Dialectical proofs accounting for strength of attacks

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Rapport conjoint IRIT–LERIA

numéro IRIT/RR–2010-7-FR

Abstract

We consider argumentation systems taking into account several attack relations of different strength. We focus on the impact of various strength attacks on the semantics of such systems, and particularly on the decision problem of credulous acceptance: namely, focussing on one particular argument, a classical issue is to compute a proof, under the form of an admissible set containing this argument. Taking into account attacks of various strength leads to search for the best proofs.

keywords: Argumentation Systems, Dialectical proofs

Contents

1 Introduction

Argumentation has become an influential approach in Artificial Intelligence to model cognitive tasks such as inconsistency handling and defeasible reasoning (e.g. [16, 20, 1]), decision making (e.g. [15]), or negotiation between agents (e.g. [3]).

Argumentation is based on the evaluation of interacting arguments, which support opinions, claims, decisions, ... Usually, the interaction takes the form of conflicts between arguments, and the fundamental issue is the selection of acceptable sets of arguments, based on the way they interact. Most of the argumentation-based proposals are instantiations of the abstract system proposed by Dung ([8]), which is reduced to a set of arguments (completely abstract entities) and a binary relation, called attack, which captures the conflicts between arguments. The increasing interest for the argumentation formalism has led to numerous extensions of the basic abstract system which are more appropriate to the applications.

A first extension of Dung's system has included a preference relation between arguments, which models their relative strength. For instance, an argument built from certain knowledge is stronger than an argument relying upon default knowledge (see e.g. [2, 4, 24, 19]). Another kind of extension is necessary to make a distinction between various types of conflict. For instance, when arguments are built from logical rules and knowledge, rebut and undercut conflict have been distinguished ([12]). More generally, symmetric attacks may be considered as weaker than non symmetric attacks. [18] has distinguished between blocking attacks and proper attacks, as a consequence of preference between arguments. In a multi-agent setting, various attack relations over a common set of arguments represent different criteria and different contexts according to which the conflicts are perceived ([22]). Moreover, it is natural to consider that not all attacks are equal in strength. [11] has proposed weighted argument systems, in which attacks are associated with a numeric weight, indicating how reluctant one would be to disregard the attack. Behind these proposals, there is a common idea that attacks may have different strength and can be compared according to their relative strength. However, there is so far no consensus about how it should be used to define extensional semantics, according to which acceptable sets of arguments are selected. A first promising work towards that direction has been proposed in [17], where an abstract argumentation system with varied-strength attacks has been defined. In that novel system, the classical concepts of defence and admissibility are revisited, in different directions, leading to several different refinements.

Our work takes place in that abstract system with attacks of various strength. Focussing on one restriction of the notion of defence, our motivation is to define extensional semantics accounting for the strength of defence, and to study the related decision problem of credulous acceptance. This problem, which consists in deciding if a given argument belongs to a preferred extension of the argumentation system, has been extensively studied within the framework of dialectical proofs (see, for instance, [14, 6, 10] and more recently [9, 21]). In this paper, we investigate the credulous acceptance problem in argumentation systems with attacks of various strength and propose associated dialectical proofs.

In Section 2, we present the fundamental notions[1]. First, we propose a restricted notion of defence, by requiring that the counter-attack is not weaker then the attack. Then, following Dung's construction, we define a restricted admissibility, called vs-admissibility, which is used for revisiting the classical preferred semantics and the associated decision problem of credulous acceptance. The next step is to compare defences collectively offered by sets of arguments. So, in Section 3, we propose a definition for sets offering a best defence for a given argument. Section 4 is devoted to the description of dialectical proof theories for credulous acceptance under vs-admissibility. A basic one is given which produces a solution under the form of a vs-admissible set containing the queried argument. Then, improvements are proposed which produce an optimized solution under the form of a best defence for the queried argument. Section 5 is devoted to concluding remarks and some related works. Proofs can be found in Appendix A.

2 Fundamentals

We consider the abstract system defined in [17]:

Definition 1 (Argumentation system with attacks of various strength – $\mathbf{AS_{vs}}$) *An* argumentation system with attacks of various strength *is a triple* $\langle \mathbf{A}, \mathbf{ATT}, \succeq \rangle$ *where*

- $\mathbf{A}$ *is a finite*[2] *set of arguments,*
- $\mathbf{ATT}$ *is a finite set of attack relations* $\langle \xrightarrow{1}, \ldots, \xrightarrow{n} \rangle$ *on* $\mathbf{A}$ *and*
- $\succeq$ *is a binary relation on* $\mathbf{ATT}$.

