $\max_{y \in Y} \kappa(y) = 1$.

The left-hand side of a rule may be empty (i.e., $m = 0$): in that case, the rule is interpreted as if it were

$$\top \Rightarrow Y \sim (\kappa_\ell(y_1), \ldots, \kappa_\ell(y_n)). \tag{8}$$

Such rules may be used to represent leak coefficients, which apply to all possible combinations of causes.

The antecedents of the rules fed into the uncertain MAX operator must cover all possible combinations $\mathbf{x} \in X_1 \times \ldots \times X_n$ of the modalities of the parent variables of Y in order to ensure that the resulting conditional possibility distribution $\pi(Y \mid X_1, \ldots, X_n)$ be normalized. We may notice that, if a leak rule of the form of Equation 8 is given, that rule alone already covers all combinations of parent variable modalities and is thus a sufficient condition for the normalization of $\pi(Y \mid X_1, \ldots, X_n)$; in that case, the parameters of the uncertain MAX may be underspecified.

The uncertain MAX operator with thresholds, whose implementation is shown in Algorithm 2, has an additional parameter, which consists of an array of thresholds $(\theta_1, \ldots, \theta_{\|Y\|})$, with $\theta_i \in \{1, 2, \ldots, \|X_1 \times \ldots \times X_n\|\}$. Each threshold is associated with one modality y of Y and represents the minimal number of combinations of the causes for which y is more possible than the baseline possibility given by the leak coefficients ($\kappa(y) > \kappa_\ell(y)$), or zero if no leak is provided.

Algorithm 1 UNCERTAIN-MAX(Y, prm).
Generate a conditional possibility table for variable Y given its causes $X_1, \ldots, X_n$ using the uncertain MAX operator with the given parameters *prm*.

Input: Y: the effect variable; $prm = \{\langle cond_i, \mathbf{k}_i \rangle\}$: a set of normalized possibility distributions $\mathbf{k}_i = (\kappa_{i1}, \ldots, \kappa_{i\|Y\|})$, $\max_{j=1,\ldots,\|Y\|}\{\kappa_{ij}\} = 1$, which apply when condition $cond_i$ holds; $cond_i = (\langle X_{ij}, x_{ij} \rangle)$, a (possibly empty) array of pairs of a cause variable X_{ij} and one of its modalities x_{ij}; $cond_i$ holds if $X_{ij} = x_{ij}$ holds for all j; an empty condition always holds.
Output: $\pi(Y \mid X_1, \ldots, X_n)$: a conditional possibility distribution of Y given its causes $X_1, \ldots, X_n$.

$\pi(Y|X_1, \ldots, X_n) \leftarrow \mathbf{0}$
for all $\mathbf{x} \in X_1 \times \ldots \times X_n$ **do**
  $K \leftarrow \{\mathbf{k} : \langle cond_i, \mathbf{k} \rangle \in prm, \mathbf{x} \models cond_i\}$ {Select the parameters that apply to $\mathbf{x}$}
  for all $\mathbf{y} = (y_1, \ldots, y_{\|K\|}) \in Y^{\|K\|}$ **do**
    $\beta \leftarrow \min_{i=1,\ldots,\|K\|}\{\kappa_{iy_i}\}$
    $\bar{y} \leftarrow \max_{i=1,\ldots,\|K\|}\{y_i\}$
    $\pi(\bar{y} \mid \mathbf{x}) \leftarrow \max\{\beta, \pi(\bar{y} \mid \mathbf{x})\}$
  end for
end for
return $\pi(Y \mid X_1, \ldots, X_n)$

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Algorithm 2 UNCERTAIN-MAX-THRESHOLD(Y, prm, thr).

Generate a conditional possibility table for variable Y given its causes $X_1, \ldots, X_n$ using the uncertain MAX operator with thresholds with the given parameters *prm* and thresholds *thr*.

Input: Y: the effect variable; $prm = \{\langle cond_i, \mathbf{k}_i \rangle\}$: a set of normalized possibility distributions $\mathbf{k}_i = (\kappa_{i1}, \ldots, \kappa_{i\|Y\|})$, $\max_{j=1,\ldots,\|Y\|}\{\kappa_{ij}\} = 1$, which apply when condition $cond_i$ holds; $cond_i = (\langle X_{ij}, x_{ij} \rangle)$, a (possibly empty) array of pairs of a cause variable X_{ij} and one of its modalities x_{ij}; $cond_i$ holds if $X_{ij} = x_{ij}$ holds for all j; an empty condition always holds; $thr = (\theta_1, \ldots, \theta_{\|Y\|})$: the minimal number of combinations of modalities of the causes for which each modality of Y is more possible than the leak.

Output: $\pi(Y \mid X_1, \ldots, X_n)$: a conditional possibility distribution of Y given its causes $X_1, \ldots, X_n$.

$\pi(Y|X_1, \ldots, X_n) \leftarrow \mathbf{0}$
$\kappa_\ell \leftarrow \mathbf{0}$
for all $\langle cond_i, \mathbf{k} \rangle \in prm : cond_i = \top$ **do**
    $\kappa_\ell \leftarrow \max\{\kappa_\ell, \mathbf{k}\}$
end for
for all $\mathbf{x} \in X_1 \times \ldots \times X_n$ **do**
    $cnt \leftarrow \mathbf{0}$ {A vector of counters, one for each $y \in Y$}
    $K \leftarrow \{\mathbf{k} : \langle cond_i, \mathbf{k} \rangle \in prm, \mathbf{x} \models cond_i\}$ {Select the parameters that apply to $\mathbf{x}$}
    for all $\mathbf{y} = (y_1, \ldots, y_{\|K\|}) \in Y^{\|K\|}$ **do**
        $\beta \leftarrow \min_{i=1,\ldots,\|K\|}\{\kappa_{iy_i}\}$
        $\bar{y} \leftarrow \max_{i=1,\ldots,\|K\|}\{y_i\}$
        if $\beta > \kappa_\ell(\bar{y})$ **then**
            $cnt_{\bar{y}} \leftarrow cnt_{\bar{y}} + 1$
        end if
        if $cnt_{\bar{y}} \geq \theta_{\bar{y}}$ **then**
            $\beta \leftarrow 1$
        end if
        $\pi(\bar{y} \mid \mathbf{x}) \leftarrow \max\{\beta, \pi(\bar{y} \mid \mathbf{x})\}$
    end for
end for
return $\pi(Y \mid X_1, \ldots, X_n)$