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POSSIBILISTIC LOGIC - AN OVERVIEW

1 INTRODUCTION

Uncertainty often pervades information and knowledge. For this reason, the handling of uncertainty in inference systems has been an issue for a long time in artificial intelligence (AI). Indeed rather early in the history of AI, in the early 1970's, the second expert system to be designed, MYCIN [Buchanan and Shortliffe, 1984], was the occasion of proposing an original setting for the representation of uncertainty in terms of degree of belief, degree of disbelief, and certainty factor, with empirical rules for combining them [Shortliffe, 1976].

Since that time, different new proposals have been developed for representing uncertainty including imprecise probabilities [Walley, 1991], belief function-based evidence theory [Shafer, 1976; Yager and Liu, 2008], possibility theory [Zadeh, 1978; Dubois and Prade, 1988], while Bayesian probabilities [Pearl, 1988; Jensen, 2001] have become prominent at the forefront of AI methods, challenging the original supremacy of logical representation settings [Minker, 2000].

The need for classical methods, well-mastered at the algorithmic level thanks to many implementation-oriented developments, has contributed to the present success of probabilistic representations. However, meanwhile, there has been a constant interest for methods providing tools oriented towards the representation of imprecise epistemic states of knowledge pervaded with uncertainty (these states may range between complete information and complete ignorance). These methods are characterized by the existence of a pair of dual measures for assessing uncertainty, leaving the freedom of neither believing p nor $\neg p$ in case of ignorance about p, as permitted in modal logic as well as in the MYCIN approach, while this is forbidden by the auto-duality of probabilities ($Prob(p) = 1 - Prob(\neg p)$).

Possibility theory has a remarkable situation among the settings devoted to the representation of imprecise and uncertain information. First, possibility theory may be numerical or qualitative [Dubois and Prade, 1998]. In the first case, possibility measures and the dual necessity measures can be regarded respectively as the upper bounds and the lower bounds of ill-known probabilities; they are also particular cases of plausibility and belief functions respectively [Shafer, 1976; Dubois and Prade, 1988]. In fact, possibility measures and necessity measures constitute the simplest, non trivial,

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imprecise probabilities system [Walley, 1996]. Second, when qualitative, possibility theory provides a natural approach to the grading of possibility and necessity modalities on finite ordinal scales. Besides, possibility theory has a logical counterpart, namely possibilistic logic [Dubois *et al.*, 1994c], which remains close to classical logic. In this overview chapter, we focus our attention on possibilistic logic.

Possibilistic logic is a weighted logic that handles uncertainty (but it also models preferences), in a qualitative way by associating certainty, or priority levels, to classical logic formulas. Moreover, possibilistic logic copes with inconsistency by taking advantage of the stratification of the set of formulas induced by the associated levels. These are the basic features of standard possibilistic logic. However, there exist further extensions of possibilistic logic that this paper also reports about. Since its introduction in the mid-eighties, multiple facets of possibilistic logic have been laid bare and various applications addressed, such as the handling of exceptions in default reasoning, the modeling of belief revision, the development of a graphical Bayesian-like network counterpart to a possibilistic logic base, or the representation of positive and negative information in a bipolar setting. The paper aims primarily at offering an introductory survey of possibilistic logic developments, but also outlines new research trends that are relevant in preference representation, or in reasoning about epistemic states. The intended purpose of this paper is to lay bare the basic ideas and the main inference methods. For more technical details and more examples, the reader is referred to the rich bibliography that is provided.

The chapter is structured as follows. We first provide a background on possibility theory. Then, the syntax and the semantics of basic possibilistic logic, where classical logic formulas are associated with lower bounds of necessity measures, are presented. Moreover, three other related formalisms, possibilistic networks, extensions of possibilistic logic for handling inconsistency, and possibilistic logic with symbolic weights are also surveyed. The next section proposes an overview of different applications of basic possibilistic logic to default reasoning, to causality ascription, to belief revision, to information fusion, to decision under uncertainty, to the handling of uncertainty in databases, and more briefly to description logic and to logic programming. The last section deals respectively i) with a multiple agent extension of possibilistic logic, ii) with bipolar representations that involve measures other than possibility and necessity and have applications in preference modeling, and finally iii) with generalized possibilistic logic, a two-tiered logic having a powerful representation power for modeling uncertain epistemic states, which can capture answer set programming.

2 QUALITATIVE POSSIBILITY THEORY - A REFRESHER

Possibility theory is an uncertainty theory devoted to the handling of incomplete information. To a large extent, it is comparable to probability theory because it is based on set-functions. It differs from the latter by the use of a pair of dual set functions (possibility and necessity measures) instead of only one. Besides, it is not additive and makes sense on ordinal structures. Before providing the basics of possibility theory, we start with a brief historical account of the idea of possibility.

2.1 Historical background

Zadeh [1978] was not the first scientist to speak about formalizing notions of possibility. The modalities *possible* and *necessary* have been used in philosophy at least since the Middle-Ages in Europe, based on Aristotle's and Theophrastus' works [Bocheński, 1947]. More recently these notions became the building blocks of modal logics that emerged at the beginning of the XX^{th} century from the works of C. I. Lewis (see [Chellas, 1980]). In this approach, possibility and necessity are all-or-nothing notions, and handled at the syntactic level. More recently, and independently from Zadeh's view, the notion of possibility, as opposed to probability, was central in the works of one economist, and in those of two philosophers.

Indeed a graded notion of possibility was introduced as a full-fledged approach to uncertainty and decision in the 1940-1970's by the English economist G. L. S. Shackle [1949; 1961], who called *degree of potential surprise* of an event its degree of impossibility, that is, retrospectively, the degree of necessity of the opposite event. Shackle's notion of possibility is basically epistemic, it is a "character of the chooser's particular state of knowledge in his present." Impossibility is understood as disbelief [Shackle, 1979]. Potential surprise is valued on a disbelief scale, namely a positive interval of the form $[0, y^*]$, where y^* denotes the absolute rejection of the event to which it is assigned. In case everything is possible, all mutually exclusive hypotheses have zero surprise. At least one elementary hypothesis must carry zero potential surprise. The degree of surprise of an event, a set of elementary hypotheses, is the degree of surprise of its least surprising realization. Shackle also introduces a notion of conditional possibility, whereby the degree of surprise of a conjunction of two events A and B is equal to the maximum of the degree of surprise of A, and of the degree of surprise of B, should A prove true. The disbelief notion introduced later by Spohn [1988; 2012] employs the same type of convention as potential surprise, but uses the set of natural integers as a disbelief scale; his conditioning rule uses the subtraction of natural integers.

In his 1973 book [Lewis, 1973] the philosopher David Lewis considers a graded notion of possibility in the form of a relation between possible worlds he calls *comparative possibility.* He connects this concept of possibility to a notion of similarity between possible worlds. This asymmetric notion of similarity is also comparative, and is meant to express statements of the form: *a world j is at least as similar to world i as world k is.* Comparative similarity of j and k with respect to i is interpreted as the comparative possibility of j with respect to k viewed from world i. Such relations are assumed to be complete pre-orderings and are instrumental in defining the truth conditions of counterfactual statements (of the form "If I were rich, I would buy a big boat"). Comparative possibility relations $\geq_\Pi$ obey the key axiom: for all events A, B, C,

$$A \geq_\Pi B \text{ implies } C \cup A \geq_\Pi C \cup B.$$

This axiom was later independently proposed by the first author [Dubois, 1986] in an attempt to derive a possibilistic counterpart to comparative probabilities. See also Grove [1988] who uses a so-called "system of spheres", with is nothing but an ordinal possibility distribution.

A framework very similar to the one of Shackle was also proposed by the philosopher L. J. Cohen [1977] who considered the problem of legal reasoning. He introduced so-called *Baconian probabilities* understood as degrees of provability (or degrees of "inductive support"). The idea is that it is hard to prove someone guilty at the court of law by means of pure statistical arguments. The basic feature of degrees of provability is that a hypothesis and its negation cannot both be provable together to any extent (the contrary being a case for inconsistency). Such degrees of provability coincide with what is known as necessity measures. I. Levi [1966; 1967], starting from Shackle's measures of surprise viewed as "measures contributing to the explication of what Keynes called 'weight of argument' " [Levi, 1979], also wrote a property identical to the minimum-based composition of necessity measures for conjunction, for so-called "degrees of confidence of acceptance".

Independently from the above works, Zadeh [1978] proposed an interpretation of membership functions of fuzzy sets as possibility distributions encoding flexible constraints induced by natural language statements. Zadeh tentatively articulated the relationship between possibility and probability, noticing that what is probable must preliminarily be possible. However, the view of possibility degrees developed in his paper refers to the idea of graded feasibility (degrees of ease, as in the example of "how many eggs can Hans eat for his breakfast") rather than to the epistemic notion of plausibility laid bare by Shackle. Nevertheless, the key axiom of "maxitivity" for possibility measures is highlighted. In two subsequent articles [Zadeh, 1979a; Zadeh, 1982], the author acknowledged the connection between possibility theory, belief functions and upper/lower probabilities, and proposed their extensions to fuzzy events and fuzzy information granules.

2.2 Basic notions of possibility theory.

The basic building blocks of possibility theory originate in Zadeh's paper [1978] and have been more extensively described and investigated in books by Dubois and Prade [1980; 1988]. See also [Dubois and Prade, 1998] for an introduction. Zadeh starts from the idea of a possibility distribution, to which he associates a possibility measure.

Possibility distributions Let U be a set of states of affairs (or descriptions thereof), or states for short. This set can be the domain of an attribute (numerical or categorical), the Cartesian product of attribute domains, the set of interpretation of a propositional language, etc. A possibility distribution is a mapping π from U to a totally ordered scale $\mathcal{S}$, with top denoted by 1 and bottom by 0. In the finite case $\mathcal{S} = \{1 = \lambda_1 > \dots \lambda_n > \lambda_{n+1} = 0\}$. The possibility scale can be the unit interval as suggested by Zadeh, or generally any finite chain, or even the set of non-negative integers[1]. It is assumed that $\mathcal{S}$ is equipped with an order-reversing map denoted by $\lambda \in \mathcal{S} \mapsto 1 - \lambda$. For a detailed discussion of different types of scales in a possibility theory perspective, the reader is referred to [Benferhat *et al.*, 2010].

The function π represents the state of knowledge of an agent (about the actual state of affairs), also called an *epistemic state* distinguishing what is plausible from what is less plausible, what is the normal course of things from what is not, what is surprising from what is expected. It represents a flexible restriction on what is the actual state of facts with the following conventions (similar to probability, but opposite to Shackle's potential surprise scale which refers to impossibility):

- $\pi(u) = 0$ means that state u is rejected as impossible;
- $\pi(u) = 1$ means that state u is totally possible (= plausible).

[1] If $\mathcal{S} = \mathbb{N}$, the conventions are opposite: 0 means possible and ∞ means impossible.

The larger $\pi(u)$, the more possible, i.e., plausible the state u is. Formally, the mapping π is the membership function of a fuzzy set [Zadeh, 1978], where membership grades are interpreted in terms of plausibility. If the universe U is exhaustive, at least one of the elements in S should be the actual world, so that $\exists u, \pi(u) = 1$ (normalization). This condition expresses the consistency of the epistemic state described by π.

Distinct values may simultaneously have a degree of possibility equal to 1. Moreover, as Shackle wrote, as early as 1949: "An outcome that we looked on as *perfectly possible* before is not rendered less possible by the fact that we have extended the list of perfectly possible outcomes" (see p. 114 in [Shackle, 1949]).

In the Boolean case, π is just the characteristic function of a subset $E \subseteq U$ of mutually exclusive states, ruling out all those states outside E considered as impossible. Possibility theory is thus a (fuzzy) set-based representation of incomplete information.

Specificity A possibility distribution π is said to be at least as specific as another π' if and only if for each state of affairs u: $\pi(u) \leq \pi'(u)$ [Yager, 1983]. Then, π is at least as restrictive and informative as π', since it rules out at least as many states with at least as much strength. In the possibilistic framework, extreme forms of partial knowledge can be captured, namely:

- *Complete knowledge*: for some $u_0, \pi(u_0) = 1$ and $\pi(u) = 0, \forall u \neq u_0$ (only u_0 is possible);
- *Complete ignorance*: $\pi(u) = 1, \forall u \in U$ (all states are possible).

Possibility theory is driven by the *principle of minimal specificity.* It states that *any hypothesis not known to be impossible cannot be ruled out.* It is a minimal commitment, cautious, information principle. Basically, we must always try to maximize possibility degrees, taking constraints into account.

Given a piece of information in the form x *is* F where F is a fuzzy set restricting the values of the ill-known quantity x, it leads to represent the knowledge by the inequality $\pi \leq \mu_F$, the membership function of F. The minimal specificity principle enforces possibility distribution $\pi = \mu_F$, if no other piece of knowledge is available. Generally there may be impossible values of x due to other piece(s) of information. Thus given several pieces of knowledge of the form x *is* F_i, for $i = 1, \ldots, n$, each of them translates into the constraint $\pi \leq \mu_{F_i}$; hence, several constraints lead to the inequality $\pi \leq \min_{i=1}^{n} \mu_{F_i}$ and on behalf of the minimal specificity principle, to the possibility distribution

$$\pi = \min_{i=1}^{n} \pi_i$$

where π_i is induced by the information item x *is* F_i ($\pi_i = \mu_{F_i}$). It justifies the use of the minimum operation for combining information items. It is noticeable that this way of combining pieces of information fully agrees with classical logic, since a classical logic base is equivalent to the logical conjunction of the logical formulas that belong to the base, and its models is obtained by intersecting the sets of models of its formulas. Indeed, in propositional logic, asserting a logical proposition a amounts to declaring that any interpretation (state) that makes a false is impossible, as being incompatible with the state of knowledge.

Possibility and necessity functions Given a simple query of the form "does event A occur?" (is the corresponding proposition a true?), where A is a subset of states, the set of models of a, the response to the query can be obtained by computing degrees of possibility [Zadeh, 1978] and necessity [Dubois and Prade, 1980], respectively (if the possibility scale $\mathcal{S} = [0, 1]$):

$$\Pi(A) = \sup_{u \in A} \pi(u); \quad N(A) = \inf_{s \notin A} 1 - \pi(u).$$

$\Pi(A)$ evaluates to what extent A is consistent with π, while $N(A)$ evaluates to what extent A is certainly implied by π. The possibility-necessity duality is expressed by $N(A) = 1 - \Pi(A^c)$, where A^c is the complement of A. Generally, $\Pi(U) = N(U) = 1$ and $\Pi(\emptyset) = N(\emptyset) = 0$ (since π is normalized to 1). In the Boolean case, the possibility distribution comes down to the disjunctive (epistemic) set $E \subseteq U$, and possibility and necessity are then as follows:

- $\Pi(A) = 1$ if $A \cap E \neq \emptyset$, and 0 otherwise: function Π checks whether proposition A is logically consistent with the available information or not.
- $N(A) = 1$ if $E \subseteq A$, and 0 otherwise: function N checks whether proposition A is logically entailed by the available information or not.

