# Symbolic possibilistic logic: completeness and inference methods

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**Abstract.** This paper studies the extension of possibilistic logic to the case when weights attached to formulas are symbolic and stand for variables that lie in a totally ordered scale, and only partial knowledge is available on the relative strength of these weights. A proof of the soundness and the completeness of this logic according to the relative certainty semantics in the sense of necessity measures is provided. Based on this result, two syntactic inference methods are presented. The first one calculates the necessity degree of a possibilistic formula using the notion of minimal inconsistent sub-base. A second method is proposed that takes inspiration from the concept of ATMS. Notions introduced in that area, such as nogoods and labels, are used to calculate the necessity degree of a possibilistic formula. A comparison of the two methods is provided, as well as a comparison with the original version of symbolic possibilistic logic.

## 1 Introduction

Possibilistic logic [1] is an approach to reason under uncertainty using totally ordered propositional bases. In this logic, each formula is assigned a degree, often encoded by a weight belonging to $(0, 1]$, seen as a lower bound on the certainty level of the formula. Such degrees of certainty obey graded versions of the principles that found the notions of belief or knowledge in epistemic logic, namely the conjunction of two formulas is not believed less than the least believed of their conjuncts. This is the basic axiom of degrees of necessity in possibility theory [2]. See [3] for a recent survey of possibilistic logic. Deduction in possibilistic logic follows the rule of the weakest link: the strength of an inference chain is that of the least certain formula involved in this chain. The weight of a formula in the deductive closure is the weight of the strongest path leading from the base to the formula. Possibilistic logic has developed techniques for knowledge representation and reasoning in various areas, such as non-monotonic reasoning, belief revision and belief merging see references in [3].

About 10 years ago, a natural extension of possibilistic logic was proposed using partially ordered symbolic weights attached to formulas [4], we call here symbolic possibilistic logic, for short. Weights represent ill-known certainty values on a totally ordered scale. Only partial knowledge on the relative strength of weights is supposed to be available, under the form of weak inequality constraints. In that paper, a possibilistic knowledge base along with the knowledge pertaining to weights is encoded in propositional logic, augmenting the atomic formulas with those pertaining to weights. They give a characterisation, and a deduction method for plausible inference in this logic using the idea of forgetting variables. This generalisation of possibilistic logic differs from other approaches that represent sets of formulas equipped with a partial order in the setting of conditional logics [5]. It also contrasts with another line of research consisting in viewing a partial order on weights as a family of total orders, thus viewing a symbolic possibilistic base as a set of usual possibilistic bases [6].

In this paper, we revisit symbolic possibilistic logic, first by assuming strict inequality constraints between weights and by focusing on the weighted completion of a possibilistic knowledge base. We provide an original completeness proof, absent from [4]. This proof is more general than the completeness proof of standard possibilistic logic as, contrary to the latter, we cannot rely on classical inference from sets of formulas having at least a given certainty degree. Specific inference methods to compute the symbolic weight attached to a conclusion are proposed, especially some inspired by the literature on abductive reasoning initiated by Reiter [7]. Our approach yields a partial order on the language, while the alternative partially ordered generalizations of possibilistic logic [4, 6] only compute a set of plausible consequences.

## 2 Symbolic possibilistic logic revisited

In this section, first we recall the construction of possibilistic logic. Then, we present symbolic possibilistic logic. In the paper, $\mathcal{L}$ denotes a propositional language. Formulas are denoted by $\phi_1 \cdots \phi_n$, and $\Omega$ is the set of interpretations. $[\phi]$ denotes the set of models of $\phi$, a subset of $\Omega$. As usual, $\vdash$ and $\models$ denote syntactic inference and semantic entailment, respectively.

### 2.1 Background on standard possibilistic logic

Possibilistic logic is an extension of classical logic which handles weighted formulas of the form $(\phi_j, p_j)$ where $\phi_j$ is a propositional formula and $p_j \in ]0, 1]$. $(\phi_j, p_j)$ is interpreted by $N(\phi_j) \geq p_j$, where $N$ is a necessity measure, the conjugate of a possibility measure. A possibility measure [2] is defined on subsets of $\Omega$ from a possibility distribution $\pi$ on $\Omega$ as $\Pi(A) = \max_{\omega \in A} \pi(\omega)$ expressing the plausibility of any proposition $\phi$, with $[\phi] = A$, and the necessity measure expressing certainty levels is defined by $N(A) = 1 - \Pi(\overline{A})$ where $\overline{A}$ is the complement of $A$.