Each $\xrightarrow{i} \subseteq \mathbf{A} X \mathbf{A}$ represents a conflict relation, and $\succeq$ represents a relative strength between these conflict relations. The relation $\succeq$ is only assumed reflexive (it may be partial, and transitive or not). The corresponding strict relation[3] will be denoted by $\succ$. If the relation $\succeq$ is empty, a classical system (in Dung's sense) is recovered with the single attack relation obtained as the union of the attack relations $\xrightarrow{i}$. In the following of this paper, $\mathbf{AS}$ will denote the classical system $\langle \mathbf{A}, \mathbf{R} = \bigcup_i \xrightarrow{i} \rangle$ associated with the argumentation system with attacks of various strength $\mathbf{AS_{vs}} = \langle \mathbf{A}, \mathbf{ATT}, \succeq \rangle$.

Example 1 *Consider the* $\mathbf{AS_{vs}}$ *defined by* $\mathbf{AS_{vs}} = \langle \mathbf{A}, \mathbf{ATT}, \succeq \rangle$ *with* $\mathbf{A} = \{A, B, C_1, C_2\}$, $\mathbf{ATT} = \langle \xrightarrow{i}, \xrightarrow{j}, \xrightarrow{k} \rangle$ *with* $\xrightarrow{i} = \{(B, A)\}$, $\xrightarrow{j} = \{(C_1, B)\}$, $\xrightarrow{k} = \{(C_2, B)\}$, *and* $\succeq$ *defined by* $\xrightarrow{j} \succeq \xrightarrow{i}$.

$\mathbf{AS_{vs}}$ *can be depicted by the graph:*

$$\begin{tikzcd} C_1 \arrow[r, "j"] & B \arrow[r, "i"'] & A \\ C_2 \arrow[ur, "k"'] & & \end{tikzcd}$$

[1] All these notions have also been introduced in [5], but with a different purpose.

[2] We assume that $\mathbf{A}$ represents the set of arguments proposed by rational agents at a given time; so it makes sense to assume that $\mathbf{A}$ is finite.

[3] $A \succ B$ iff ($A \succeq B$ and not ($B \succeq A$)). $\succ$ is irreflexive and asymmetric.

Our purpose is to study the impact of these attacks of various strength on the notion of defence, which is a key concept in argumentation. The first level to be considered is the individual defence level. In classical systems, an argument A attacked by an argument B is said defended (against B) as soon as there exists an argument C attacking B. Indeed, any attack on B is relevant for inhibiting the attack from B to A. Now, if attacks may have different strengths, it is natural to compare the attack on B with the attack from B to A. The idea is that some attacks on B will be too weak to inhibit the attack on A and thus will not be relevant for reinstating A. Let $\mathbf{AS_{vs}} = \langle \mathbf{A}, \mathbf{ATT}, \succeq \rangle$, the following definition captures the idea of relevant defender:

Definition 2 (vs-defence – vs means "various-strength") *Let A, B, $C \in \mathbf{A}$ such that $C \xrightarrow{j} B$ and $B \xrightarrow{i} A$. C* vs-defends *A against B (or C is a* vs-defender *of A against B) iff $\xrightarrow{i} \not\succ \xrightarrow{j}$ (i.e. the attack from B to A is not strictly stronger than the one from C to B).*

Note that the same kind of definition is encountered in works about preference-base argumentation, for combining attack relation and preference relation (see [2, 24]): the idea is that an attack from B to A is relevant if A is not strictly preferred to B; otherwise, it can be considered that A defends itself against B.

So, in a similar way, Definition 2 states that the attack from B to A is overruled by the attack from C to B if the attack from B to A is not strictly stronger than the attack from C to B.

Note that Definition 2 can be restated in the system proposed by [17], where defenders are classified in four categories: strong, weak, normal and unqualified defenders. Let A, B, $C \in \mathbf{A}$ such that $C \xrightarrow{j} B$ and $B \xrightarrow{i} A$. According to [17]:

- C is a *strong defender* of A against B iff $\xrightarrow{j} \succ \xrightarrow{i}$;
- C is a *weak defender* of A against B iff $\xrightarrow{i} \succ \xrightarrow{j}$;
- C is a *normal defender* of A against B iff $\xrightarrow{j} \succeq \xrightarrow{i}$ and $\xrightarrow{i} \succeq \xrightarrow{j}$;
- C is an *unqualified defender* of A against B iff $\xrightarrow{i}$ and $\xrightarrow{j}$ are uncomparable.