Possibility measures satisfy the characteristic "maxitivity" property

$$\Pi(A \cup B) = \max(\Pi(A), \Pi(B)).$$

Necessity measures satisfy an axiom dual to that of possibility measures, namely

$$N(A \cap B) = \min(N(A), N(B)).$$

On infinite spaces, these axioms must hold for infinite families of sets. As a consequence, of the normalization of π, $\min(N(A), N(A^c)) = 0$ and $\max(\Pi(A), \Pi(A^c)) = 1$, where A^c is the complement of A, or equivalently $\Pi(A) = 1$ whenever $N(A) > 0$, which totally fits the intuition behind this formalism, namely that something somewhat certain should be first fully possible, i.e. consistent with the available information. Moreover, one cannot be somewhat certain of both A and A^c, without being inconsistent. Note also that we only have $N(A \cup B) \geq \max(N(A), N(B))$. This goes well with the idea that one may be certain about the event $A \cup B$, without being really certain about more specific events such as A and B.

Certainty qualification Human knowledge is often expressed in a declarative way using statements to which belief degrees are attached. Certainty-qualified pieces of uncertain information of the form "A is certain to degree α" can then be modeled by the constraint $N(A) \geq \alpha$. It represents a family of possible epistemic states π that obey this constraint. The least specific possibility distribution among them exists and is defined by [Dubois and Prade, 1988]:

$$\pi_{(A,\alpha)}(u) = \left\{ \begin{array}{cc} 1 & \text{if } u \in A, \\ 1-\alpha & \text{otherwise.} \end{array} \right\}$$

If $\alpha = 1$ we get the characteristic function of A. If $\alpha = 0$, we get total ignorance. This possibility distribution is a key building-block to construct possibility distributions from several pieces of uncertain knowledge. It is instrumental in possibilistic logic semantics. Indeed, e.g. in the finite case, any possibility distribution can be viewed as a collection of nested certainty-qualified statements. Let $E_i = \{u : \pi(u) \geq \lambda_i \in L\}$ be the λ_i-cut of π. Then it is easy to check that $\pi(u) = \min_{i: s \notin E_i} 1 - N(E_i)$ (with convention $\min_{\emptyset} = 1$).

We can also consider possibility-qualified statements of the form $\Pi(A) \geq \beta$; however, the least specific epistemic state compatible with this constraint is trivial and expresses total ignorance.

Two other measures A measure of *guaranteed possibility* or *strong* possibility can be defined, that differs from the functions Π (*weak* possibility) and N (*strong* necessity) [Dubois and Prade, 1992; Dubois *et al.*, 2000]:

$$\Delta(A) = \inf_{u \in A} \pi(u).$$

It estimates to what extent *all* states in A are actually possible according to evidence. $\Delta(A)$ can be used as a degree of evidential support for A. Of course, this function possesses a dual conjugate ∇ such that $\nabla(A) = 1 - \Delta(A^c) = \sup_{u \notin A} 1 - \pi(u)$. Function $\nabla(A)$ evaluates the degree of potential or *weak* necessity of A, as it is 1 only if some state s out of A is impossible. It follows that the functions Δ and ∇ are *decreasing* with respect to set inclusion, and that they satisfy the characteristic properties $\Delta(A \cup B) = \min(\Delta(A), \Delta(B))$ and $\nabla(A \cap B) = \max(\nabla(A), \nabla(B))$ respectively.

Uncertain statements of the form "A is possible to degree β" often mean that any realization of A are possible to degree β (e.g. "it is possible that the museum is open this afternoon"). They can then be modeled by a constraint of the form $\Delta(A) \geq \beta$. It corresponds to the idea of observed evidence.

This type of information is better exploited by assuming an informational principle opposite to the one of minimal specificity, namely, *any situation not yet observed is tentatively considered as potentially impossible.* This is similar to the closed-world assumption. The *most specific* distribution $\delta_{(A,\beta)}$ in agreement with $\Delta(A) \geq \beta$ is:

$$\pi_{[A,\beta]}(u) = \left\{ \begin{array}{cc} \beta & \text{if } u \in A, \\ 0 & \text{otherwise.} \end{array} \right\}$$

Note that while possibility distributions induced from certainty-qualified pieces of knowledge combine conjunctively, by discarding possible states, evidential support distributions induced by possibility-qualified pieces of evidence combine disjunctively, by accumulating possible states. Given several pieces of knowledge of the form *x is F_i is possible* (in the sense of guaranteed or strong possibility), for $i = 1, \ldots, n$, each of them translates into the constraint $\pi \geq \mu_{F_i}$; hence, several constraints lead to the inequality $\pi \geq \max_{i=1}^n \mu_{F_i}$ and on behalf of a closed-world assumption-like principle based on maximal specificity, expressed by the possibility distribution

$$\pi = \max_{i=1}^{n} \pi_i$$

where π_i is induced by the information item *x is F_i is possible.* This principle justifies the use of the maximum operation for combining evidential support functions. Acquiring pieces of possibility-qualified evidence leads to updating $\pi_{[A,\beta]}$ into some wider distribution $\pi > \pi_{[A,\beta]}$. Any possibility distribution can be represented as a collection of nested possibility-qualified statements of the form $(E_i, \Delta(E_i))$, with $E_i = \{u : \pi(u) \geq \lambda_i\}$, since $\pi(u) = \max_{i:u \in E_i} \Delta(E_i)$, dually to the case of certainty-qualified statements.

The possibilistic cube of opposition Interestingly enough, it has been shown [Dubois and Prade, 2012] that the four set function evaluations of an event A and the four evaluations of its opposite A^c can be organized in a cube of opposition (see Figure 1), whose front and back facets are graded extension of the traditional square of opposition [Parsons, 1997].

Counterparts of the characteristic properties of the square of opposition do hold. First, the diagonals (in dotted lines) of these facets link dual measures through the involutive order-reversing function $1-(\cdot)$. The vertical edges of the cube, as well as the diagonals of the side facets, which are bottom-oriented arrows, correspond to entailments here expressed by inequalities. Indeed, provided that π and $1-\pi$ are both normalized, we have for all A,

$$\max(N(A), \Delta(A)) \leq \min(\Pi(A), \nabla(A)).$$

The thick black lines of the top facets express mutual exclusiveness under the form $\min(N(A), N(A^c)) = \min(\Delta(A), \Delta(A^c)) = \min(N(A), \Delta(A^c)) = \min(\Delta(A), N(A^c)) = 0$. Dually, the double lines of the bottom facet correspond to $\max(\Pi(A), \Pi(A^c)) = \max(\nabla(A), \nabla(A^c)) = \max(\Pi(A), \nabla(A^c)) = \max(\nabla(A), \Pi(A^c)) = 1$. Thus, the cube in Figure 1 summarizes the interplay between the different measures in possibility theory.

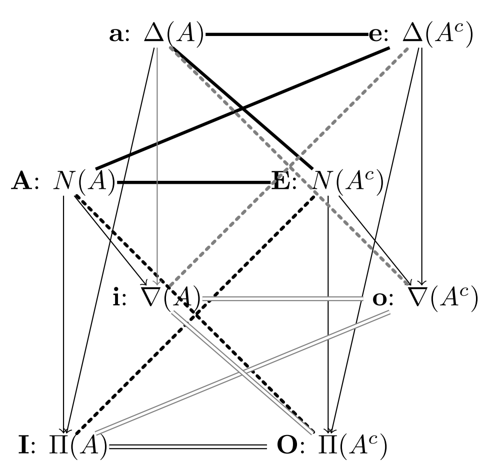

Figure 1. The cube of opposition of possibility theory

3 NECESSITY-BASED POSSIBILISTIC LOGIC AND RELATED ISSUES

Possibilistic logic has been developed for about thirty years; see [Dubois and Prade, 2004] for historical details. Basic possibilistic logic (also called standard possibilistic logic) has been first introduced in artificial intelligence as a tool for handling uncertainty in a qualitative way in a logical setting. Later on, it has appeared that basic possibilistic logic can also be used for representing preferences [Lang, 1991]. Then, each logic formula represents a goal to be reached with its priority level (rather than a statement that is believed to be true with some certainty level). Possibilistic logic heavily relies on the notion of necessity measure, but may be also related [Dubois *et al.*, 1991a] to Zadeh's theory of approximate reasoning [Zadeh, 1979b].

A basic possibilistic logic formula is a pair (a, α) made of a classical logic formula a associated with a certainty level $\alpha \in (0, 1]$, viewed as a lower bound of a *necessity measure*, i.e., (a, α) is understood as $N(a) \geq \alpha$. Formulas of the form $(a, 0)$, which do not contain any information ($N(a) \geq 0$ always holds), are not part of the possibilistic language. As already said, the interval [0,1] can be replaced by any linearly ordered scale. Since necessity measures N are monotonic functions w. r. t. entailment, i.e. if $a \models b$ then $N(a) \leq N(b)$. The min decomposability property of necessity measures for conjunction, i.e., $N(a \wedge b) = \min(N(a), N(b))$, expresses that to be certain about $a \wedge b$, one has to be certain about a and to be certain about b. Thanks to this decomposability property, a possibilistic logic base can be always put in a clausal equivalent form.

An interesting feature of possibilistic logic is its ability to deal with inconsistency. Indeed a possibilistic logic base Γ, i.e. a set of possibilistic logic formulas, viewed as a conjunction thereof, is associated with an inconsistency level $inc\text{-}l(\Gamma)$, which is such that the formulas associated with a level strictly greater than $inc\text{-}l(\Gamma)$ form a consistent subset of formulas. A possibilistic logic base is semantically equivalent to a possibility distribution that restricts the set of interpretations (w. r. t. the considered language) that are more or less compatible with the base. Instead of an ordinary subset of models as in classical logic, we have a fuzzy set of models, since the violation by an interpretation of a formula that is not fully certain (or imperative) does not completely rule out the interpretation.

The certainty-qualified statements of possibilistic logic have a clear modal flavor. Possibilistic logic can also be viewed as a special case of a labelled deductive system [Gabbay, 1996]. Inference in possibilistic logic propagates certainty in a qualitative manner, using the law of the weakest link, and is inconsistency-tolerant, as it enables non-trivial reasoning to be performed from the largest consistent subset of most certain formulas. A characteristic feature of this uncertainty theory is that a set of propositions $\{a \in \mathcal{L} : N(a) \geq \alpha\}$, in a propositional language $\mathcal{L}$, that are believed at least to a certain extent is deductively closed (thanks to the min-decomposability of necessity measures with respect to conjunction). As a consequence, possibilistic logic remains very close to classical logic.

From now on, we shall use letters such as a, b, c for denoting propositional formulas, and letters such as p, q, r will denote atomic formulas.

This section is organized in four main subparts. First, the syntactic and semantic aspects of basic possibilistic logic are presented. Then we briefly survey extended inference machineries that take into account formulas that are "drowned" under the inconsistency level, as well as an extension of possibilistic logic that handles certainty levels in a symbolic manner, which allows for a partially known ordering of these levels. Lastly, we review another noticeable, Bayesian-like, representation framework, namely possibilistic networks. They are associated to a possibility distribution decomposed thanks to conditional independence information, and provide a graphical counterpart to possibilistic logic bases to which they are semantically equivalent.

3.1 Basic possibilistic logic

Basic possibilistic logic [Dubois *et al.*, 1994c] has been mainly developed as a formalism for handling qualitative uncertainty (or preferences) with an inference mechanism that is a simple extension of classical logic. A possibilistic logic formula is a pair made of i) any well-formed classical logic formula, propositional or first-ordered, and ii) a weight expressing its certainty or priority. Its classical logic component can be only true or false: fuzzy statements with intermediary degrees of truth are not allowed in standard possibilistic logic (although extensions exist for handling fuzzy predicates [Dubois *et al.*, 1998b; Alsinet and Godo., 2000; Alsinet *et al.*, 2002]).

Syntactic aspects In the following, we only consider the case of (basic) possibilistic *propositional* logic, ΠL for short, i.e., possibilistic logic formulas (a, α) are such that a is a formula in a propositional language; for (basic) possibilistic *first order* logic, the reader is referred to [Dubois *et al.*, 1994c].

Axioms and inference rules. The axioms of ΠL, [Dubois *et al.*, 1994c], are those of propositional logic, *PL* for short, where each axiom schema is now supposed to hold with the maximal certainty, i.e. is associated with level 1. It has two inference rules:

- if $\beta \leq \alpha$ then $(a, \alpha) \vdash (a, \beta)$ (certainty weakening)
- $(\neg a \vee b, \alpha), (a, \alpha) \vdash (b, \alpha)$, $\forall \alpha \in (0, 1]$ (modus ponens)

We may equivalently use the certainty weakening rule with the ΠL counterpart of the resolution rule:

$$(\neg a \vee b, \alpha), (a \vee c, \alpha) \vdash (b \vee c, \alpha),\ \forall \alpha \in (0, 1] \text{ (resolution)}$$

Using certainty weakening, it is then easy to see that the following inference rule is valid

$$(\neg a \vee b, \alpha), (a \vee c, \beta) \vdash (b \vee c, \min(\alpha, \beta)) \text{ (weakest link resolution)}$$

The idea that in a reasoning chain, the certainty level of the conclusion is the smallest of the certainty levels of the formulas involved in the premises is at the basis of the syntactic approach proposed by [Rescher, 1976] for plausible reasoning, and would date back to Theophrastus, an Aristotle's follower.

The following inference rule we call *formula weakening* holds also as a consequence of α-β-resolution.

$$\text{if } a \vdash b \text{ then } (a, \alpha) \vdash (b, \alpha),\ \forall \alpha \in (0, 1] \text{ (formula weakening)}$$

Indeed $a \vdash b$ expresses that $\neg a \vee b$ is valid in *PL* and thus $(\neg a \vee b, 1)$ holds, which by applying the α-β-resolution rule with (a, α) yields the result.

It turns out that any valid deduction in propositional logic is valid in possibilistic logic as well where the corresponding propositions are associated with any level $\alpha \in (0, 1]$. Thus since $a, b \vdash a \wedge b$, we have $(a, \alpha), (b, \alpha) \vdash (a \wedge b, \alpha)$. Note that we also have $(a \wedge b, \alpha) \vdash (a, \alpha)$ and $(a \wedge b, \alpha) \vdash (b, \alpha)$ by the formula weakening rule. Thus, stating $(a \wedge b, \alpha)$ is equivalent to stating (a, α) and (b, α). Thanks to this property, it is always possible to rewrite a ΠL base under the form of a collection of weighted *clauses*.

Note also that if we assume that for any propositional tautology t, i.e., such that $t \equiv \top$, (t, α) holds with any certainty level, which amounts to saying that each *PL* axiom holds with any certainty level, then the α-β-resolution rule entails the level weakening rule, since $(\neg a \vee a, \beta)$ together with $(a \vee c, \alpha)$ entails $(a \vee c, \beta)$ with $\beta \leq \alpha$.

Inference and consistency. Let $\Gamma = \{(a_i, \alpha_i), i = 1, ..., m\}$ be a set of possibilistic formulas. Inference in ΠL from a base Γ is quite similar to the one in *PL*. We may either use the ΠL axioms, certainty weakening and modus ponens rules, or equivalently proceed by refutation (proving

$\Gamma \vdash (a, \alpha)$ amounts to proving $\Gamma, (\neg a, 1) \vdash (\bot, \alpha)$ by repeated application of the weakest link-resolution rule, where Γ stands for a collection of ΠL formulas $(a_1, \alpha_1), ..., (a_m, \alpha_m)$. Moreover, note that

$$\Gamma \vdash (a, \alpha) \text{ if and only if } \Gamma_\alpha \vdash (a, \alpha) \text{ if and only if } (\Gamma_\alpha)^* \vdash a$$

where $\Gamma_\alpha = \{(a_i, \alpha_i) \in \Gamma, \alpha_i \geq \alpha\}$ and $\Gamma^* = \{a_i \mid (a_i, \alpha_i) \in \Gamma\}$. The certainty levels stratify the knowledge base Γ into nested level cuts Γ_α, i.e. $\Gamma_\alpha \subseteq \Gamma_\beta$ if $\beta \leq \alpha$. A consequence (a, α) from Γ can only be obtained from formulas having a certainty level at least equal to α, so from formulas in Γ_α; then a is a classical consequence from the *PL* knowledge base $(\Gamma_\alpha)^*$, and $\alpha = \max\{\beta | (\Gamma_\beta)^* \vdash a\}$.