A possibilistic base is a finite set of weighted formulas $\Sigma = \{(\phi_j, p_j), j = 1 \cdots m\}$. It can be associated with a possibility distribution $\pi_\Sigma$ on $\Omega$ in the following way:

$$\forall j, \pi_j(\omega) = \begin{cases} 1 \text{ if } \omega \in [\phi_j], \\ 1 - p_j \text{ if } \omega \notin [\phi_j] \end{cases} \qquad \pi_\Sigma(\omega) = \min_j \pi_j(\omega). \tag{1}$$

Note that $\pi_j$ is the least informative possibility distribution among those such that $N(\phi_j) \geq p_j$, where a possibility distribution $\pi$ is less informative than $\rho$ if and only if $\pi \geq \rho$. Likewise $\pi_\Sigma$ is the least informative possibility distribution compatible with the base $\Sigma$, on behalf of *the principle of minimal specificity*. It can be checked that $N_\Sigma(\phi_j) = \min_{\omega \notin [\phi_j]}(1 - \pi_\Sigma(\omega)) \geq p_j$ is the least necessity degree in agreement with $\Sigma$. However, it may occur that $N_\Sigma(\phi_j) > p_j$. The (semantic) closure of $\Sigma$ is then defined by $\{(\phi, N_\Sigma(\phi)) : \phi \in \mathcal{L} : N_\Sigma(\phi) > 0\}$, which simply corresponds to a ranking on the language. The semantics of possibilistic logic allows to replace weighted conjunctions $(\bigwedge_i \phi_i, p)$ by a set of formulas $(\phi_i, p)$ without altering the underlying possibility distribution, since $N(\phi \wedge \psi) = \min(N(\phi), N(\psi))$: from the minimal specificity principle, we can associate the same weight to each sub-formula in the conjunction. Therefore, we can turn any possibilistic base into a semantically equivalent weighted clausal base.

**Syntactic inference in possibilistic logic** A sound and complete syntactic inference $\vdash_\pi$ for possibilistic logic can be defined using axioms of classical logic turned into formulas weighted by 1 and inference rules [1]:

- Weakening rule: If $p_i > p_j$ then $(\phi, p_i) \vdash_\pi (\phi, p_j)$
- Modus Ponens : $\{(\phi \rightarrow \psi, p), (\phi, p)\} \vdash_\pi (\psi, p)$

This Modus Ponens rule embodies the law of accepted beliefs at any level, assumed they form a deductively closed set [8]. It is related to axiom K of modal logic. The soundness and completeness of possibilistic logic for the above proof theory can be translated by the following equality [1]: $N_\Sigma(\phi) = \max\{p : \Sigma \vdash_\pi (\phi, p)\}$

Note that we can also express inference in possibilistic logic by classical inference on $p$-cuts $\Sigma_p^{\geq} = \{\phi_j : p_j \geq p\}$[1]: $N_\Sigma(\phi) = \max\{p : (\Sigma_p^{\geq}) \vdash \phi\}$.

Inconsistency degree $Inc(\Sigma)$ of a possibilistic base $\Sigma$ is defined as follows: $Inc(\Sigma) = \max\{p | \Sigma \vdash_\pi (\bot, p)\}$. It can be proved that $N_\Sigma(\phi) = Inc(\Sigma \cup (\neg\phi, 1))$ [1, 9].

### 2.2 Symbolic possibilistic logic (SPL)

In symbolic possibilistic logic (SPL), only partial knowledge is available on the relative strength of weights attached to formulas. So, weights are symbolic expressions taking values on a totally ordered necessity scale (such as ]0, 1]), and there is a set of constraints over these weights, describing their relative strength. The name "symbolic possibilistic logic" indicates that we shall perform symbolic computations on the weights. The set $\mathcal{P}$ of symbolic weights $p_j$ is recursively obtained using a finite set of variables (called elementary weights) $H = \{a_1, \ldots, a_k\}$ taking values on the scale $]0, 1]$ and $\max / \min$ expressions built on $H$: $H \subseteq \mathcal{P},\ 1 \in \mathcal{P}$, and if $p_i, p_j \in \mathcal{P}$, then $\max(p_i, p_j), \min(p_i, p_j) \in \mathcal{P}$.

Let $\Sigma = \{(\phi_j, p_j), j = 1, \cdots, m\}$ be a symbolic possibilistic base where $p_j$ is a $\max / \min$ expression built on $H$. A formula $(\phi_j, p_j)$ is still interpreted as $N(\phi_j) \geq p_j$ [4]. The knowledge about weights is encoded by a finite set $\mathcal{C} = \{p_i > p_j\}$ of constraints between $\max / \min$ expressions, a partial ordering on symbolic expressions. We can prove $p > q$, denoted by $\mathcal{C} \vDash p > q$ if and only if every valuation of symbols appearing in $p, q$ (on $]0, 1]$) which satisfies the constraints in $\mathcal{C}$ also satisfies $p > q$.