Our notion of vs-defender exactly corresponds to "not weak defender", or equivalently to "strong or normal or unqualified defender". In the particular case of a complete relation on $\mathbf{ATT}$, $\xrightarrow{i} \not\succ \xrightarrow{j}$ means that $\xrightarrow{j} \succeq \xrightarrow{i}$. Then, "$C$ vs-defends A against B" exactly corresponds to "C is a strong defender or a normal defender of A against B".

As required above, the notion of vs-defence refines the classical notion of defence.

Property 1 *Let A, B, $C \in \mathbf{A}$. If C vs-defends A against B then C defends A against B in Dung's sense.*

Then, following the classical construction of acceptability (or collective defence), we propose to refine Dung's classical notions of acceptability and admissibility[4].

[4] In Dung's work [8]: a set $S \subseteq \mathbf{A}$ is *conflict-free* if $\forall A, B \in S$, B does not attack A; S is *admissible* if it is conflict-free and defends all its elements against all the possible attacks.

Definition 3 (vs-acceptability and vs-admissibility) *Let $S \subseteq \mathbf{A}$ and $A \in \mathbf{A}$.*

- *A is* vs-acceptable *with regard to (wrt) S (or S* collectively vs-defends *A against any attack) iff $\forall B \in \mathbf{A}$, if B attacks A then $\exists C \in S$ such that C vs-defends A against B.*
- *S is* vs-admissible *iff S is conflict-free in $\mathbf{AS}$*[5] *and $\forall A \in S$, A is vs-acceptable wrt S.*

The notion of vs-acceptability is a particular case of the notion of "constrained acceptability" of [17]. Indeed, it is sufficient to consider the defence profile containing the three levels strong, normal and unqualified. Moreover, the vs-admissibility requires the classical strict notion of conflict-free. That ensures that, for any vs-admissible set S, no attack may occur between elements of S. Note also that vs-acceptability and vs-admissibility refine the corresponding classical notions (this is a direct consequence of Property 1):

Property 2 *If $S \subseteq \mathbf{A}$ is vs-admissible, then S is admissible in Dung's sense.*

The converse is false: Let $C \xrightarrow{j} B \xrightarrow{i} A$, with $\xrightarrow{i}\succ\xrightarrow{j}$; $\{C, A\}$ is admissible but not vs-admissible. Note also that the empty set is vs-admissible.

Admissibility is the basis of most of the classical semantics, in Dung's abstract system. So, it is straightforward to revisit some classical semantics, using the notion of vs-admissibility. For instance, let us consider the preferred semantics which produces maximal (for set-inclusion) admissible sets of arguments.

Definition 4 (preferred vs-extension) *Let $S \subseteq \mathbf{A}$ be a vs-admissible set. S is a* preferred vs-extension *of $\mathbf{AS_{vs}}$ iff $\nexists S' \subseteq \mathbf{A}$ such that $S \subset S'$ and S' is vs-admissible.*

Example 1 (cont) *With $\xrightarrow{j}\succ\xrightarrow{i}$ and $\xrightarrow{k}\succ\xrightarrow{i}$, C_1 and C_2 are vs-defenders of A. With $\xrightarrow{j}\succ\xrightarrow{i}\succ\xrightarrow{k}$, C_1 is the only vs-defender of A. In both cases, the set $\{A, C_1, C_2\}$ is the only preferred vs-extension.*

Preferred vs-extensions have nice properties:

Property 3 *Preferred vs-extensions satisfy the 3 following points:*

1. *Let $S \subseteq \mathbf{A}$ be a vs-admissible set, there exists a preferred vs-extension E of $\mathbf{AS_{vs}}$ such that $S \subseteq E$.*
2. *There always exists at least one preferred vs-extension of $\mathbf{AS_{vs}}$.*
3. *For each preferred vs-extension E of $\mathbf{AS_{vs}}$, there exists a preferred extension E' of $\mathbf{AS}$ such that $E \subseteq E'$.*

[5] *i.e.*, $\forall A, B \in S$, $\nexists \xrightarrow{i}\in \mathbf{ATT}$, s.t. $B \xrightarrow{i} A$.

Finally, we consider the well-known decision problem called credulous acceptance problem [10], that is deciding if a given argument belongs to (at least) one preferred extension. So, under vs-admissibility, it comes to decide if a given argument belongs to (at least) one preferred vs-extension, or equivalently, due to Property 3:

Definition 5 (Credulous vs-acceptability) *Under vs-admissibility, an argument is* credulously accepted *iff it belongs to (at least) one vs-admissible set.*

In order to be able to solve this credulous vs-acceptance problem, we define in Section 4 proof theories, inspired by the work of [6]. We want to propose several kinds of proof exploiting the idea of minimality and the notion of "best defence". So, the next step given in Section 3 is the comparison of defences.