The *inconsistency level* of Γ is defined by

$$\textit{inc-l}(\Gamma) = \max\{\alpha \mid \Gamma \vdash (\bot, \alpha)\}.$$

The possibilistic formulas in Γ whose level is strictly above $\textit{inc-l}(\Gamma)$ are safe from inconsistency, namely $\textit{inc-l}(\{(a_i, \alpha_i) \mid (a_i, \alpha_i) \in \Gamma \text{ and } \alpha_i > \textit{inc-l}(\Gamma)\}) = 0$. Indeed, if $\alpha > \textit{inc-l}(\Gamma)$, $(\Gamma_\alpha)^*$ is consistent. In particular, we have the remarkable property that the classical consistency of Γ^* is equivalent to saying that Γ has a level of inconsistency equal to 0. Namely,

$$\textit{inc-l}(\Gamma) = 0 \text{ if and only if } \Gamma^* \text{ is consistent.}$$

Semantic aspects The semantics of ΠL [Dubois *et al.*, 1994c] is expressed in terms of possibility distributions, (weak) possibility measures and (strong) necessity measures. Let us first consider a ΠL formula (a, α) that encodes the statement $N(a) \geq \alpha$. Its semantics is given by the following possibility distribution $\pi_{(a,\alpha)}$ defined, in agreement with the formula of the certainty qualification in Section 2.2, by:

$$\pi_{(a,\alpha)}(\omega) = 1 \text{ if } \omega \vDash a \text{ and } \pi_{(a,\alpha)}(\omega) = 1 - \alpha \text{ if } \omega \vDash \neg a$$

Intuitively, the underlying idea is that any model of a should be fully possible, and that any interpretation that is a counter-model of a, is all the less possible as a is more certain, i.e. as α is higher. When $\alpha = 0$, the (trivial) information $N(a) \geq 0$ is represented by $\pi_{(a,0)} = 1$, and the formula $(a, 0)$ can be ignored. It can be easily checked that the associated necessity measure is such that $N_{(a,\alpha)}(a) = \alpha$, and $\pi_{(a,\alpha)}$ is the least informative possibility distribution (i.e. maximizing possibility degrees) such that this constraint holds. In fact, any possibility distribution π such that $\forall \omega, \pi(\omega) \leq \pi_{(a,\alpha)}(\omega)$ is such that its associated necessity measure N satisfies $N(a) \geq N_{(a,\alpha)}(a) = \alpha$ (hence is more committed).

Due to the min-decomposability of necessity measures, $N(\bigwedge_i a_i) \geq \alpha \Leftrightarrow \forall i, N(a_i) \geq \alpha$, and then any possibilistic propositional formula can be put in clausal form. Let us now consider a ΠL knowledge base $\Gamma = \{(a_i, \alpha_i), i = 1, ..., m\}$, thus corresponding to the conjunction of ΠL formulas (a_i, α_i), each representing a constraint $N(a_i) \geq \alpha_i$. The base Γ is semantically associated with the possibility distribution:

$$\pi_\Gamma(\omega) = \min_{i=1,\ldots,m} \pi_{(a_i,\alpha_i)}(\omega) = \min_{i=1,\ldots,m} \max([a_i](\omega), 1 - \alpha_i)$$

where $[a_i]$ is the characteristic function of the models of a_i, namely $[a_i](\omega) = 1$ if $\omega \vDash a_i$ and $[a_i](\omega) = 0$ otherwise. Thus, the least informative induced possibility distribution π_Γ is obtained as the min-based conjunction of the fuzzy sets of interpretations (with membership functions $\pi_{(a_i,\alpha_i)}$), representing each formula. It can be checked that

$$N_\Gamma(a_i) \geq \alpha_i \text{ for i=1,\ldots,m},$$

where N_Γ is the necessity measure defined from π_Γ. Note that we may only have an inequality here since Γ may, for instance, include two formulas associated to equivalent propositions, but with distinct certainty levels.

Remark 1. Let us mention that a similar construction can be made in an additive setting where each formula is associated with a cost (in $\mathbb{N} \cup \{+\infty\}$), the weight (cost) attached to an interpretation being the sum of the costs of the formulas in the base violated by the interpretation, as in penalty logic [Dupin de Saint Cyr *et al.*, 1994; Pinkas, 1991]. The so-called "cost of consistency" of a formula is then defined as the minimum of the weights of its models (which is nothing but a ranking function in the sense of Spohn [1988], or the counterpart of a possibility measure defined on $\mathbb{N} \cup \{+\infty\}$ where now 0 expresses full possibility, and $+\infty$ complete impossibility since it is a cost that cannot be paid).

So a ΠL knowledge base is understood as a set of constraints $N(a_i) \geq \alpha_i$ for $i = 1, \ldots, m$, and the set of possibility distributions π associated with N that are compatible with this set of constraints has a largest element which is nothing but π_Γ, i.e. we have $\forall \omega, \pi(\omega) \leq \min_{i=1,\ldots,m} \pi_{(a_i,\alpha_i)} = \pi_\Gamma(\omega)$. Thus, the possibility distribution π_Γ representing semantically a ΠL base Γ, is the one assigning the largest possibility degree to each interpretation, in agreement with the semantic constraints $N(a_i) \geq \alpha_i$ for $i = 1, \ldots, m$ that are associated with the formulas (a_i, α_i) in Γ. Thus, any possibility distribution $\pi \leq \pi_\Gamma$ semantically agrees with Γ, which can be written $\pi \vDash \Gamma$.

The semantic entailment is defined by

$$\Gamma \vDash (a, \alpha) \text{ if and only if } \forall \omega, \pi_\Gamma(\omega) \leq \pi_{\{(a,\alpha)\}}(\omega).$$

It can be shown [Dubois *et al.*, 1994c] that possibilistic logic is sound and complete wrt this semantics, namely

$$\Gamma \vdash (a, \alpha) \text{ if and only if } \Gamma \models (a, \alpha).$$

Moreover, we have $$inc\text{-}l(\Gamma) = 1 - \max_{\omega \in \Omega} \pi_\Gamma(\omega),$$

which acknowledges the fact that the normalization of π_Γ is equivalent to the classical consistency of Γ^*. Thus, an important feature of possibilistic logic is its ability to deal with inconsistency. The consistency of Γ is estimated by the extent to which there is at least one completely possible interpretation for Γ, i.e. by the quantity $cons\text{-}l(\Gamma) = 1 - inc\text{-}l(\Gamma) = \max_{\omega \in \Omega} \pi_\Gamma(\omega) = \max_{\pi \models \Gamma} \max_{\omega \in \Omega} \pi(\omega)$ (where $\pi \models \Gamma$ if and only if $\pi \le \pi_\Gamma$).

EXAMPLE 1. Let us illustrate the previously introduced notions on the following ΠL base Γ, which is in clausal form (p, q, r are atoms): $\{(\neg p \vee q, 0.8), (\neg p \vee r, 0.9), (\neg p \vee \neg r, 0.1), (p, 0.3), (q, 0.7), (\neg q, 0.2), (r, 0.8)\}$.

First, it can be checked that $inc\text{-}l(\Gamma) = 0.2$.

Thus, the sub-base $\Gamma_{0.3} = \{(\neg p \vee q, 0.8), (\neg p \vee r, 0.9), (p, 0.3), (q, 0.7), (r, 0.8)\}$ is safe from inconsistency, and its deductive closure is consistent, i.e. $\nexists\, a$, $\nexists\, \alpha > 0, \nexists\, \beta > 0$ such that $\Gamma_{0.3} \vdash (a, \alpha)$ and $\Gamma_{0.3} \vdash (\neg a, \beta)$. By contrast, $\Gamma_{0.1} \vdash (\neg r, 0.1)$ and $\Gamma_{0.1} \vdash (r, 0.8)$. Note also that, while $(\neg p \vee r, 0.9), (p, 0.3) \vdash (r, 0.3)$, we clearly have $\Gamma \vdash (r, 0.8)$ also. This illustrates the fact that in possibilistic logic, we are interested in practice in the proofs having the strongest weakest link, and thus leading to the highest certainty levels. Besides, in case Γ contains $(r, 0.2)$ rather than $(r, 0.8)$, then $(r, 0.2)$ is of no use, since subsumed by $(r, 0.3)$. Indeed, it can be checked that $\Gamma \setminus \{(r, 0.8)\}$ and $\Gamma \setminus \{(r, 0.8)\} \cup \{(r, 0.2)\}$ are associated to the same possibility distribution.

The possibility distribution associated with Γ, whose computation is detailed in Table 1, is given by $\pi_\Gamma(pqr) = 0.8$; $\pi_\Gamma(\neg pqr) = 0.7$; $\pi_\Gamma(\neg p \neg qr) = 0.3$; $\pi_\Gamma(p \neg qr) = \pi_\Gamma(\neg pq \neg r) = \pi_\Gamma(\neg p \neg q \neg r) = 0.2$; $\pi_\Gamma(pq \neg r) = \pi_\Gamma(p \neg q \neg r) = 0.1$.

ω	$\pi_{(\neg p \vee q, .8)}$	$\pi_{(\neg p \vee r, .9)}$	$\pi_{(\neg p \vee \neg r, .1)}$	$\pi_{(p, .3)}$	$\pi_{(q, .7)}$	$\pi_{(\neg q, .2)}$	$\pi_{(r, .8)}$	π_Γ
pqr	1	1	0.9	1	1	0.8	1	0.8
$pq\neg r$	1	0.1	1	1	1	0.8	0.2	0.1
$p\neg qr$	0.2	1	0.9	1	0.3	1	1	0.2
$p\neg q\neg r$	0.2	0.1	1	1	0.3	1	0.2	0.1
$\neg pqr$	1	1	1	0.7	1	0.8	1	0.7
$\neg pq\neg r$	1	1	1	0.7	1	0.8	0.2	0.2
$\neg p\neg qr$	1	1	1	0.7	0.3	1	1	0.3
$\neg p\neg q\neg r$	1	1	1	0.7	0.3	1	0.2	0.2

Table 1.Detailed computation of the possibility distribution in the example

As can be seen $cons\text{-}l(\Gamma) = \max_{\omega\in\Omega} \pi_\Gamma(\omega) = 0.8$ and $inc\text{-}l(\Gamma) = 1-0.8 = 0.2$. Similarly, we have $inc\text{-}l(\Gamma \setminus \{(\neg q, 0.2)\}) = 0.1$
and $inc\text{-}l(\Gamma \setminus \{(\neg q, 0.2), (\neg p \vee \neg r, 0.1)\}) = 0$. □

Remark 2. Using the weakest link-resolution rule repeatedly, leads to a refutation-based proof procedure that is sound and complete w. r. t. the semantics exists for propositional possibilistic logic [Dubois *et al.*, 1994c]. It exploits an adaptation of an A* search algorithm in to order to reach the empty clause with the greatest possible certainty level [Dubois *et al.*, 1987]. Algorithms and complexity evaluation (similar to the one of classical logic) can be found in [Lang, 2001].

Remark 3. It is also worth pointing out that a similar approach with lower bounds on probabilities would not ensure completeness [Dubois *et al.*, 1994a]. Indeed the repeated use of the probabilistic counterpart of the above resolution rule, namely $(\neg a \vee b, \alpha); (a \vee c, \beta) \models (b \vee c, \max(0, \alpha + \beta - 1))$ (where (d, α) here means $Probability(d) \geq \alpha$), is not always enough for computing the best probability lower bounds on a formula, given a set of probabilistic constraints of the above form. This is due to the fact that a set of formulas all having a probability at least equal to α is not deductively closed in general (except if $\alpha = 1$).

Remark 4. Moreover, a formula such as $(\neg a \vee b, \alpha)$ can be rewritten under the semantically equivalent form $(b, \min({}^t(a), \alpha))$, where ${}^t(a) = 1$ if a is true and ${}^t(a) = 0$ if p is false. This latter formula now reads "b is α-certain, provided that a is true" and can be used in hypothetical reasoning in case (a, γ) is not deducible from the available information (for some $\gamma > 0$) [Benferhat *et al.*, 1994a; Dubois and Prade, 1996].

Formulas associated with lower bounds of possibility A piece of information of the form (a, α) (meaning $N(a) \geq \alpha$) is also semantically equivalent to $\Pi(\neg a) \leq 1 - \alpha$, i.e. in basic possibilistic logic we are dealing with upper bounds of possibility measures. Formulas associated with lower bounds of possibility measures (rather than necessity measures) have been also introduced [Dubois and Prade, 1990b; Lang *et al.*, 1991; Dubois *et al.*, 1994c]. A possibility-necessity resolution rule then governs the inference:

$$N(a \vee b) \geq \alpha, \Pi(\neg a \vee c) \geq \beta \vDash \Pi(b \vee c) \geq \alpha * \beta$$

with $\alpha * \beta = \alpha$ if $\beta > 1 - \alpha$ and $\alpha * \beta = 0$ if $1 - \alpha \geq \beta$.[2]

[2]Noticing that $\Pi(\neg a \wedge \neg b) \leq 1-\alpha$, and observing that $\neg a \vee c \equiv (\neg a \wedge \neg b) \vee [(\neg a \wedge b) \vee c]$, we have $\beta \leq \Pi(\neg a \vee c) \leq \max(1 - \alpha, \Pi(b \vee c))$. Besides, $\beta \leq \max(1 - \alpha, x) \Leftrightarrow x \geq \alpha * \beta$, where $*$ is a (non-commutative) conjunction operator associated by residuation to the mul tiple-valued implication $\alpha \rightarrow x = \max(1-\alpha, x) = \inf\{\beta \in [0,1] | \alpha * \beta \leq x\}$. Hence the result.

This rule and the weakest link-resolution rule of basic possibilistic logic are graded counterparts of two inference rules well-known in modal logic [Fariñas del Cerro, 1985].

The possibility-necessity resolution rule can be used for handling partial ignorance, where fully ignoring a amounts to writing that $\Pi(a) = 1 = \Pi(\neg a)$. This expresses "alleged ignorance" and corresponds more generally to the situation where $\Pi(a) \geq \alpha > 0$ and $\Pi(\neg a) \geq \beta > 0$. This states that both a and $\neg a$ are somewhat *possible*, and contrasts with the type of uncertainty encoded by (a, α), which expresses that $\neg a$ is rather *impossible*. Alleged ignorance can be transmitted through equivalences. Namely from $\Pi(a) \geq \alpha > 0$ and $\Pi(\neg a) \geq \beta > 0$, one can deduce $\Pi(b) \geq \alpha > 0$ and $\Pi(\neg b) \geq \beta > 0$ provided that we have $(\neg a \vee b, 1)$ and $(a \vee \neg b, 1)$ [Dubois and Prade, 1990b; Prade, 2006].