At the semantic level, $N_\Sigma(\phi)$ is now a symbolic $\max / \min$ expression of the form

$$N_\Sigma(\phi) = \min_{\omega \not\vDash \phi} \max_{j:\omega \not\vDash \phi_j} p_j. \tag{2}$$

We directly use the expression defined in standard possibilistic logic. The main difference with standard possibilistic logic is that we cannot simplify this expression down to a single weight. To perform inference at the syntax level, one must slightly reformulate the inference rules of possibilistic logic in order to account for the symbolic nature of weights:

- Fusion rule: $\{(\phi, p), (\phi, p')\} \vdash_\pi (\phi, \max(p, p'))$
- Weakening rule: $(\phi, p_i) \vdash_\pi (\phi, \min(p_j, p)), \forall p$
- Modus Ponens : $\{(\phi \to \psi, p), (\phi, p)\} \vdash_\pi (\psi, p)$

We call skeleton of a possibilistic base $\Sigma$ the set of propositional formulas appearing in it, and denote it by $\Sigma^*$. If $B$ is a subset of the skeleton $\Sigma^*$ of $\Sigma$ that implies $\phi$, it is clear that $(\Sigma, \mathcal{C}) \vdash_\pi (\phi, \min_{\phi_j \in B} p_j)$. Using syntactic inference, we can compute the expression representing the strength of deduction of $\phi$ from $\Sigma$:

$$N_\Sigma^\vdash(\phi) = \max_{B \subseteq \Sigma^*, B \vdash \phi} \min_{j : \phi_j \in B} p_j. \tag{3}$$

Note that in the above expression, it suffices to take $\max$ on all minimal subsets $B$ for inclusion that imply $\phi$. The aim of SPL is to compare the strength degrees of any two formulas in the language via their resulting weights.

**Definition 1** $(\Sigma, \mathcal{C})$ *implies that* $\phi$ *is more certain than* $\psi$ *(*$(\Sigma, \mathcal{C}) \models \phi > \psi$*) if and only if* $\mathcal{C} \vDash N_\Sigma^\vdash(\phi) > N_\Sigma^\vdash(\psi)$.

**Example 1** *Let* $\Sigma = \{(x, p), (\neg x \vee y, q), (\neg x, r), (\neg y, s)\}$, $\mathcal{C} = \{p > q, q > r, q > s\}$. *Then,* $N_\Sigma^\vdash(y) = \max(\min(p, q), \min(p, r)) = q$ *and* $N_\Sigma^\vdash(x) = p$. *So,* $x > y$.

Note that in SPL, comparing the certainty degrees of formulas as per Definition 1 requires that the set of constraints $\mathcal{C}$ be not empty. Otherwise, no strict inequalities can be inferred between formula weights.

## 3 The completeness of symbolic possibilistic logic

The completeness of SPL comes down to proving that the two following expressions are equal : $N_\Sigma(\phi) = N_\Sigma^\vdash(\phi), \forall \phi \in \mathcal{L}$. This proof does not appear in [4], where the focus is on plausible inference.

**Proposition 1** *SPL is sound and complete for the above inference system.*

The proof cannot rely on cuts, like for standard possibilistic logic, due to the fact that the weights are partially ordered. So we provide the sketch of a direct proof that the two expressions of $N_\Sigma(\phi)$ and $N_\Sigma^\vdash(\phi)$ coincide independently of constraints in $\mathcal{C}$. In this proof, we use the notion of hitting-set [7]:

**Definition 2 (Hitting-set)** *Let* $\mathcal{S}$ *be a collection of sets. A hitting-set of* $\mathcal{S}$ *is a set* $H \subseteq \cup_{S_i \in \mathcal{S}} S_i$ *such that* $H \cap S_i \neq \varnothing$ *for each* $S_i \in \mathcal{S}$. *A hitting-set* $H$ *of* $\mathcal{S}$ *is minimal if and only if no strict subset of* $H$ *is a hitting-set of* $\mathcal{S}$.

**Proof of Proposition 1:** Due to the lack of space, we only give the list of steps and results needed. Let $\Sigma_\omega^-$ be the subset of formulas in $\Sigma^*$ falsified by $\omega$, and $\Sigma_\omega^+$ be the subset of formulas in $\Sigma^*$ satisfied by $\omega$. We have to prove that $\min_{\omega \not\models \phi} \max_{j : \phi_j \in \Sigma_\omega^-} p_j = \max_{B \subseteq \Sigma^*, B \vdash \phi} \min_{\phi_j \in B} p_j$. We distinguish cases according to whether $\Sigma^*$ is consistent or not.