3.2 Reasoning under inconsistency

As already emphasized, an important feature of possibilistic logic is its ability to deal with inconsistency [Dubois and Prade, 2011b]. Indeed all formulas whose level is strictly greater than *inc*-$l(\Gamma)$ are safe from inconsistency in a possibilistic logic base Γ. But, any formula in Γ whose level is less or equal to *inc*-$l(\Gamma)$ is ignored in the standard possibilistic inference process; these formulas are said to be "drowned". However, other inferences that salvage formulas that are below the level of inconsistency, but are not involved in some inconsistent subsets of formulas, have been defined and studied; see [Benferhat *et al.*, 1999a] for a survey. One may indeed take advantage of the weights for handling inconsistency in inferences, while avoiding the drowning effect (at least partially). The main approaches are now reviewed.

Degree of paraconsistency and safely supported-consequences An extension of the possibilistic inference has been proposed for handling paraconsistent information [Dubois *et al.*, 1994b; Benferhat *et al.*, 1999a]. It is defined as follows. First, for each formula a such that (a, α) is in Γ, we extend the language and compute triples (a, β, γ) where β (resp. γ) is the highest degree with which a (resp. $\neg a$) is supported in Γ. More precisely, a is said to be *supported in* Γ *at least at degree* β if there is a *consistent* sub-base of $(\Gamma_\beta)^*$ that entails a, where $\Gamma_\beta = \{(a_i, \alpha_i) | \alpha_i \geq \beta\}$. Let Γ^o denote the set of bi-weighted formulas which is thus obtained.

EXAMPLE 2. Take

$\Gamma = \{(p, 0.8), (\neg p \vee q, 0.6), (\neg p, 0.5), (\neg r, 0.3), (r, 0.2), (\neg r \vee q, 0.1)\}$.

Then, $\Gamma^o =$
$\{(a,0.8,0.5), (\neg p,0.5,0.8), (\neg r,0.3,0.2), (r,0.2,0.3), (\neg p \vee q,0.6,0), (\neg r \vee q,0.6,0)\}$.

Indeed consider, e.g., $(\neg r \vee q, 0.6, 0)$. Then we have that $(p, 0.8)$ and $(\neg p \vee q, 0.6)$ entail $(q, 0.6)$ (modus ponens), which implies $(\neg r \vee q, 0.6, 0)$ (logical weakening); it only uses formulas above the level of inconsistency 0.5; but there is no way to derive $\neg q$ from any consistent subset of Γ^*; so $\gamma = 0$ for $\neg r \vee q$. □

A formula (a, β, γ) is said to have a *paraconsistency degree* equal to $\min(\beta, \gamma)$. In particular, the formulas of interest are such that $\beta \geq \gamma$, i.e. the formula is at least as certain as it is paraconsistent. Clearly the set of formulas of the form $(a, \beta, 0)$ in Γ^o has an inconsistency level equal to 0, and thus leads to safe conclusions. However, one may obtain a larger set of consistent conclusions from Γ^o as explained now.

Defining an inference relation from Γ^o requires two evaluations:
- the *undefeasibility* degree of a consistent set A of formulas:

$$UD(A) = \min\{\beta \mid (a, \beta, \gamma) \in \Gamma^o \text{ and } a \in A\}$$

- the *unsafeness* degree of a consistent set A of formulas:

$$US(A) = \max\{\gamma | (a, \beta, \gamma) \in \Gamma^o \text{ and } a \in A\}$$

We say that A is a reason (or an argument) for b if A is a minimal (for set inclusion) consistent subset of Γ that implies b, i.e.,

- $A \subseteq \Gamma$
- $A^* \nvdash_{PL} \bot$
- $A^* \vdash_{PL} b$
- $\forall B \subset A, B^* \nvdash_{PL} b$

Let $label(b) = \{(A, UD(A), US(A)) \mid A$ is a reason for $b\}$, and $label(b)^* = \{A \mid (A,\ UD(A), US(A)) \in label(b)\}$. Then $(b, UD(A'), US(A'))$ is said to be a DS-consequence of Γ^o (or Γ), denoted by $\Gamma^o \vdash_{DS} (b, UD(A'), US(A'))$, if and only if $UD(A') > US(A')$, where A' is maximizing $UD(A)$ in $label(b)^*$ and in case of several such A', the one which minimizes $US(A')$. It can be shown that $\vdash_{DS}$ extends the entailment in possibilistic logic [Benferhat *et al.*, 1999a].

EXAMPLE 3. (Example 2 continued) In the above example, $label(b) = \{(A, 0.6, 0.5), (B, 0.2, 0.3)\}$ with $A = \{(p, 0.8, 0.5), (\neg p \vee q, 0.6, 0)\}$ and $B = \{(r, 0.2, 0.3), (\neg r \vee q, 0.6, 0)\}$. Then, $\Gamma^o \vdash_{DS} (q, 0.6, 0.5)$. □

But, if we rather first minimize $US(A')$ and then maximize $UD(A')$, the entailment would not extend the possibilistic entailment. Indeed in the above example, we would select $(B, 0.2, 0.3)$ but $0.2 > 0.3$ does not hold, while $\Gamma \vdash (q, 0.6)$ since $0.6 > inc\text{-}l(\Gamma) = 0.5$.

Note also that $\vdash_{DS}$ is more productive than the possibilistic entailment, as seen on the example, e.g., $\Gamma^o \vdash_{DS} (\neg r, 0.3, 0.2)$, while $\Gamma \vdash (\neg r, 0.3)$ does not hold since $0.3 < inc\text{-}l(\Gamma) = 0.5$.

Another entailment denoted by $\vdash_{SS}$, named safely supported consequence relation, less demanding than $\vdash_{DS}$, is defined by $\Gamma^o \vdash_{SS} b$ if and only $\exists A \in label(b)$ such that $UD(A) > US(A)$. It can be shown that the set $\{b \mid \Gamma^o \vdash_{SS} b\}$ is classically consistent [Benferhat *et al.*, 1999a].

Another proposal that sounds natural, investigated in [Benferhat *et al.*, 1993a] is the idea of argued consequence $\vdash_{\mathcal{A}}$, where $\Gamma \vdash_{\mathcal{A}} b$ if there exists a reason A for b stronger than any reason B for $\neg b$ in the sense that the possibilistic inference from A yields b with a level strictly greater than the level obtained for $\neg b$ from any reason B. $\vdash_{\mathcal{A}}$ is more productive than $\vdash_{SS}$. Unfortunately, $\vdash_{\mathcal{A}}$ may lead to classically inconsistent of conclusions. There are several other inference relations that have been defined, in particular using a selection of maximal consistent sub-bases based on the certainty or priority levels. However these approaches are more adventurous than $\vdash_{SS}$ and may lead to debatable conclusions. See for details [Benferhat *et al.*, 1999a].

From quasi-classical logic to quasi-possibilistic logic Besnard and Hunter [1995] [Hunter, 2000] have defined a new paraconsistent logic, called quasi-classical logic. This logic has several nice features, in particular the connectives behave classically, and when the knowledge base is classically consistent, then quasi-classical logic gives almost the same conclusions as classical logic (with the exception of tautologies or formulas containing tautologies). Moreover, the inference in quasi-classical logic has a low computational complexity. The basic ideas behind this logic is to use all rules of classical logic proof theory, but to forbid the use of resolution after the introduction of a disjunction (it allows to get rid of the *ex falso quodlibet sequitur*). So the rules of quasi-classical logic are split into two classes: composition and decomposition rules, and the proofs cannot use decomposition rules once a composition rule has been used. Intuitively speaking, this means that we may have resolution-based proofs both for a and $\neg a$. We also derive the disjunctions built from such previous consequences (e.g. $\neg a \vee b$) as additional valid consequences. But it is forbidden to reuse such additional consequences for building further proofs [Hunter, 2000].

Although possibilistic logic takes advantage of its levels for handling inconsistency, there are situations where it offers no useful answers, while quasi-classical logic does. This is when formulas involved in inconsistency have the same level, especially the highest one, 1. Thus, consider the example $\Gamma = \{(p, 1), (\neg p \vee q, 1), (\neg p, 1)\}$, where quasi-classical logic infers a, $\neg a$, b from Γ^*, while everything is drowned in possibilistic logic, and nothing can be derived by the safely supported consequence relation. This has led to the preliminary proposal of a quasi-possibilistic logic [Dubois *et al.*, 2003]. It is to be compared to the generalized inference rule, applicable to formulas in Γ^o, $(\neg p \vee q, \beta, \gamma)$ $(p \vee r, \beta', \gamma') \vdash (q \vee r, \min(\beta, \beta'), \max(\gamma, \gamma'))$, proposed in [Dubois *et al.*, 1994b]. Note that in the above example, we obtain $(p, 1, 1)$, $(\neg p, 1, 1)$ and $(q, 1, 1)$, as expected, by applying this rule. Such concerns should also be related to Belnap's four-valued logic which leaves open the possibility that a formula and its negation be supported by distinct sources [Belnap, 1977; Dubois, 2012].

3.3 Possibilistic logic with symbolic weights

In basic possibilistic logic, the certainty levels associated to formulas are assumed to belong to a totally ordered scale. In some cases, their value and their relative ordering may be unknown, and it may be then of interest to process them in a purely symbolic manner, i.e. computing the level from a derived formula as a symbolic expression. For instance, $\Gamma = \{(p, \alpha), (\neg p \vee q, \beta), (q, \gamma)\} \vdash (q, \max(\min(\alpha, \beta), \gamma))$. This induces a partial order between formulas based on the partial order between symbolic weights (e.g., $\max(\min(\alpha, \beta), \alpha, \gamma) \geq \min(\alpha, \delta)$ for any values of $\alpha, \beta, \gamma, \delta$).

Possibilistic logic formulas with symbolic weights have been used in preference modeling [Dubois *et al.*, 2006; Dubois *et al.*, 2013b; Dubois *et al.*, 2013c]. Then, interpretations (corresponding to the different alternatives) are compared in terms of vectors acknowledging the satisfaction or the violation of the formulas associated with the different (conditional) preferences, using suitable order relations. Thus, partial orderings of interpretations can be obtained, and may be refined in case some additional information on the relative priority of the preferences is given.

Possibilistic formulas with symbolic weights can be reinterpreted as *two-sorted* classical logic formulas. Thus, the formula (a, α) can be re-encoded by the formula $a \vee A$. Such a formula can be intuitively thought as expressing that a should be true if the situation is not abnormal ($a \vee A \equiv \neg A \rightarrow a$). Then it can be seen that $\{a \vee A, \neg a \vee b \vee B\} \vdash b \vee A \vee B$ is the counterpart of $\{(a, \alpha), (\neg a \vee b, \beta)\} \vdash (b, \min(\alpha, \beta))$ in possibilistic logic, as $\{a \vee A, a \vee A'\} \vdash$

$a \vee (A \wedge A')$ is the counterpart of $\{(a, \alpha), (a, \alpha')\} \vdash (a, \max(\alpha, \alpha'))$.

Partial information about the ordering between levels associated to possibilistic formulas can be also represented by classical logic formulas pertaining to symbolic levels. Thus, the constraint $\alpha \geq \beta$ translates into formula $\neg A \vee B$[3]. This agrees with the ideas that "the more abnormal 'a false' is, the more certain a", and that "if it is very abnormal, it is abnormal". It can be checked that $\{a \vee A, \neg a \vee b \vee B, \neg A \vee B\} \vdash a \vee B$ and $b \vee B$, i.e., we do obtain the counterpart of (b, β), while $\beta = \min(\alpha, \beta)$. The possibilistic logic inference machinery can be recast in this symbolic setting, and efficient computation procedures can be developed taking advantage of the compilation of the base in a dNNF format [Benferhat and Prade, 2005], including the special case where the levels are totally ordered [Benferhat and Prade, 2006]. In this latter case, it provides a way for compiling a possibilistic logic base and then process inference from it in polynomial time.

One motivation for dealing with a partial order on formulas relies on the fact that possibilistic logic formulas coming from different sources, may not always be stratified according to a complete preorder. Apart from the above one, several other extensions of possibilistic logic have been proposed when the total order on formulas is replaced by a partial preorder [Benferhat *et al.*, 2004b; Cayrol *et al.*, 2014]. The primary focus is usually on semantic aspects, namely the construction of a partial order on interpretations from a partial order on formulas and conversely. The difficulty lies in the fact that equivalent definitions in the totally ordered case are no longer equivalent in the partially ordered one, and that a partial ordering on subsets of a set cannot be expressed by means of a single partial order on the sets of elements in general.

3.4 Possibilistic networks

(Basic) possibilistic logic bases provide a compact representation of possibility distributions involving a finite number of possibility levels. Another compact representation of such qualitative possibility distributions is in terms of possibilistic directed graphs, which use the same conventions as Bayesian nets, but rely on conditional possibility [Benferhat *et al.*, 2002a]. An interesting feature of possibilistic logic is then that a possibilistic logic base has thus graphical representation counterparts to which the base is semantically equivalent. We start with a brief reminder on the notions of conditioning and independence in possibility theory.

[3]Note that one cannot express *strict* inequalities ($\alpha > \beta$) in this way (except on a finite scale).

Conditioning Notions of conditioning exist in possibility theory. Conditional possibility can be defined similarly to probability theory using a Bayesian-like equation of the form [Dubois and Prade, 1990a]

$$\Pi(B \cap A) = \Pi(B \mid A) \star \Pi(A)$$

where $\Pi(A) > 0$ and $\star$ may be the minimum or the product; moreover $N(B \mid A) = 1 - \Pi(B^c \mid A)$. The above equation makes little sense for necessity measures, as it becomes trivial when $N(A) = 0$, that is under lack of certainty, while in the above definition, the equation becomes problematic only if $\Pi(A) = 0$, which is natural as then A is considered impossible (see [Coletti and Vantaggi, 2009] for the handling of this situation). If operation $\star$ is the minimum, the equation $\Pi(B \cap A) = \min(\Pi(B \mid A), \Pi(A))$ fails to characterize $\Pi(B \mid A)$, and we must resort to the minimal specificity principle to come up with a qualitative conditioning of possibility [Dubois and Prade, 1988]:

$$\Pi(B \mid A) = \begin{cases} 1 \text{ if } \Pi(B \cap A) = \Pi(A) > 0, \\ \Pi(B \cap A) \text{ otherwise.} \end{cases}$$

It is clear that $N(B \mid A) > 0$ if and only if $\Pi(B \cap A) > \Pi(B^c \cap A)$. Note also that $N(B \mid A) = N(A^c \cup B)$ if $N(B \mid A) > 0$. Moreover, if $\Pi(B \mid A) > \Pi(B)$ then $\Pi(B \mid A) = 1$, which points out the limited expressiveness of this qualitative notion (no gradual positive reinforcement of possibility). However, it is possible to have that $N(B) > 0, N(B^c \mid A_1) > 0, N(B \mid A_1 \cap A_2) > 0$ (i.e., oscillating beliefs).

In the numerical setting, we must choose $\star$ = product that preserves continuity, so that $\Pi(B \mid A) = \frac{\Pi(B \cap A)}{\Pi(A)}$ which makes possibilistic and probabilistic conditionings very similar [De Baets *et al.*, 1999] (now, gradual positive reinforcement of possibility is allowed).

Independence There are also several variants of possibilistic independence between events. Let us mention here the two basic approaches:

- *Unrelatedness*: $\Pi(A \cap B) = \min(\Pi(A), \Pi(B))$. When it does not hold, it indicates an epistemic form of mutual exclusion between A and B. It is symmetric but sensitive to negation. When it holds for all pairs made of A, B and their complements, it is an epistemic version of logical independence.

- *Causal independence*: $\Pi(B \mid A) = \Pi(B)$. This notion is different from the former one and stronger. It is a form of directed epistemic independence whereby learning A does not affect the plausibility of B. It is neither symmetric not insensitive to negation: in particular, it is not equivalent to $N(B \mid A) = N(B)$.

Generally, independence in possibility theory is neither symmetric, nor insensitive to negation. For Boolean variables, independence between events is not equivalent to independence between variables. But since the possibility scale can be qualitative or quantitative, and there are several forms of conditioning, there are also various possible forms of independence. For studies of these different notions and their properties see [De Cooman, 1997; De Campos and Huete, 1999; Dubois *et al.*, 1997; Dubois *et al.*, 1999a; Ben Amor *et al.*, 2002].

Graphical structures Like joint probability distributions, joint possibility distributions can be decomposed into a conjunction of conditional possibility distributions (using $\star =$ minimum, or product), once an ordering of the variables is chosen, in a way similar to Bayes nets [Benferhat *et al.*, 2002a]. A joint possibility distribution associated with ordered variables $X_1, \ldots, X_n$, can be decomposed by the chain rule

$$\pi(X_1, \ldots, X_n) = \pi(X_n \mid X_1, \ldots, X_{n-1}) \star \cdots \star \pi(X_2 \mid X_1) \star \pi(X_1).$$

Such a decomposition can be simplified by assuming conditional independence relations between variables, as reflected by the structure of the graph. The form of independence between variables at work here is conditional non-interactivity: Two variables X and Y are independent in the context Z, if for each instance (x, y, z) of (X, Y, Z) we have: $\pi(x, y \mid z) = \pi(x \mid z) \star \pi(y \mid z)$.

Possibilistic networks are thus defined as counterparts of Bayesian networks [Pearl, 1988] in the context of possibility theory. They share the same basic components, namely:
(i) a *graphical component* which is a DAG (Directed Acyclic Graph) $\mathcal{G} = (V, E)$ where V is a set of nodes representing variables and E a set of edges encoding conditional (in)dependencies between them.
(ii) a *valued component* associating a local normalized conditional possibility distribution to each variable $V_i \in V$ in the context of its parents. The two definitions of possibilistic conditioning lead to two variants of possibilistic networks: in the numerical context, we get *product-based* networks, while in the ordinal context, we get *min-based* networks (also known as qualitative possibilistic networks). Given a possibilistic network, we can compute its joint possibility distribution using the above chain rule. Counterparts of

product-based numerical possibilistic nets using ranking functions exist as well [Spohn, 2012].

Ben Amor and Benferhat [2005] have investigated the properties of qualitative independence that enable local inferences to be performed in possibilistic nets. Uncertainty propagation algorithms suitable for possibilistic graphical structures have been studied [Ben Amor *et al.*, 2003; Benferhat *et al.*, 2005], taking advantage of the idempotency of min operator in the qualitative case [Ben Amor *et al.*, 2003]. Such graphical structures may be also of particular interest for representing preferences [Ben Amor *et al.*, 2014].

Possibilistic nets and possibilistic logic Since possibilistic nets and possibilistic logic bases are compact representations of possibility distributions, it should not come as a surprise that possibilistic nets can be directly translated into possibilistic logic bases and vice-versa, both when conditioning is based on minimum or on product [Benferhat *et al.*, 2002a; Benferhat *et al.*, 2001b]. Hybrid representations formats have been introduced where local possibilistic logic bases are associated to the nodes of a graphical structure rather than conditional possibility tables [Benferhat and Smaoui, 2007a]. An important feature of the possibilistic logic setting is the existence of such equivalent representation formats: set of prioritized logical formulas, preorders on interpretations (possibility distributions) at the semantical level, possibilistic nets, but also set of conditionals of the form $\Pi(p \wedge q) > \Pi(p \wedge \neg q)$, and there are algorithms for translating one format in another [Benferhat *et al.*, 2001a].

4 APPLICATIONS OF BASIC POSSIBILISTIC LOGIC

Basic possibilistic logic has found many applications in different reasoning tasks, beyond the simple deductive propagation of certainty levels or priority levels. In the following, we survey its use in default reasoning (and its utilization in causality ascription), in belief revision and information fusion, in decision under uncertainty, and in the handling of uncertainty in information systems.

In a computational perspective, possibilistic logic has also impacted logic programming [Dubois *et al.*, 1991c; Benferhat *et al.*, 1993b; Alsinet and Godo., 2000; Alsinet *et al.*, 2002; Nicolas *et al.*, 2006; Nieves *et al.*, 2007; Confalonieri *et al.*, 2012; Bauters *et al.*, 2010; Bauters *et al.*, 2011; Bauters *et al.*, 2012]. It has also somewhat influenced the handling of soft constraints in constraint satisfaction problems [Schiex, 1992; Schiex *et al.*, 1995]. Let us also mention applications to diagnosis and recognition problems [Dubois *et al.*, 1990; Benferhat *et al.*, 1997a; Dubois and Prade, 2000; Grabisch and Prade, 2001; Grabisch, 2003], and to the encoding of control access policies [Benferhat *et al.*, 2003].

4.1 Default reasoning and causality

Possibilistic logic can be used for describing the normal course of things and a possibilistic logic base reflects how entrenched are the beliefs of an agent. This is why possibilistic logic is of interest in default reasoning, but also in causality ascription, as surveyed in this subsection.

Default reasoning Nonmonotonic reasoning has been extensively studied in AI in relation with the problem of reasoning under incomplete information with rules having potential exceptions [Léa Sombé Group, 1990], or for dealing with the frame problem in dynamic worlds [Brewka *et al.*, 2011]. In the following, we recall the possibilistic approach [Benferhat *et al.*, 1998b; Dubois and Prade, 2011c], which turned out [Benferhat *et al.*, 1997b] to provide a faithful representation of the postulate-based approach proposed by Kraus, Lehmann and Magidor [1990], and completed in [Lehmann and Magidor, 1992].

A default rule "if a then b, generally", denoted $a \rightsquigarrow b$, is then understood formally as the constraint

$$\Pi(a \wedge b) > \Pi(a \wedge \neg b)$$

on a possibility measure Π describing the semantics of the available knowledge. It expresses that in the context where a is true, there exists situations where having b true is strictly more satisfactory than any situations where b is false in the same context. As already said, this constraint is equivalent to $N(b \mid a) = 1 - \Pi(\neg b \mid a) > 0$, when $\Pi(b \mid a)$ is defined as the greatest solution of the min-based equation $\Pi(a \wedge b) = \min(\Pi(b \mid a), \Pi(a))$.

The above constraint can be equivalently expressed in terms of a comparative possibility relation, namely $a \wedge b >_{\Pi} a \wedge \neg b$. Any finite consistent set of constraints of the form $a_i \wedge b_i \geq_{\Pi} a_i \wedge \neg b_i$, representing a set of defaults $\Delta = \{a_i \rightsquigarrow b_i, i = 1, \cdots, n\}$, is compatible with a non-empty family of relations $>_{\Pi}$, and induces a partially defined ranking $>_{\pi}$ on Ω, that can be completed according to the principle of minimal specificity, e.g. [Benferhat *et al.*, 1999b]. This principle assigns to each world ω the highest possibility level (in forming a well-ordered partition of Ω) without violating the constraints. This defines a unique complete preorder. Let $E_1, \ldots, E_m$ be the obtained partition. Then $\omega >_\pi \omega'$ if $\omega \in E_i$ and $\omega' \in E_j$ with $i < j$, while $\omega \sim_\pi \omega'$ if $\omega \in E_i$ and $\omega' \in E_i$ (where $\sim_\pi$ means $\geq_\pi$ and $\leq_\pi$).

A numerical counterpart to $>_\pi$ can be defined by $\pi(\omega) = \frac{m+1-i}{m}$ if $\omega \in E_i, i = 1, \ldots, m$. Note that this is purely a matter of convenience to use a numerical scale, and any other numerical counterpart such that $\pi(\omega) > \pi(\omega')$ iff $\omega >_\pi \omega'$ will work as well. Namely the range of π is used as an ordinal scale.

EXAMPLE 4. Let us consider the following classical example with default rules $d1$: "birds fly", $d2$: "penguins do not fly", $d3$: "penguins are birds", symbolically written

$$d1 : b \rightsquigarrow f;\ d2 : p \rightsquigarrow \neg f;\ d3 : p \rightsquigarrow b.$$

The set of three defaults is thus represented by the following set $\mathcal{C}$ of constraints:

$$b \wedge f \geq_\Pi b \wedge \neg f\ ;\ p \wedge \neg f \geq_\Pi p \wedge f\ ;\ p \wedge b \geq_\Pi p \wedge \neg b.$$

Let Ω be the finite set of interpretations of the considered propositional language, generated by b, f, p in the example. If this language is made of the literals $p_1, \ldots, p_n$, these interpretations correspond to the possible worlds (i.e., the completely described situations) where the conjunctions $*p_1 \wedge \ldots \wedge *p_n$ are true, where $*$ stands for the presence of the negation sign $\neg$ or its absence. In our example, $\Omega = \{\omega_0 : \neg b \wedge \neg f \wedge \neg p$, $\omega_1 : \neg b \wedge \neg f \wedge p$, $\omega_2 : \neg b \wedge f \wedge \neg p$, $\omega_3 : \neg b \wedge f \wedge p$, $\omega_4 : b \wedge \neg f \wedge \neg p$, $\omega_5 : b \wedge \neg f \wedge p$, $\omega_6 : b \wedge f \wedge \neg p$, $\omega_7 : b \wedge f \wedge p\}$. Any interpretation ω thus corresponds to a particular proposition. One can compute the possibility. For instance, $\Pi(b \wedge f) = \max(\pi(\omega_6), \pi(\omega_7))$.

Then the set of constraints $\mathcal{C}$ on interpretations can be written as:

$C1 : \max(\pi(\omega_6), \pi(\omega_7)) > \max(\pi(\omega_4), \pi(\omega_5))$,

$C2 : \max(\pi(\omega_5), \pi(\omega_1)) > \max(\pi(\omega_3), \pi(\omega_7))$,

$C3 : \max(\pi(\omega_5), \pi(\omega_7)) > \max(\pi(\omega_1), \pi(\omega_3))$.

The well ordered partition of Ω which is obtained in this example is

$\{\omega_0, \omega_2, \omega_6\} >_\pi \{\omega_4, \omega_5\} >_\pi \{\omega_1, \omega_3, \omega_7\}$.

In the example, we have $m = 3$ and $\pi(\omega_0) = \pi(\omega_2) = \pi(\omega_6) = 1$;
$\pi(\omega_4) = \pi(\omega_5) = 2/3$; $\pi(\omega_1) = \pi(\omega_3) = \pi(\omega_7) = 1/3$. □

From the possibility distribution π associated with the well ordered partition, we can compute the necessity level $N(a)$ of any proposition a. The method then consists in turning each default $p_i \rightsquigarrow q_i$ into a possibilistic clause $(\neg p_i \vee q_i,\ N(\neg p_i \vee q_i))$, where N is computed from the greatest possibility distribution π induced by the set of constraints corresponding to the default knowledge base, as already explained. We thus obtain a possibilistic logic base K. This encodes the generic knowledge embedded in the default rules. Then we apply the possibilistic inference for reasoning with the formulas in K encoding the defaults together with the available factual knowledge encoded as fully certain possibilistic formulas in a base F.

However, the conclusions that can be obtained from $K \cup F$ with a certainty level strictly greater than the level of inconsistency of this base are safe. Roughly speaking, it turns out that in this approach, the most specific rules w.r.t. a given context remain above the level of inconsistency.

EXAMPLE 5. Example 4 continued. Using the possibility distribution obtained at the previous step, we compute:

$$\begin{aligned} N(\neg p \vee \neg f) &= \min\{1 - \pi(\omega) | \omega \models p \wedge f\} \\ &= \min(1 - \pi(\omega_3), 1 - \pi(\omega_7)) = 2/3, \end{aligned}$$

$N(\neg b \vee f) = \min\{1 - \pi(\omega) \mid \omega \models b \wedge \neg f\} = \min(1 - \pi(\omega_4), 1 - \pi(\omega_5)) = 1/3$, and $N(\neg p \vee b) = \min(1 - \pi(\omega_1), 1 - \pi(\omega_3)) = 2/3$.

Thus, we have the possibilistic logic base

$$K = \{(\neg p \vee \neg f, 2/3), (\neg p \vee b, 2/3), (\neg b \vee f, 1/3)\}.$$

Suppose that all we know about the factual situation under consideration is that "Tweety" is a bird, which is encoded by $F = \{(b, 1)\}$. Then we apply the weakest link resolution rule, and we can check that $K \cup \{(b, 1)\} \vdash (f, 1/3)$, i.e., we conclude that if all we know about "Tweety" is that it is a bird, then it flies. If we are said that "Tweety" is in fact a penguin, i.e., $F = \{(b, 1), (p, 1)\}$, then $K \cup F \vdash (\bot, 1/3)$, which means that $K \cup \{(b, 1)\}$ augmented with the new piece of factual information $\{(p, 1)\}$ is now inconsistent (at level 1/3). But the following inference $K \cup F \vdash (\neg f, 2/3)$ is valid (since $2/3 > inc\text{-}l(K \cup F) = 1/3$). Thus, knowing that "Tweety" is a penguin, we now conclude that it does not fly. □

This encoding takes advantage of the fact that when a new piece of information is received, the level of inconsistency of the base cannot decrease, and if it strictly increases, some inferences that were safe before are now drowned in the new inconsistency level of the base and are thus no longer allowed, hence a nonmonotonic consequence mechanism takes place.

Such an approach has been proved to be in full agreement with the Kraus-Lehmann-Magidor postulates-based approach to nonmonotonic reasoning

[Kraus *et al.*, 1990]. More precisely, two nonmonotonic entailments can be defined in the possibilistic setting, the one presented above, based on the less specific possibility distribution compatible with the constraints encoding the set of defaults, and another one more cautious, where one considers that b can be deduced in the situation where all we know is $F = \{a\}$ if and only if the inequality $\Pi(a \wedge b) > \Pi(a \wedge \neg b)$ holds true for *all* the Π compatible with the constraints encoding the set of defaults. The first entailment coincides with the rational closure inference [Lehmann and Magidor, 1992], while the later corresponds to the (cautious) preferential entailment [Kraus *et al.*, 1990]; see [Dubois and Prade, 1995a; Benferhat *et al.*, 1997b]. Besides, the ranking of the defaults obtained above from the well-ordered partition [Benferhat *et al.*, 1992] is the same as the Z-ranking introduced by Pearl [1990].

While the consequences obtained with the preferential entailment are hardly debatable, the ones derived with the rational closure are more adventurous. However these latter consequences can be always modified if necessary by the *addition* of further defaults. These added defaults may express independence information [Dubois *et al.*, 1999a] of the type "in context c, the truth or the falsity of a has no influence on the truth of b" [Benferhat *et al.*, 1994b; Benferhat *et al.*, 1998b]. Lastly, a default rule may be itself associated with a certainty level; in such a case each formula will be associated with two levels, namely a priority level reflecting its relative specificity in the base, and its certainty level respectively [Dupin de Saint Cyr and Prade, 2008].

Let us also mention an application to possibilistic inductive logic programming. Indeed learning a stratified set of first-order logic rules as an hypothesis in inductive logic programming has been shown of interest for learning both rules covering normal cases and more specific rules that handle more exceptional cases [Serrurier and Prade, 2007].