1. Suppose that $\Sigma^*$ is consistent. Then all $B$'s implying $\phi$ are consistent.
   We note that:
   - For $N_\Sigma^{\vdash}(\phi)$, it is sufficient to consider the minimal (for set-inclusion) subsets of $\Sigma^*$, say $B_i, i = 1, n$, that imply $\phi$: $N_\Sigma^{\vdash}(\phi) = \max_{i=1,\cdots,n} \quad \min_{\phi_j \in B_i} p_j$.
   - For $N_\Sigma(\phi)$, it is sufficient to consider the interpretations $\omega$ such that $\omega \not\models \phi$ and $\Sigma_\omega^-$ is minimal (for set inclusion) : $N_\Sigma(\phi) = \min_{\omega \not\models \phi, \Sigma_\omega^- \text{ minimal}} \max_{j:\phi_j \in \Sigma_\omega^-} p_j$.

   **Lemma 1** *If $\Sigma^*$ is a minimal (for set inclusion) base that implies $\phi$, $N_\Sigma(\phi) = N_\Sigma^{\vdash}(\phi)$.*

   We conclude that $N_\Sigma(\phi) \geq N_\Sigma^{\vdash}(\phi)$ since for each $B \subseteq \Sigma, N_\Sigma(\phi) \geq N_B(\phi) = N_B^{\vdash}(\phi)$. Using distributivity, we can rewrite the syntactic necessity degree in terms of the minimal hitting-sets of the set $\{B_1, \ldots, B_n\}$. By indexing all the minimal hitting-sets $H_s$ of $\{B_1, \ldots, B_n\}$ by $s \in \mathcal{S}$ we obtain:

   $$N_\Sigma^{\vdash}(\phi) = \max_{B \subseteq \Sigma^*, B \vdash \phi} \min_{\phi_j \in B} p_j = \min_{s \in \mathcal{S}} \max_{\phi_j \in H_s} p_j.$$

   **Lemma 2** $\forall \omega \not\models \phi, \Sigma_\omega^-$ *is a hitting-set of* $\{B_1, \ldots B_n\}$ *(that is* $\forall i, B_i \cap \Sigma_\omega^- \neq \varnothing$*).*

   Note that the above result holds in particular when $\Sigma_\omega^-$ is minimal. The sub-bases $\Sigma_\omega^-$ such that $\omega \not\models \phi$ that are minimal are the complements of the maximal sub-bases $M_{\neg\phi}$ of $\Sigma^*$ consistent with $\neg\phi$, the set of which we denote by $\mathcal{M}_{\neg\phi}$. Notice that:

   **Lemma 3** *The complement of each minimal hitting-set $H_s$ of $\{B_1, \ldots B_n\}$ is a maximal sub-base of $\Sigma^*$ consistent with $\neg\phi$.*

   Then we can obtain the converse inequality $N_\Sigma(\phi) \leq N_\Sigma^{\vdash}(\phi)$ since:
   $N_\Sigma^{\vdash}(\phi) = \min_{s \in \mathcal{S}} \max_{\phi_j \in H_s} p_j = \min_{M_{\neg\phi} = \overline{H_s}, s \in \mathcal{S}} \max_{\phi_j \notin M_{\neg\phi}} p_j$
   $\geq \min_{M_{\neg\phi} \in \mathcal{M}_{\neg\phi}} \max_{\phi_j \notin M_{\neg\phi}} p_j = N_\Sigma(\phi)$.
2. Suppose that $\Sigma^*$ is inconsistent with no constraint on the weights. Then, some of the minimal sub-bases that imply $\phi$ may be inconsistent. We have the following results:
   - Let $I_1, \ldots, I_p$ be the minimal inconsistent sub-bases of $\Sigma^*$ (smallest inconsistent sub-bases in the sense of inclusion). The inconsistency degree of $\Sigma$ is $Inc(\Sigma) = N_\Sigma^{\vdash}(\bot) = \max_{k=1}^{p} \min_{\phi_j \in I_k} p_j$, and $N_\Sigma^{\vdash}(\phi) = \max(Inc(\Sigma),$
     $\max_{i=1}^{n} \min_{\phi_j \in B_i} p_j)$, $B_i$ being the minimal consistent sub-bases that imply $\phi$ (if any).
   - $N_\Sigma^{\vdash}(\phi) \geq Inc(\Sigma)$. However there is never strict inequality if $\mathcal{C} = \varnothing$.
   - The definition of $N_\Sigma(\phi)$ is the same as in the consistent case. However, $\forall \omega, \Sigma_\omega^+ \subset \Sigma$ (since $\Sigma_\omega^+$ is consistent).

   Now, we are able to prove completeness:
   - Lemma 1 can be used. Now, $\Sigma^*$ is a minimal inconsistent base implying $\phi$, and none of its sub-bases implies $\phi$. The inequality $N_\Sigma(\phi) \geq N_\Sigma^{\vdash}(\phi)$ still holds (note that minimality does not exclude inconsistency).
   - For Lemma 2, $\Sigma_\omega^+$ is always consistent. So in the case of an inconsistent set $I_i$, we cannot have $I_i \subset \Sigma_\omega^+$. The proof of Lemma 2 still holds, since the sets $\overline{H_s}$ are consistent, as the $M_{\neg\phi}$.

   So completeness has been proved even if the base $\Sigma^*$ is inconsistent. □

**Remark**: However, it may happen that some minimal inconsistent subset $I_i$ of $\Sigma^*$ is not a minimal sub-base implying $\phi$. For instance, if $\Sigma = \{(\phi, a), (\neg\phi, b)\}$ the unique minimal sub-base implying $\phi$ is $\{\phi\}$. In that case, $N_\Sigma^{\vdash}(\phi) = \max_{B \subseteq \Sigma^*, B \vdash \phi} \min_{\phi_j \in B} p_j = \max(\min(a, b), a) = a = N_\Sigma(\phi)$. Similarly, $N_\Sigma^{\vdash}(\neg\phi) = b$. So we have $N_\Sigma^{\vdash}(\bot) = \min(a, b) \leq N_\Sigma^{\vdash}(\phi)$ and $N_\Sigma^{\vdash}(\bot) \leq N_\Sigma^{\vdash}(\neg\phi)$. We have $\{a\} \subset \{a, b\}$ but it cannot be concluded that $N_\Sigma^{\vdash}(\bot) < N_\Sigma^{\vdash}(\neg\phi)$.

## 4 Toward inference methods in symbolic possibilistic logic

In this section, we will present two syntactic inference methods that calculate the necessity degree $N^{\vdash}_{\Sigma}(\phi)$ of a possibilistic formula. The first method is based on the use of the notion of minimal inconsistent sub-base. The second one is inspired by abductive reasoning. We assume that the weights bearing on formulas of the original SPL base are elementary, with possibility of assigning the same weight to different formulas.

### 4.1 Syntactic inference based on minimal inconsistent sub-bases

Given a formula $\phi$, computing the expression in equation (3) requires the determination of all minimal sub-bases $B_i$ such that $B_i \vdash \phi$. Some of the minimal sub-bases that imply $\phi$ may be inconsistent. In that case, they are minimal inconsistent in $\Sigma^*$.

**Lemma 4** *Let $B \subseteq \Sigma^*$ inconsistent and minimal implying $\phi$. Then $B$ is minimal inconsistent in $\Sigma^*$.*

So, if $B \subseteq \Sigma^*$ is a minimal sub-base implying $\phi$, either $B$ is consistent or $B$ is a minimal inconsistent sub-base of $\Sigma^*$. However, it may happen that some minimal inconsistent sub-base in $\Sigma^*$ is not a minimal sub-base implying $\phi$. It follows easily:

**Proposition 2** *Let $B_1, \cdots, B_k$ be the minimal consistent sub-bases of $\Sigma^*$ implying $\phi$. Let $I_1, \cdots, I_l$ be the minimal inconsistent sub-bases in $\Sigma^*$ which do not contain any $B_j$, $j = 1 \cdots k$, $N^{\vdash}_{\Sigma}(\phi) = \max(\max_{i=1}^{k} \min_{\phi_j \in B_i} p_j, \max_{i=1}^{l} \min_{\phi_j \in I_i} p_j)$*

Besides, we know that $B \subseteq \Sigma^*$ is minimal implying $\phi$ if and only if $B$ is minimal such that $B \cup \{\neg\phi\}$ is inconsistent. We can prove even more:

**Proposition 3** *Let $(\Sigma, \mathcal{C})$ be an SPL base, and $B$ a sub-base of $\Sigma^*$.*

- *If $B$ is consistent and minimal implying $\phi$ then $B \cup \{\neg\phi\}$ is a minimal inconsistent sub-base of $\Sigma^* \cup \{\neg\phi\}$.*
- *If $K$ is a minimal inconsistent sub-base of $\Sigma^* \cup \{\neg\phi\}$ containing $\neg\phi$, then $K \setminus \{\neg\phi\}$ is consistent, minimal implying $\phi$.*

Due to Proposition 2 and Proposition 3, computing $N^{\vdash}_{\Sigma}(\phi)$ amounts to determining:

- the set of minimal inconsistent subsets $K_i$ of $\Sigma^* \cup \{\neg\phi\}$ containing $\neg\phi$;
- the minimal inconsistent sub-bases of $\Sigma^*$ which do not contain any of the $B_i = K_i \setminus \{\neg\phi\}$'s obtained in the previous step.