Causality Our understanding of sequences of reported facts depends on our own beliefs on the normal course of things. We have seen that $N(b|a) > 0$ can be used for expressing that in context a, b is normally true. Then qualitative necessity measures may be used for describing how (potential) causality is perceived in relation with the advent of an abnormal event that precedes a change.

Namely, if on the one hand an agent holds the two following beliefs represented by $N(b|a) > 0$ and $N(\neg b|a \wedge c) > 0$ about the normal course of things, and if on the other hand it has been reported that we are in context a, and that b, which was true, has become false after c takes place, then the agent will be led to think that "a caused $\neg b$". See [Bonnefon *et al.*, 2008] for a detailed presentation and discussion, and [Bonnefon *et al.*, 2012] for a study of the very restricted conditions under which causality is transitive in this approach. The theoretical consequences of this model have been validated from a cognitive psychology point of view.

Perceived causality may be badly affected by spurious correlations. For a proper assessing of causality relations, Pearl [2000] has introduced the notion of *intervention* in Bayesian networks, which comes down to enforcing the values of some variables so as to lay bare their influence on other ones. Following the same line, possibilistic networks have been studied from the standpoint of causal reasoning, using the concept of intervention; see [Benferhat and Smaoui, 2007b; Benferhat, 2010; Benferhat and Smaoui, 2011], where tools for handling interventions in the possibilistic setting have been developed. Finally, a counterpart of the idea of intervention has been investigated in possibilistic logic knowledge bases, which are non-directed structures (thus contrasting with Bayesian and possibilistic networks) [Benferhat *et al.*, 2009].

4.2 Belief revision and information fusion

In belief revision, the new input information that fires the revision process has priority on the information contained in the current belief set. This contrasts with information fusion where sources play symmetric roles (even if they have different reliability levels). We briefly survey the contributions of possibilistic logic to these two problems.

Belief revision and updating Keeping in mind that nonmonotonic reasoning and belief revision can be closely related [Gärdenfors, 1990], it should not be a surprize that possibilistic logic finds application also in belief revision. In fact, comparative necessity relations (which can be encoded by necessity measures) [Dubois, 1986] are nothing but the epistemic entrenchment relations [Dubois and Prade, 1991] that underly well-behaved belief revision processes [Gardenförs, 1988]. This enables the possibilistic logic setting to provide syntactic revision operators that apply to possibilistic knowledge bases, including the case of uncertain inputs [Dubois and Prade, 1997a; Benferhat *et al.*, 2002c; Benferhat *et al.*, 2010; Qi, 2008; Qi and Wang, 2012]. Note that in possibilistic logic, the epistemic entrenchment of the formulas is made explicit through the certainty levels. Formulas (a, α) are viewed as pieces of belief that are more or less certain. Moreover, in a revision process it is expected that all formulas independent of the validity of the input information should be retained in the revised state of belief; this intuitive idea may receive a precise meaning using a suitable definition of possibilistic independence between events [Dubois *et al.*, 1999a]. See [Dubois *et al.*, 1998a] for a comparative overview of belief change operations in the different representation settings (including possibilistic logic).

Updating in a dynamic environment obeys other principles than the revision of a belief state by an input information in a static world, see, e.g., [Léa Sombé Group (ed.), 1994]. It can be related to the idea of Lewis'*imaging* [1976], whose a possibilistic counterpart has been proposed in [Dubois and Prade, 1993]. A possibilistic logic transposition of Kalman filtering that combines the ideas of updating and revision can be found in [Benferhat *et al.*, 2000b].

Information fusion Information fusion can take place in the different representation formats of the possibilistic setting. In particular, the combination of possibility distributions can be equivalently performed in terms of possibilistic logic bases. Namely, the syntactic counterpart of the pointwise combination of two possibility distributions π_1 and π_2 into a distribution $\pi_1 \oplus \pi_2$ by any monotonic combination operator $\oplus$[4] such that $1 \oplus 1 = 1$, can be computed, following an idea first proposed in [Boldrin, 1995; Boldrin and Sossai, 1997]. Namely, if the possibilistic logic base Γ_1 is associated with π_1 and the base Γ_2 with π_2, a possibilistic base that is semantically equivalent to $\pi_1 \oplus \pi_2$ can be obtained in the following way [Benferhat *et al.*, 1998a]:

$$\Gamma_{1\oplus 2} = \{(a_i, 1 - (1 - \alpha_i) \oplus 1) \text{ s.t. } (a_i, \alpha_i) \in \Gamma_1\},$$

$$\cup \{(b_j, 1 - 1 \oplus (1 - \beta_j)) \text{ s.t. } (b_j, \beta_j) \in \Gamma_2\},$$

$$\cup \{(a_i \vee b_j, 1 - (1 - \alpha_i) \oplus (1 - \beta_j)) \text{ s.t. } (a_i, \alpha_i) \in \Gamma_1, (b_j, \beta_j) \in \Gamma_2\}.$$

$$\text{For} \quad \oplus = \min, \text{we get} \quad \Gamma_{1\oplus 2} = \Gamma_1 \cup \Gamma_2 \quad \text{with} \quad \pi_{\Gamma_1 \cup \Gamma_2} = \min(\pi_1, \pi_2)$$

as expected (conjunctive combination). For $\oplus = \max$ (disjunctive combination), we get

$$\Gamma_{1\oplus 2} = \{(a_i \vee b_j, \min(\alpha_i, \beta_j)) \text{ s.t. } (a_i, \alpha_i) \in \Gamma_1, \text{ and } (b_j, \beta_j) \in \Gamma_2\}.$$

[4] $\oplus$ is supposed to be monotonic in the wide sense for each of its arguments: $\alpha \oplus \beta \geq \gamma \oplus \delta$ as soon as $\alpha \geq \gamma$ and $\beta \geq \delta$. Examples of such combination operators $\oplus$ are triangular norms (a non-decreasing semi-group of the unit interval having identity 1 and absorbing element 0) and the dual triangular co-norms that respectively extend conjunction and disjunction to multiple-valued settings [Klement *et al.*, 2000].

With non idempotent $\oplus$ operators, some reinforcement effects may be obtained. Moreover, fusion can be applied directly to qualitative or quantitative possibilistic networks [Benferhat and Titouna, 2005; Benferhat and Titouna, 2009]. See [Benferhat *et al.*, 1999c; Benferhat *et al.*, 2001c; Kaci *et al.*, 2000; Qi *et al.*, 2010b; Qi *et al.*, 2010a] for further studies on possibilistic logic merging operators.

Besides, this approach has been also applied to the syntactic encoding of the merging of *classical* logic bases based on Hamming distance (where distances are computed between each interpretation and the different classical logic bases, thus giving birth to counterparts of possibility distributions) [Benferhat *et al.*, 2002b].

4.3 Qualitative handling of uncertainty in decision and information systems

Uncertainty often pervades the available information. Possibility theory offers an appropriate setting for the representation of incomplete and uncertain epistemic information in a qualitative manner. In this subsection, we provide a brief presentation of the possibilistic logic approach to decision under uncertainty and to the management of uncertain databases.

Qualitative decision under uncertainty Possibility theory provides a valuable setting for qualitative decision under uncertainty where a pessimistic and an optimistic decision criteria have been axiomatized [Dubois and Prade, 1995b; Dubois *et al.*, 2001a; Benferhat *et al.*, 2000a]. The exact counterpart of these pessimistic and optimistic criteria, when the knowledge and the preferences are respectively expressed under the form of *two distinct* possibilistic logic bases, have been shown in [Dubois *et al.*, 1999b] to correspond to the following definitions:

- The pessimistic utility $u_*(d)$ of a decision d is the maximal value of $\alpha \in \mathcal{S}$ such that
$$K_\alpha \wedge d \vdash_{PL} P_{\underline{\nu(\alpha)}}$$
- The $u^*(d)$ of a decision d is the maximal value of $n(\alpha) \in \mathcal{S}$ such that
$$K_{\underline{\alpha}} \wedge d \wedge P_{\underline{\alpha}} \not\equiv \bot$$

where $\mathcal{S}$ denotes a finite bounded totally ordered scale, ν is the ordered reversing map of this scale, K_α is a set of classical logic formulas gathering the pieces of knowledge that are certain at a level at least equal to α, and where $P_{\underline{\beta}}$ is a set of classical logic formulas made of a set of goals (modeling preferences) whose priority level is *strictly* greater than β.

As can be seen, an optimal pessimistic decision leads for sure to the satisfaction of all the goals in $P_{\underline{\nu(\alpha)}}$ whose priority is greater than a level as low as possible, according to a part K_α of our knowledge which is as certain as possible. An optimal optimistic decision maximizes only the consistency of all the more or less important goals with all the more or less certain pieces of knowledge. Optimal pessimistic or optimistic decisions can then be computed in an answer set programming setting [Confalonieri and Prade, 2014]. Besides, this possibilistic treatment of qualitative decision can be also related to an argumentative view of decision [Amgoud and Prade, 2009].

Handling uncertainty in information systems In the possibilistic approach to the handling of uncertainty in databases, the available information on the value of an attribute A for an item x is usually represented by a possibility distribution defined on the domain of attribute A. Then, considering a classical query, we can compute two sets of answers, namely the set of items that more or less *certainly* satisfy the query (this corresponds to the above pessimistic viewpoint), and the *larger* set of items that more or less *possibly* satisfy the query (this corresponds to the above optimistic viewpoint) [Dubois and Prade, 1988].

Computation may become tricky for some basic relational operations such as the join of two relations, for which it becomes necessary to keep track that some uncertain values should remain equal in any extension. As in the probabilistic case, methods based on *lineage* have been proposed to handle such problems [Bosc and Pivert, 2005]. Their computational cost remain heavy in practice.

However, uncertain data can be processed at a much more affordable cost provided that we restrict ourselves to pieces of information of the form $(a(x), \alpha)$ expressing that it is certain at level α that $a(x)$ is the value of attribute A for the item x. More generally, $a(x)$ can be replaced by a disjunction of values. Then, a possibilistic logic-like treatment of uncertainty in databases can take place in a relational database framework. It can be shown that such an uncertainty modeling is a representation system for the whole relational algebra. An important result is that the data complexity associated with the extended operators in this context is the same as in the classical database case [Bosc *et al.*, 2009; Pivert and Prade, 2014]. An additional benefit of the possibilistic setting is an easier elicitation of the certainty levels. We illustrate the idea with the following simple example.

R	*Name*	*Married*	*City*
1	John	(yes, α)	$(\text{Toulouse}, \mu)$
2	Mary	$(\text{yes}, 1)$	(Albi, ρ)
3	Peter	(no, β)	$(\text{Toulouse}, \phi)$

S	*City*	*Flea Market*
1	Albi	(yes, γ)
2	Toulouse	(yes, δ)

Table 2. A database with possibilistic uncertainty

EXAMPLE 6. Let us consider a database example with two relations R and S containing uncertain pieces of data. See Table 2. If we look here for the persons who are married and leave in a city with a flea market, we shall retrieve *John* with certainty $\min(\alpha, \mu, \delta)$ and *Mary* with certainty $\min(\rho, \gamma)$.

It is also possible to accommodate disjunctive information in this setting. Assume for instance that the third tuple of relation R is now (Peter, (no, β), $(\text{Albi} \ \vee \ \text{Toulouse}, \phi)$). Then, if we look for persons who are not married and leave in a city with a flea market, one retrieve *Peter* with certainty $\min(\beta, \phi, \gamma, \delta)$. Indeed we have in possibilistic logic that $(\neg\text{Married}, \beta)$ and $(\text{Albi} \vee \text{Toulouse}, \phi)$, $(\neg\text{Albi} \vee \text{Flea Market}, \gamma)$, $(\neg\text{Toulouse} \vee \text{Flea Market}, \delta)$ entail $(\neg\text{Married}, \beta)$ and $(\text{Flea Market}, \min(\phi, \gamma, \delta))$. □

This suggests the potentials of a necessity measure-based approach to the handling of uncertain pieces of information. Clearly, the limited setting of certainty-qualified information is less expressive than the use of general possibility distributions (we cannot here retrieve items that are just somewhat possible without being somewhat certain), but this framework seems to be expressive enough for being useful in practice. Let us also mention a possibilistic modeling of the validity and of the completeness of the information (pertaining to a given topic) in a database [Dubois and Prade, 1997b].

Besides, the possibilistic handling of uncertainty in description logic [Qi *et al.*, 2011; Zhu *et al.*, 2013] has also computational advantages, in particular in the case of the *possibilistic DL-Lite* family [Benferhat and Bouraoui, 2013; Benferhat *et al.*, 2013]. Lastly, possibilistic logic has been recently shown to be of interest in database design [Koehler *et al.*, 2014a; Koehler *et al.*, 2014b].

5 EXTENSIONS OF POSSIBILISTIC LOGIC

Possibilistic logic has been extended in different manners. In this section, we consider three main types of extension: i) replacing the totally ordered scale of the certainty levels by a partially ordered structure; ii) dealing with logical formulas that are weighted in terms of lower bounds of a strong (guaranteed) possibility measure Δ (see subsection 2.2); iii) allowing for negation of basic possibilistic logic formulas, or for their disjunction (and no longer only for their conjunction), which leads to *generalized* possibilistic logic.

5.1 Lattice-based possibilistic logics

Basically, a possibilistic formula is a pair made of a classical logic formula and a label that qualifies in what conditions or in what manner the classical logic formula is regarded as true. One may think of associating "labels" other than certainty levels. It may be lower bounds of other measures in possibility theory, such as in particular strong possibility measures, as reviewed in the next subsection. It may be also labels taking values in partially ordered structures, such as lattices. This can be motivated by different needs, as briefly reviewed now.

Different intended purposes Timed possibilistic logic [Dubois *et al.*, 1991b] has been the first proposed extension of this kind. In timed possibilistic logic, logical formulas are associated with sets of time instants where the formula is known as being certainly true. More generally certainty may be graded as in basic possibilistic logic, and then formulas are associated with *fuzzy* sets of time instants where the grade attached to a time instant is the certainty level with which the formula is true at that time. At the semantic level, it leads to an extension of necessity (and possibility) measures now valued in a distributive lattice structure.

Taking inspiration of possibilistic logic, Lafage, Lang and Sabbadin [1999] have proposed a *logic of supporters*, where each formula a is associated with a set of logical arguments in favor of a. More recently, an *interval-based possibilistic logic* has been presented [Benferhat *et al.*, 2011] where classical logic formulas are associated with intervals, thought as imprecise certainty levels.

Another early proposed idea, in an information fusion perspective, is to associate each formula with a set of distinct explicit sources that support its truth [Dubois *et al.*, 1992]. Again, a certainty level may be attached to each source, and then formulas are associated with fuzzy sets of sources. This has led to the proposal of a "multiple agent" logic where formulas are of the form (a, A), where A denotes a subset of agents that are known to believe that a is true. In contrast with timed possibilistic logic where it is important to make sure that the knowledge base remains consistent over time, what matters in multiple agent logic is the collective consistency of *subsets* of agents (while the collection of the beliefs held by the whole set of agents may be inconsistent. We now indicate the main features of this latter logic.

Multiple agent logic Multiple agent possibilistic logic was outlined in [Dubois and Prade, 2007], but its underlying semantics has been laid bare more recently [Belhadi *et al.*, 2013]. A multiple agent propositional formula is a pair (a, A), where a is a classical propositional formula of $\mathcal{L}$ and A is a non-empty subset of All, i.e., $A \subseteq All$ (All denote the finite set of all considered agents). The intuitive meaning of formula (a, A) is that *at least all* the agents in A believe that a is true. In spite of the obvious parallel with possibilistic logic (where propositions are associated with levels expressing the strength with which the propositions are believed to be true), (a, A) should not be just used as another way of expressing the strength of the support in favor of a (the larger A, the stronger the support), but rather as a piece of information linking a proposition with a group of agents.

Multiple agent logic has two inference rules:

- if $B \subseteq A$ then $(a, A) \vdash (a, B)$ (subset weakening)
- $(\neg a \vee b, A), (a, A) \vdash (b, A)$, $\forall A \in 2^{ALL} \setminus \emptyset$ (modus ponens)

As a consequence, we also have the resolution rule

$$\text{if } A \cap B \neq \emptyset, \text{ then } (\neg a \vee b, A), (a \vee c, B) \vdash (b \vee c, A \cap B).$$

If $A \cap B = \emptyset$, the information resulting from applying the rule does not belong to the language, and would make little sense: it is of no use to put formulas of the form $(a, \emptyset)$ in a base as it corresponds to information possessed by no agent.

Since 2^{ALL} is not totally ordered as in the case of certainty levels, we cannot "slice" a multiple agent knowledge base $\Gamma = \{(a_i, A_i), i = 1, \ldots, m\}$ into layers as in basic possibilistic logic. Still, one can define the restriction of Γ to a subset $A \subseteq All$ as

$$\Gamma_A = \{(a_i, A_i \cap A) \mid A_i \cap A \neq \emptyset \text{ and } (a_i, A_i) \in \Gamma\}.$$

Moreover, an *inconsistency subset* of agents can be defined for Γ as

$$inc\text{-}s(\Gamma) = \bigcup\{A \subseteq All \mid \Gamma \vdash (\bot, A)\} \text{ and } inc\text{-}s(\Gamma) = \emptyset \text{ if } \nexists A \text{ s.t. } \Gamma \vdash (\bot, A).$$

Note that in this definition $A = \emptyset$ is not forbidden. For instance, let $\Gamma = \{(p, A), (q, B), (\neg p \vee q, C), (\neg q, D)\}$, then $inc\text{-}s(\Gamma) = (A \cap C \cap D) \cup (B \cap D)$, and obviously $inc\text{-}s(\Gamma_{A \cap B \cap C \cap \overline{D}}) = \emptyset$.

Clearly, *it is not the case* that the consistency of Γ ($inc\text{-}s(\Gamma) = \emptyset$) implies that Γ° is consistent. This feature contrasts with possibilistic logic. Just consider the example $\Gamma = \{(a, A), (\neg a, \overline{A})\}$, then $inc\text{-}s(\Gamma) = A \cap \overline{A} = \emptyset$ while $\Gamma^\circ = \{a_i \mid (a_i, A_i) \in \Gamma, i = 1, \ldots, m\}$ is inconsistent. This is compatible with situations where agents contradict each other. Yet, the consistency of Γ° does entail $inc\text{-}s(\Gamma) = \emptyset$.

The semantics of *ma-L* is expressed in terms of set-valued possibility distributions, set-valued possibility measures and set-valued necessity measures. Namely, the semantics of formula (a, A) is given by set-valued distribution $\boldsymbol{\pi}_{\{(a,A)\}}$:

$$\forall \omega \in \Omega, \boldsymbol{\pi}_{\{(a,A)\}}(\omega) = \begin{cases} All & \text{if } \omega \models a \\ A^c & \text{if } \omega \models \neg a \end{cases}$$

where $A^c = All \setminus A$, and the formula (a, A) is understood as expressing the constraints $\mathbf{N}(a) \supseteq A$ where $\mathbf{N}$ is a set-valued necessity measure. Soundness and completeness results can be established with respect to this semantics [Belhadi *et al.*, 2013].

Basic possibilistic logic and multiple agent logic may then be combined in a possibilistic multiple agent logic. Formulas are pairs (a, F) where F is now a fuzzy subset of All. One may in particular consider the fuzzy sets $F = (\alpha/A)$ such that $(\alpha/A)(k) = \alpha$ if $k \in A$, and $(\alpha/A)(k) = 0$ if $k \in \overline{A}$, i.e., we restrict ourselves to formulas of the form $(a, \alpha/A)$ that encode the piece of information "at least all agents in A believe a at least at level α". Then the resolution rule becomes

$$(\neg p \vee q, \alpha/A); (p \vee r, \beta/B) \vdash (q \vee r, \min(\alpha, \beta)/(A \cap B)).$$

5.2 Uses of the strong possibility set function in possibilistic logic

As recalled in subsection 2.2, a possibility distribution can be associated not only with the increasing set functions Π and N, but also with the decreasing set functions Δ and ∇. As we are going to see, this enables a double reading, from above and from below, of a possibility distribution. This double reading may be of interest in preference representation by allowing the use of different but equivalent representation formats. Moreover a Δ-based possibilistic logic, handling formulas associated with lower bounds of a Δ-set function, can be developed. This is reviewed first.

The last part of this subsection is devoted to a different use of Δ-set functions. Namely, the modeling of bipolar information. Then one distinguishes between positive information (expressed by means of Δ-based possibilistic logic formulas) and negative information (expressed by means of N-based possibilistic logic formulas), these two types of information being associated with *two* distinct possibility distributions.

Double reading of a possibility distribution In basic possibilistic logic, a base $\Gamma^N = \{(a_i, \alpha_i), i = 1, ..., m\}$ is semantically associated with the possibility distribution $\pi^N_\Gamma(\omega) = \min_{i=1,\ldots,m} \max([a_i](\omega), 1 - \alpha_i)$, where $[a_i]$ is the characteristic function of the models of a_i. As being the result of a min-combination, this corresponds to a reading "from above" of the possibility distribution.

Let us consider another type of logical formula (now denoted between brackets rather than parentheses) as a pair $[b, \beta]$, expressing the constraint $\Delta(b) \geq \beta$, where Δ is a *guaranteed* or *strong* possibility measure. Then, a Δ-base $\Gamma^\Delta = \{[b_j, \beta_j] \mid j = 1, \ldots, n\}$ is associated to the distribution

$$\pi^\Delta_\Gamma(\omega) = \max_{j=1,\ldots,n} \pi_{[b_j,\beta_i]}(\omega) \text{ with } \pi_{[b_j,\beta_i]}(\omega) = \begin{cases} \beta_j \text{ if } \omega \in [b_j] \\ 0 \text{ otherwise.} \end{cases}$$

As being the result of a max-combination, this corresponds to a reading "from below" of the possibility distribution. It can be proved [Dubois *et al.*, 2014b] that the N-base $\Gamma^N = \{(a_i, \alpha_i) \mid i = 1, \ldots, m\}$ is semantically equivalent to the Δ-base

$$\Gamma^\Delta = \{[\wedge_{i \in J} a_i, \min_{k \notin J}(1 - \alpha_k)] : J \subseteq \{1, \ldots, m\}\}.$$

Although it looks like the translated knowledge base is exponentially larger than the original one, it can be simplified. Indeed, suppose, without loss of generality that $1 = \alpha_1 > \alpha_2 > \cdots > \alpha_m$ and $\alpha_{m+1} = 0$ by convention (we combine conjunctively all formulas with the same level). Then it is easy to check that,

$$\max_{J \subseteq \{1,\ldots,m\}} \min(\min_{k \notin J}(1-\alpha_k), [\wedge_{j \in J} a_j](\omega)) = \max_{k=1,\ldots,m+1} \min(1-\alpha_k, [\wedge_{j=1}^{k-1} a_j](\omega))$$

which corresponds to the Δ-base

$$\Gamma^\Delta = \{[\wedge_{j=1}^{k-1} a_j, 1 - \alpha_k] : k = 1, \ldots, m + 1\},$$

with $\wedge_{j=1}^{0} a_j = \top$ (tautology). Of course, likewise the Δ-base $\Gamma^{\Delta} = \{[b_j, \beta_j] \mid j = 1, \ldots, n\}$ is semantically equivalent to the N-base

$$\Gamma^N = \{(\vee_{j \in J} b_j, \max_{k \notin J}(1 - \beta_k)) : J \subseteq \{1, \ldots, n\}\},$$

which can be simplified as

$$\Gamma^N = \{(\vee_{j=1}^{k-1} b_j, 1 - \beta_k) : k = 1, \ldots, n+1\},$$

with $\beta_1 > \beta_2 > \cdots > \beta_n > \beta_{n+1} = 0, \vee_{j=1}^{0} \psi_j = \bot$ (contradiction), by convention.

Thus, a possibilistic logic base Γ^{Δ} expressed in terms of a strong possibility measure can always be rewritten equivalently in terms of a standard possibilistic logic base Γ^N using necessity measures and conversely, enforcing the equality $\pi_{\Gamma}^N = \pi_{\Gamma}^{\Delta}$. The transformation from π_{Γ}^N to π_{Γ}^{Δ} corresponds to writing the min-max expression of π_{Γ}^N as a max-min expression (applying the distributivity of min over max) and conversely. This is now illustrated on a preference example.

Preference representation As already emphasized, possibilistic logic applies to the representation of both knowledge and preferences . In case of preferences, the level α associated a formula a in (a, α) is understood as a priority.

EXAMPLE 7. Thus, a piece of preference such as “I prefer p to q and q to r” (where p, q, r may not be mutually exclusive) can be represented by the possibilistic base $\Gamma^N = \{(p \vee q \vee r, 1), (p \vee q, 1 - \gamma), (p, 1 - \beta)\}$ with $\gamma < \beta < 1$, by translating the preference into a set of more or less imperative goals. Namely, Γ states that p is somewhat imperative, that $p \vee q$ is more imperative, and that $p \vee q \vee r$ is compulsory. Note that the preferences are here expressed negatively: “nothing is possible outside p, q, or r”, “nothing is really possible outside p, or q”, and “nothing is strongly possible outside p”. The possibilistic base Γ^N is associated with the possibility distribution π_{Γ}^N which rank-orders the alternatives: $\pi_{\Gamma}^N(pqr) = 1$, $\pi_{\Gamma}^N(p\neg qr) = 1$, $\pi_{\Gamma}^N(pq\neg r) = 1$, $\pi_{\Gamma}^N(p\neg q\neg r) = 1$, $\pi_{\Gamma}^N(\neg pqr) = \beta$, $\pi_{\Gamma}^N(\neg pq\neg r) = \beta$, $\pi_B(\neg p\neg qr) = \gamma$, $\pi_{\Gamma}^N(\neg p\neg p\neg r) = 0$.

From this possibility distribution, one can compute the associated measure of strong possibility for some events of interest:

$$\Delta(p) = min(\pi_{\Gamma}^N(pqr), \pi_{\Gamma}^N(p\neg qr), \pi_{\Gamma}^N(pq\neg r), \pi_{\Gamma}^N(p\neg b\neg r)) = 1$$

$$\Delta(q) = min(\pi_{\Gamma}^N(pqr), \pi_{\Gamma}^N(\neg pqr), \pi_{\Gamma}^N(pq\neg r), \pi_{\Gamma}^N(\neg pq\neg r)) = \beta$$

$$\Delta(r) = min(\pi_\Gamma^N(pqr), \pi_\Gamma^N(\neg pqr), \pi_\Gamma^N(p\neg qr), \pi_\Gamma^N(\neg p\neg qr)) = \gamma.$$

It gives birth to the positive base

$$\Gamma^\Delta = \{[p, 1], [q, \beta], [r, \gamma]\},$$

itself associated with a possibility distribution

$\pi_\Gamma^\Delta(pqr) = 1$, $\pi_\Gamma^\Delta(p\neg qr) = 1$, $\pi_\Gamma^\Delta(pq\neg r) = 1$, $\pi_\Gamma^\Delta(p\neg q\neg r) = 1$,
$\pi_\Gamma^\Delta(\neg pqr) = \beta$, $\pi_\Gamma^\Delta(\neg pq\neg r) = \beta$, $\pi_\Gamma^\Delta(\neg p\neg qr) = \gamma$, $\pi_\Gamma^\Delta(\neg p\neg q\neg r) = 0$.

It can be checked that $\pi_\Gamma^N = \pi_\Gamma^\Delta$. Thus, the preferences are here equivently expressed in a positive manner as a "weighted" disjunction of the three choices p, q and r, stating that p is fully satisfactory, q is less satisfactory, and that r is still less satisfactory. □

This shows that the preferences here can be equivalently encoded under the form of the positive base Γ^Δ, or of the negative base Γ^N [Benferhat *et al.*, 2001d]. Let us mention the representational equivalence [Benferhat *et al.*, 2004a] between qualitative choice logic [Brewka *et al.*, 2004; Benferhat and Sedki, 2008] and Δ-based possibilistic logic, which can be viewed itself as a kind of DNF-like counterpart of standard (CNF-like) possibilistic logic at the representation level.

The above ideas have been applied to preference queries to databases [Bosc *et al.*, 2010; Dubois and Prade, 2013] for modeling the connectives "and if possible" and "or at least" in queries. Besides, it has been shown that the behavior of Sugeno integrals, a well-known family of qualitative multiple criteria aggregation operators, can be described under the form of possibilistic logic bases (of the N-type, or of the Δ-type) [Dubois *et al.*, 2014b]. It is also possible to represent preferences with an additive structure in the possibilistic setting thanks to appropriate fusion operators as noticed in [Prade, 2009].

Inference in Δ-based possibilistic logic While in basic possibilistic logic formulas, the certainty level assesses the certainty that the interpretations violating the formulas are excluded as possible worlds, Δ-based formulas rather express to what extent the models of the formulas are actually possible in the real world. This is a consequence of the decreasingness of set functions Δ, which leads to a non standard behavior with respect to inference. Indeed, the following cut rule can be established [Dubois *et al.*, 2000; Dubois and Prade, 2004] (using the notation of Δ-based formulas):

$$[a \wedge b, \alpha], [\neg a \wedge c, \beta] \vdash [b \wedge c, \min(\alpha, \beta)]$$

This is due to the fact that in terms of models, we have $[b \wedge c] \subseteq [a \wedge b] \cup [\neg a \wedge c]$. Thus, if both any model of $[a \wedge b]$ any model of $[\neg a \wedge c]$ are satisfactory, it should be also the case of any model of $[b \wedge c]$. Moreover, there is also an inference rule mixing strong possibility and weak necessity, established in [Dubois *et al.*, 2013a]:

$$\Delta([a \wedge b]) \geq \alpha \text{ and } \nabla([\neg a \wedge c]) \geq \beta \text{ entails } \nabla([b \wedge c]) \geq \alpha * \beta$$

where $\alpha * \beta = \alpha$ if $\alpha > 1 - \beta$ and $\alpha * \beta = 0$ if $1 - \beta \geq \alpha$.

Besides, it has been advocated in [Casali *et al.*, 2011; Dubois *et al.*, 2013a] that desires obey the characteristic postulate of set functions Δ, namely $\Delta(a \vee b) = \min(\Delta(a), \Delta(b))$. Indeed, all the models of $a \vee b$ are satisfactory (or desirable), if both all the models of a and all the models of b are actually satisfactory.

Then desiring a amounts to find satisfactory *any* situation where a is true. However, this may be a bit too strong since there may exist some exceptional situations that are not satisfactory although a is true. This calls for a *nonmonotonic* treatment of desires in terms of Δ function. This is outlined in [Dubois *et al.*, 2014a].