The above computation comes down to the well-known problem of determining the minimal inconsistent sub-bases, forming a set $MIS(S)$, of a given set of formulas $S$. Let $\mathcal{B}^{\vdash}(\phi) = \{B \subseteq \Sigma^* | B \cup \{\neg\phi\} \in MIS(\Sigma^* \cup \{\neg\phi\})\}$ and $\mathcal{B}_i(\phi) = \{B \in MIS(\Sigma^*) | B$ does not contain any base from $\mathcal{B}^{\vdash}(\phi)\}$. Then let $\mathcal{B}(\phi) = \mathcal{B}^{\vdash}(\phi) \bigcup \mathcal{B}_i(\phi)$. So, the necessity degree of a formula $\phi$ can be computed as follows:

$$N^{\vdash}_{\Sigma}(\phi) = \max_{B_i \in \mathcal{B}(\phi)} \min_{\phi_j \in B_i} p_j \quad (4)$$

The most efficient method for solving the $MIS$ problem exploits the duality between minimal inconsistent subsets $MIS(S)$, and maximal consistent subsets $MCS(S)$, and the fact that checking the consistency of a base is less time-consuming than checking its inconsistency [10]. Given a propositional base $S$, $MIS(S)$ is obtained from $MCS(S)$ using hitting-sets [10, 11].

Once we are able to compute the necessity degree of a formula, according to definition 1, we can compare two SPL formulas by comparing their necessity degrees which are $\max$ / $\min$ expressions. So we have to check whether $\mathcal{C} \vDash N_{\Sigma}^{\vdash}(\phi) > N_{\Sigma}^{\vdash}(\psi)$ that is $\mathcal{C} \vDash \max_{B \in \mathcal{B}(\phi)} \min_{i:\phi_i \in B} a_i > \max_{C \in \mathcal{B}(\psi)} \min_{j:\phi_j \in C} b_j$ . That amounts to finding an expression $\min(a_1, \cdots a_n)$ in $N_{\Sigma}^{\vdash}(\phi)$ which dominates all expressions $\min(b_1, \cdots, b_m)$ in $N_{\Sigma}^{\vdash}(\psi)$. Rather than applying this test in a brute force way, it is natural to use sets of elementary weights instead of formulas, and to simplify the expressions of $N_{\Sigma}^{\vdash}(\phi)$ and $N_{\Sigma}^{\vdash}(\psi)$ using $\mathcal{C}$ prior to comparing them. The inference technique proposed next is useful to that effect.

### 4.2 Syntactic inference based on ATMSs

In this section, we present another syntactic method for SPL inference, based on abductive reasoning. Namely, consider the weights involved in the computation of $N_{\Sigma}^{\vdash}(\phi)$ as assumptions that explain the certainty of $\phi$. It suggests to use an Assumption-based Truth-Maintenance System (ATMS [12]), in which a distinction is made between two kinds of data, the data representing knowledge and the data representing assumptions. We first recall the basic definitions of ATMS, then we show how we encode an SPL base in order to use an ATMS for computing the necessity degree of a formula.

**Definition 3** *Let ($\mathcal{J}$, $\mathcal{A}$) be an ATMS base where $\mathcal{J}$ is a consistent base of propositional formulas, and $\mathcal{A}$ is a set of propositional variables (the assumptions).*

- *Any subset $E$ of $\mathcal{A}$ is called an environment*
- *An environment $E$ is $\mathcal{J}$-incoherent if and only if $E \cup \mathcal{J}$ is inconsistent*
- *A nogood is a minimal $\mathcal{J}$-incoherent environment*
- *An environment $E$ supports $\phi$ if and only if $E$ is not $\mathcal{J}$-incoherent and $E \cup \mathcal{J} \vdash \phi$*

*Given ($\mathcal{J}$, $\mathcal{A}$) and a formula $\phi$, the ATMS is able to provide all the minimal environments that support $\phi$, under the form of a set $Label(\phi)$.*

Given an SPL base $(\Sigma, \mathcal{C})$, the possibilistic base $\Sigma$ is encoded by a pair ($\mathcal{J}$, $\mathcal{A}$) as follows : each elementary weight $a_i$ is associated with a propositional variable (for simplicity we keep $a_i$ as propositional variable) and each SPL formula $(\phi_i, a_i)$ is encoded by the propositional formula $\neg a_i \vee \phi_i$.

**Definition 4** *Let $(\Sigma, \mathcal{C})$ be an SPL base. The associated ATMS base ($\mathcal{J}_{\Sigma}, \mathcal{A}$) is defined by : $\mathcal{J}_{\Sigma} = \{\neg a_i \vee \phi_i | (\phi_i, a_i) \in \Sigma\}$ and $\mathcal{A} = \{a_i | (\phi_i, a_i) \in \Sigma\}$*

As shown in Section 4.1, in order to compute $N_{\Sigma}^{\vdash}(\phi)$, we have to consider the sub-bases of $\Sigma^*$ which are minimal implying $\phi$ and consistent, and then some of the minimal inconsistent sub-bases of $\Sigma^*$. Moreover, for computing $N_{\Sigma}^{\vdash}(\phi)$, we only need the weights associated with the formulas belonging to these sub-bases. With the encoding of Definition 4, it is easy to see that each consistent sub-base of $\Sigma^*$ which is minimal implying $\phi$ exactly corresponds to an environment in $Label(\phi)$ with respect to the ATMS base ($\mathcal{J}_\Sigma$, $\mathcal{A}$). And each minimal inconsistent sub-base of $\Sigma^*$ exactly corresponds to a nogood with respect to the ATMS base ($\mathcal{J}_\Sigma$, $\mathcal{A}$). So, it follows easily from Proposition 2 that :

**Proposition 4** *Given an SPL base $(\Sigma, \mathcal{C})$ and the associated ATMS base ($\mathcal{J}_\Sigma$, $\mathcal{A}$), let $\mathcal{U}(\phi) = \{U_1, \cdots, U_k\}$ be the so-called* useful nogoods *for $\phi$, i.e. the nogoods which do not contain any environment of $Label(\phi)$. Then we have:*
$N^{\vdash}_{\Sigma}(\phi) = \max(\max_{E \in Label(\phi)} \min_{a \in E} a, \max_{i=1}^{k} \min_{a \in U_i} a)$.

See [13] for further details on calculating labels and nogoods.

**Example 2** *Let $\Sigma = \{(\neg x \vee y, a), (x, b), (\neg y, c), (\neg x, e)\}$. This SPL base is encoded by the ATMS base: $\mathcal{J}_\Sigma = \{\neg a \vee \neg x \vee y, \neg b \vee x, \neg c \vee \neg y, \neg e \vee \neg x\}$ and $\mathcal{A} = \{a, b, c, e\}$.*

*We obtain $Label(y) = \{\{a, b\}\}$. The nogoods are $\{a, b, c\}, \{b, e\}$, hence only the second one is useful for y. So, $N^{\vdash}_{\Sigma}(y) = \max(\min(a, b), \min(b, e))$.*

### 4.3 Comparing complex symbolic weights

One of the benefits of the last method lies in the fact that everything is computed only in terms of weights (in the label of the formula and the useful nogoods). Then constraints on weights can be used to simplify the $\max / \min$ expressions, while in the previous method, we use all formulas in the symbolic possibilistic base. Moreover, in the ATMS method, one can think of exploiting constraints and simplify the sets of weights involved in the comparison of the necessity degrees at the moment we are producing them.So it is natural to simplify the expressions of $N^{\vdash}_{\Sigma}(\phi)$ and $N^{\vdash}_{\Sigma}(\psi)$ prior to comparing them,

– first by replacing each set of weights $B \in Label(\phi) \cup \mathcal{U}(\phi)$ by the reduced set of weights $W = \min_{\mathcal{C}}(B)$ consisting of the least elementary weights in $B$ according to the partial order defined by the constraints in $\mathcal{C}$.
– Then by deleting the dominated sets $W$ in the resulting family in the sense that $\mathcal{C} \vDash \min\{a \in W'\} > \min\{a \in W\}$ for some other set $W'$, using Algorithm 1.

Of course we can apply these simplifications as soon as elements of the labels or useful nogoods are produced.

**Example 2 (continued)** Consider again $\Sigma = \{(\neg x \vee y, a), (x, b), (\neg y, c), (\neg x, e)\}$ with $\mathcal{C} = \{a > b, a > c, b > e, c > e\}$. We want to check if $\mathcal{C} \vDash N^{\vdash}_{\Sigma}(y) > N^{\vdash}_{\Sigma}(\neg x)$. Note that $Label(y) = \{\{a, b\}\}$ and $\mathcal{U}(y) = \{\{b, e\}\}$. Likewise $Label(\neg x) = \{\{a, c\}, \{e\}\}$, and there is no useful nogood for $\neg x$.
Using $\mathcal{C}$ we can reduce $\{a, b\}$ to $\{b\}$ and $\{b, e\}$ to $\{e\}$ and the necessity degree of $y$ to $b$, since $b > e \in \mathcal{C}$. Likewise we can reduce $\{a, c\}$ to $\{c\}$ and the necessity degree of $\neg x$ to $c$ since $c > e \in \mathcal{C}$. Now, $b > c \notin \mathcal{C}$, so we cannot conclude $y > \neg x$ (nor the converse).