Bipolar representation The representation capabilities of possibilistic logic are suitable for expressing bipolar information [Dubois and Prade, 2006; Benferhat *et al.*, 2008]. Indeed this setting allows the representation of both negative information and positive information. The bipolar setting is of interest for representing observations and knowledge, or for representing positive and negative preferences. Negative information reflects what is not (fully) impossible and thus remains potentially possible (non impossible). It induces constraints restricting where the real world is (when expressing knowledge), or delimiting the potentially satisfactory choices (when dealing with preferences).

Negative information can be encoded by basic (i.e., necessity-based) possibilistic logic formulas. Indeed, (a, α) encodes $N(a) \geq \alpha$, which is equivalent to $\Pi(\neg a) \leq 1 - \alpha$, and thus reflects the impossibility of $\neg a$, which is all the stronger as α is high. Positive information expressing what is actually possible, or what is really desirable, is encoded by Δ-based formulas $[b, \beta]$, which expresses the constraint $\Delta(b) \geq \beta$. Positive information and negative information are not necessarily provided by the same sources: in other words, they may rely on two different possibility distributions.

The modeling of beliefs and desires provides another example where two possibility distributions are needed, one for restricting the more or less plausible states of the world according to the available knowledge, another for describing the more or less satisfactory states according to the expressed desires [Dubois *et al.*, 2013a]

Fusion operations can be defined at the semantic and at the syntactic level in the bipolar setting [Benferhat and Kaci, 2003; Benferhat *et al.*, 2006]. The fusion of the negative part of the information is performed by using the formulas of subsection 4.2 for basic possibilistic logic. Their counterpart for *positive* information is

$$\Gamma^{\Delta}_{1\oplus 2} = \left| \begin{array}{lll} & \{[a_i, \alpha_i \oplus 0] & \text{s.t. } [a_i, \alpha_i] \in \Gamma^{\Delta}_1\}, \\ \cup & \{[b_j, 0 \oplus \beta_j] & \text{s.t. } [b_j, \beta_j] \in \Gamma^{\Delta}_2\}, \\ \cup & \{[a_i \wedge b_j, \alpha_i \oplus \beta_j] & \text{s.t. } [a_i, \alpha_i] \in \Gamma^{\Delta}_1, [b_j, \beta_j] \in \Gamma^{\Delta}_2\}, \end{array} \right.$$

while $\pi_{\Gamma^{\Delta}_{1\oplus 2}} = \pi_{\Gamma^{\Delta}_1} \oplus \pi_{\Delta^{\Delta}_2}$.

This may be used for aggregating positive (together with negative) preferences given by different agents who state what would be really satisfactory for them (and what they reject more or less strongly). This may also be used for combining positive (together with negative) knowledge. Then positive knowledge is usually made of reported cases that testify what is actually possible, while negative knowledge excludes what is (more or less certainly) impossible.

A consistency condition is natural between positive and negative information, namely what is actually possible (positive information) should be *included* in what is not impossible (complement of the negative information). Since positive information is combined disjunctively (the more positive information we have, the more the interpretations that are actually possible), and negative information conjunctively in a fusion process (the more negative information we have, the less the worlds that are non impossible), this consistency condition should be enforced in the result. This can be done by a revision step that gives priority either to the negative side (in general when handling preferences, where rejections are more important), or to the positive side (it may apply for knowledge when reliable observations are conflicting with general beliefs) [Dubois *et al.*, 2001b].

5.3 Generalized possibilistic logic

In basic possibilistic logic, only conjunctions of possibilistic logic formulas are allowed (since a conjunction is equivalent to the conjunction of its conjuncts, due to the min-decomposability of necessity measures). However, the negation and the disjunction of possibilistic logic formulas make sense as well. Indeed, the pair (a, α) is both a possibilistic logic formula at the object level, and a classical formula at the meta level. Since (a, α) is semantically interpreted as $N(a) \geq \alpha$, a possibilistic formula can be manipulated as a formula that is true (if $N(a) \geq \alpha$) or false (if $N(a) < \alpha$). Then possibilistic formulas can be combined with all propositional connectives. We are then in the realm of *generalized possibilistic logic* (GPL) [Dubois and Prade, 2011a], first suggested in [Dubois and Prade, 2007]. Note that for disjunction, the set of possibility distributions representing the disjunctive constraint '$N(a) \geq \alpha$ or $N(b) \geq \beta$' has no longer a unique extremal element in general, as it is the case for conjunction. Thus the semantics of GPL is in terms of *set* of possibility distributions rather than given by a unique possibility distribution as in basic possibilistic logic.

More precisely GPL is a two-tier propositional logic, in which propositional formulas are encapsulated by modal operators that are interpreted in terms of uncertainty measures from possibility theory. Let $\mathcal{S}_k = \{0, \frac{1}{k}, \frac{2}{k}, ..., 1\}$ with $k \in \mathbb{N} \setminus \{0\}$ be the finite set of certainty degrees under consideration, and let $\mathcal{S}_k^+ = \mathcal{S}_k \setminus \{0\}$. Let $\mathcal{L}$ be the language of all propositional formulas. The language of GPL $\mathcal{L}_{\mathbf{N}}^k$ with $k+1$ certainty levels is as follows:

- If $a \in \mathcal{L}$ and $\alpha \in \mathcal{S}_k^+$, then $\mathbf{N}_\alpha(a) \in \mathcal{L}_{\mathbf{N}}^k$.
- If $\varphi \in \mathcal{L}_{\mathbf{N}}^k$ and $\psi \in \mathcal{L}_{\mathbf{N}}^k$, then $\neg\varphi$ and $\varphi \wedge \psi$ are also in $\mathcal{L}_{\mathbf{N}}^k$.

Here we use the notation $\mathbf{N}_\alpha(a)$, instead of (a, α), emphasizing the closeness with modal logic calculus and allowing the introduction of other associated modalities. So, an agent asserting $\mathbf{N}_\alpha(a)$ has an epistemic state π such that $N(a) \geq \alpha > 0$. Hence $\neg\mathbf{N}_\alpha(a)$ stands for $N(a) < \alpha$, which, given the finiteness of the set of considered certainty degrees, means $N(a) \leq \alpha - \frac{1}{k}$ and thus $\Pi(\neg a) \geq 1 - \alpha + \frac{1}{k}$. Let $\nu(\alpha) = 1 - \alpha + \frac{1}{k}$. Then, $\nu(\alpha) \in \mathcal{S}_k^+$ iff $\alpha \in \mathcal{S}_k^+$, and $\nu(\nu(\alpha)) = \alpha, \forall \alpha \in \mathcal{S}_k^+$. Thus, we can write $\mathbf{\Pi}_\alpha(a) \equiv \neg\mathbf{N}_{\nu(\alpha)}(\neg a)$. Thus, in GPL, one can distinguish between the absence of certainty that a is true ($\neg\mathbf{N}_\alpha(a)$) and the (stronger) certainty statement that a is false ($\mathbf{N}_\alpha(\neg a)$).

The semantics of GPL is defined in terms of normalized possibility distributions over propositional interpretations, where possibility degrees are limited to $\mathcal{S}_k$. A model of a GPL formula is any $\mathcal{S}_k$-valued possibility distribution which satisfies:

- π is a model of $\mathbf{N}_\alpha(a)$ iff $N(a) \geq \alpha$;
- π is a model of $\varphi_1 \wedge \varphi_2$ iff π is a model of φ_1 and a model of φ_2;
- π is a model of $\neg\varphi$ iff π is not a model of φ;

where N is the necessity measure induced by π. As usual, π is called a model of a set of GPL formulas K, written $\pi \models K$, if π is a model of each formula in K. We write $K \models \Phi$, for K a set of GPL formulas and Φ a GPL formula, iff every model of K is also a model of Φ.

The soundness and completeness of the following axiomatization of GPL has been established with respect to the above semantics [Dubois *et al.*, 2012; Dubois *et al.*, 2014c]:

(PL) The Hilbert axioms of classical logic

(K) $\mathbf{N}_\alpha(a \to b) \to (\mathbf{N}_\alpha(a) \to \mathbf{N}_\alpha(b))$

(N) $\mathbf{N}_1(\top)$

(D) $\mathbf{N}_\alpha(a) \to \mathbf{\Pi}_1(a)$

(W) $\mathbf{N}_{\alpha_1}(a) \to \mathbf{N}_{\alpha_2}(a)$, if $\alpha_1 \geq \alpha_2$

with modus ponens as the only inference rule.

Note in particular that when α is fixed we get a fragment of the modal logic KD. See [L. Fariñas del Cerro, 1991; Dubois *et al.*, 1988; Dubois *et al.*, 2000] for previous studies of the links between modal logics and possibility theory. The case where $k = 1$ coincides with the Meta-Epistemic Logic (MEL) that was introduced by Banerjee and Dubois [2009; 2014]. This simpler logic, a fragment of KD with no nested modalities nor objective formulas, can express full certainty and full ignorance only and its semantics is in terms of non-empty subsets of interpretations. Moreover, an extension of MEL [Banerjee *et al.*, 2014] to a language containing modal formulas of depth 0 or 1 only has been shown to be in some sense equivalent to S5 with a restricted language, but with the same expressive power, the semantics being based on pairs made of an interpretation (representing the real world) and a non-empty set of possible interpretations (representing an epistemic state). Note that in MEL, we have $\mathbf{\Pi}_1(a) \equiv \neg\mathbf{N}_1(\neg a)$ whereas in general we only have $\mathbf{\Pi}_1(a) \equiv \neg\mathbf{N}_{\frac{1}{k}}(\neg a)$.

GPL is suitable for reasoning about the revealed beliefs of another agent. It captures the idea that while the consistent epistemic state of an agent about the world is represented by a normalized possibility distribution over possible worlds, the meta-epistemic state of another agent about the former's epistemic state is a family of possibility distributions.

Modalities associated with set functions Δ and ∇ can also be introduced in the GPL language [Dubois *et al.*, 2014c]. For a propositional interpretation ω let us write $conj_\omega$ for the conjunction of all literals made true by ω, i.e. $conj_\omega = \bigwedge_{\omega \models a} a \wedge \bigwedge_{\omega \models \neg a} \neg a$. Since $\Delta(a) = \min_{\omega \in [a]} \Pi(\{\omega\})$, we define:

$$\mathbf{\Delta}_\alpha(a) = \bigwedge_{\omega \in [a]} \mathbf{\Pi}_\alpha(conj_\omega) \ ; \quad \nabla_\alpha(a) = \neg \mathbf{\Delta}_{\nu(\alpha)}(\neg \alpha)$$

Using the modality Δ, for any possibility distribution π over the set of interpretations Ω, we can easily define a GPL theory which has π as its *only model* [Dubois *et al.*, 2014c]. In particular, let $a_1, ..., a_k$ be propositional formulas such that $[a_i] = \{\omega \mid \pi(\omega) \geq \frac{i}{k}\}$. Then we define the theory Φ_π as:

$$\Phi_\pi = \bigwedge_{i=1}^{k} \mathbf{N}_{\nu(\frac{i}{k})}(a_i) \wedge \mathbf{\Delta}_{\frac{i}{k}}(a_i).$$

In this equation, the degree of possibility of each $\omega \in [a_i]$ is defined by inequalities from above and from below. Indeed, $\mathbf{\Delta}_{\frac{i}{k}}(a_i)$ means that $\pi(\omega) \geq \frac{i}{k}$ for all $\omega \in [a_i]$, whereas, $\mathbf{N}_{\nu(\frac{i}{k})}(a_i)$ means $\pi(\omega) \leq \frac{i-1}{k}$ for all $\omega \notin [a_i]$. It follows that $\pi(\omega) = 0$ if $\omega \notin [a_1]$, $\pi(\omega) = \frac{i}{k}$ if $\omega \in [a_i] \setminus [a_{i+1}]$ (for $i < k$) and $\pi(\omega) = 1$ if $\omega \in [a_k]$. In other words, π is indeed the only model of Φ_π. If we view the epistemic state of an agent as a possibility distribution, this means that every epistemic state can be modeled using a GPL theory. Conceptually, the construction of Φ_π relates to the notion of "only knowing " from Levesque [1990]. See [Dubois *et al.*, 2014c] for a detailed study.

Another remarkable application of generalized possibilistic logic is its capability to encode any Answer Sets Programs , choosing $\mathcal{S}_2^+ = 1/2, 1$. In this case, we can discriminate between propositions in which we are fully certain and propositions which we consider to be more plausible than not. This is sufficient to enable us to capture the semantics of rules (with negation as failure) within GPL. See [Dubois *et al.*, 2011] for the introduction of basic ideas in a possibility theory and approximate reasoning perspective, and [Dubois *et al.*, 2012] for theoretical results (including the encoding of equilibrium logic [Pearce, 2006]).

In GPL modalities cannot be nested. Still, it seems possible to give a meaning in the possibility theory setting to a formula of the form $((a, \alpha), \beta)$. Its semantics, viewing (a, α) as a true or false statement, is given by a possibility distribution over the possibility distributions π such that $\pi \leq \pi_{(a,\alpha)}$ (that makes $N(a) \geq \alpha$ true) and all the other possibility distributions, with respective weights 1 and $1 - \beta$. This may reduce to one possibility distribution corresponding to the semantics of $(a, \min(\alpha, \beta))$, via the disjunctive weighted aggregation $max(min(\pi_{(a,\alpha)}, 1), min(1, 1 - \beta)$, which expresses that either it is the case that $N(a) \geq \alpha$ with a possibility level equal to 1, or one knows nothing with possibility $1 - \beta$. Nested modalities are in particular of interest for expressing mutual beliefs of multiple agents.

This suggests to hybridize GPL with possibilistic multiple agent logic, and to study if the Booleanization of possibilistic formulas may give us the capability of expressing mutual beliefs between agents in a proper way, as well as validating inferences with nested modalities such that $(\neg(a,1),\alpha),((a,1) \vee b,\beta) \vdash (b, min(\alpha,\beta))$, following ideas suggested in [Dubois and Prade, 2007; Dubois and Prade, 2011a].

6 CONCLUSION

Possibilistic logic is thirty years old. Although related to the idea of fuzzy sets through possibility measures, possibilistic logic departs from other fuzzy logics [Dubois *et al.*, 2007], since it primarily focuses on classical logic formulas pervaded with qualitative uncertainty. Indeed basic possibilistic logic, as well as generalized possibilistic logic remain close to classical logic, but still allow for a sophisticated and powerful treatment of modalities.

The chapter is an attempt at offering a broad overview of the basic ideas underlying the possibilistic logic setting, through the richness of its representation formats, and its various applications to many AI problems, in relation with the representation of epistemic states and their handling when reasoning from and about them. In that respect possibilistic logic can be compared to other approaches including nonmonotonic logics, modal logics, or Bayesian nets.

Directions for further research in possibilistic logic includes theoretical issues and application concerns. On the theoretical side, extensions to nonclassical logics [Besnard and Lang, 1994], to the handling of fuzzy predicates [Dellunde *et al.*, 2011; El-Zekey and Godo, 2012], to partially ordered sets of logical formulas [Cayrol *et al.*, 2014] are worth continuing, relations with conditional logics [Halpern, 2005; Lewis, 1973; Hájek, 1998] worth investigating. On the applied side, it seems that the development of efficient implementations, of applications to information systems, and of extensions of possibilistic logic to multiple agent settings and to argumentation [Chesñevar *et al.*, 2005; Alsinet *et al.*, 2008; Nieves and Cortés, 2006; Godo *et al.*, 2012; Amgoud and Prade, 2012] would be of particular interest.

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