In general, the deletion of dominated sets of weights can be achieved by means of Algorithm 1 applied to all pairs of reduced sets in $Label(\phi) \cup \mathcal{U}(\phi)$.
Finally we can compare the set of non-dominated reduced subsets from $Label(\phi) \cup \mathcal{U}(\phi)$ with the one for $Label(\psi) \cup \mathcal{U}(\psi)$, in order to decide if $\phi > \psi$, using Algorithm 2.

**Algorithm 1:** $Comp\_Min$

**Data**: $F$ and $G$ two sets of weights, $\mathcal{C}$ a set of constraints.
**Result**: $\min F > \min G$ ?

Dec:=false;
**while** *Dec=false and* $b_i \in G$ **do**
  Dec:=true;
  **while** *Dec=true and* $a_i \in F$ **do**
    Dec:=Dec $\wedge\ a_i > b_i \in \mathcal{C}$;
**return** *Dec;*

**Algorithm 2:** $Comp\_Max$

**Data**: $\mathcal{F}$ and $\mathcal{G}$ two families of sets of weights, $\mathcal{C}$ a set of constraints
**Result**:
$\max_{F\in\mathcal{F}} \min F > \max_{G\in\mathcal{G}} \min G$ ?

Dec:=false;
**while** *Dec=false and* $E_j \in \mathcal{F}$ **do**
  Dec:=true;
  **while** *Dec=true and* $E_i \in \mathcal{G}$ **do**
    Dec:=Dec
    $\wedge\ Comp\_Min(E_i, E_j, \mathcal{C})$;
**return** *Dec;*

## 5 Related works

The question of reasoning with a partially ordered knowledge base encoded in a symbolic possibilisitic logic has been addressed previously in [4]. These authors have proposed to encode symbolic possibilistic pairs in propositional logic like in section 4.2. However there are several differences:

- A possibilistic formula $(\phi, a)$ in [4] is encoded as a formula $A \vee \phi$ where $A$ is a variable supposed to mean "$\geq a$", i.e. $[a, 1]$ (while we use $\neg a \vee \phi \in \mathcal{J}_\Sigma$).
- Constraints between weights in [4] are reflexive (not strict), of the form $p \geq q$ with complex max-min weights. It allows them to be encoded also as propositional formulas (for elementary constraints, $\neg A \vee B$ encodes $a \geq b$). It is then possible to express all pieces of information (formulas and weights) about an SPL base in a single propositional base containing only clauses, which makes it natural to use the variable forgetting technique so as to deduce the necessity degree of a formula. We cannot encode strict constraints using a material implication, hence the use of the ATMS approach. We must encode the SPL base in two parts and we thus apply techniques such as MIS, and ATMS notions plus specific algorithms to compare complex weights.
- In [4], $\mathcal{C} \vDash p > q$ means $\mathcal{C} \vDash p \geq q$ and $\mathcal{C} \nvDash q \geq p$ and is somewhat analogous to strict Pareto order between vectors. With this vision, from $\Sigma = \{(\phi, a), (\psi, b)\}$ and $\mathcal{C} = \varnothing$ we could infer infer $N_\Sigma(\phi \vee \psi) > N_\Sigma(\phi)$. Indeed, one has $N_\Sigma(\phi) = a, N_\Sigma(\phi \vee \psi) = \max(a, b)$ $\mathcal{C} \vDash \max(a, b) \geq a$ but not $\mathcal{C} \vDash a \geq \max(a, b)$. This is problematic because it amounts to interpreting strict inequality as the impossibility of proving a weak one, which is non-monotonic. In our method, $p > q$ holds provided that it holds for all instantiations of $p, q$ in accordance with the constraints. Only such strict constraints appear in $\mathcal{C}$.

In the future it should be interesting to handle both strict and loose inequality constraints, since loose constraints between formula weights can be derived in our setting just by means of the weakening inference rule.

## 6 Conclusion

This paper is another step in the study of inference from a partially ordered propositional base. We present a version of possibilistic logic with partially ordered symbolic weights. It differs from conditional logic frameworks [5] by the use of the minimal specificity principle which is not at work in such logical frameworks. We provide a proof of the soundness and completeness of this logic. Two syntactic inference methods are defined which allow us to infer new formulas with complex symbolic weights (necessity degrees of formulas): One that requires the enumeration of minimal inconsistent subsets to calculate necessity degrees. The other use results from the ATMS formalism. It enables constraints over weights to be taken into account so as to simplify the comparison of symbolic necessity degrees. This work has potential applications for the revision and the fusion of beliefs, as well as preference modeling [14].

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