3 Comparison of defences

As stated in Section 2, a vs-admissible set proposes a valuable defence for each of its elements. This is a basic requirement. The next step is to take into account the existence of attacks of various strength for evaluating the quality of a valuable defence, and for selecting vs-admissible sets which offer a best defence. Clearly, if an argument B attacks an argument A, comparing two attacks against B enables to compare two defenders of A against B. Starting from that remark, we propose to compare defences at different levels:

- comparing two vs-defenders of a same argument,
- comparing two sets which collectively vs-defend a same argument.

As said above, two vs-defenders of A can be compared by considering the relative strength of the attacks they put on an attacker of A. Let $\mathbf{AS_{vs}}$ be an argumentation system with attacks of various strength, we can define:

Definition 6 (Comparison of vs-defenders) *Let $A, B, C_1, C_2 \in \mathbf{A}$ such that both C_1 and C_2 are vs-defenders of A against B with $C_1 \xrightarrow{j} B$ and $C_2 \xrightarrow{k} B$.*

- C_1 is better than C_2 *iff* $\xrightarrow{j}\succeq\xrightarrow{k}$.
- C_1 *is* strictly better than C_2 *iff* $\xrightarrow{j}\succ\xrightarrow{k}$.

Now, let S_1 and S_2 be two sets which collectively vs-defend A (that is A vs-acceptable wrt S_1 and wrt S_2). In order to determine whether S_1 offers a better defence for A than S_2, it is sufficient to compare the subset of S_1 containing the vs-defenders of A with the subset of S_2 containing the vs-defenders of A. So, we need the following definition:

Definition 7 (Set of defenders) *Let $S \subseteq \mathbf{A}$ and $A \in \mathbf{A}$ such that A is vs-acceptable wrt S. $\mathbf{Def}(A, S)$ is the set of the vs-defenders of A which belong to S.*

Note that, if $C \in \mathbf{Def}(A, S)$, there exists B attacking A such that C attacks B. So, if A is not attacked, $\mathbf{Def}(A, S) = \varnothing$.

Example 1 (cont) *Assuming that* $\xrightarrow{j}\succ\xrightarrow{i}$ *and* $\xrightarrow{k}\succ\xrightarrow{i}$ *implies* $\mathbf{Def}(A, \{C_1, C_2, A\}) = \{C_1, C_2\}$. *Assuming that* $\xrightarrow{j}\succ\xrightarrow{i}\succ\xrightarrow{k}$ *implies* $\mathbf{Def}(A, \{C_1, C_2, A\}) = \{C_1\}$.

Given S_1 and S_2 two sets of arguments which collectively vs-defend A, we have to compare $\mathbf{Def}(A, S_1)$ and $\mathbf{Def}(A, S_2)$. The idea is the following: given E_1 and E_2 two sets of vs-defenders of A, E_2 is at least as strong as E_1 if E_2 improves the defence of A offered by E_1 on at least one defender. That idea can be formalized using the following *general scheme of set-comparison*:

Definition 8 (Scheme of set-comparison) *Let $\mathcal{E}$ be a set of elements. Let $E_1 \subseteq \mathcal{E}$ and $E_2 \subseteq \mathcal{E}$. Let* `is-strictly-better-than` *be a strict binary relation on $\mathcal{E}$. E_2* strictly better than *E_1 will be denoted by $E_2 \sqsupset E_1$ and defined by:*

- *$E_2 \sqsupseteq E_1$ iff there exist C_1 belonging to E_1 and C_2 belonging to $E_2 \setminus E_1$ such that C_2* `is-strictly-better-than` *C_1.*
- *$E_2 \sqsupset E_1$ iff ($E_2 \sqsupseteq E_1$ and not $E_1 \sqsupseteq E_2$).*

Note that there exist many schemes for defining a comparison of subsets from the comparison of their elements (see for instance [7, 13]). The choice depends on the concerned application and it is generally guided by some criteria to satisfy. Note that Definition 8 supports *a priori* neither the minimal sets for set-inclusion, nor the maximal sets for set-inclusion[6].

The scheme of Definition 8 obviously satisfies:

Property 4

1. *If $E_2 \subseteq E_1$, either E_1 and E_2 are uncomparable, or $E_1 \sqsupset E_2$.*
2. *If $E_2 \subseteq E_3$ and $E_2 \sqsupseteq E_1$ then $E_3 \sqsupseteq E_1$.*

Our aim is to use the scheme of Definition 8 for comparing sets of vs-defenders of a given argument A.
So, we instantiate this scheme by replacing the relation `is-strictly-better-than` by the strict relation given in Definition 6.

Example 1 (cont) *Assume that C_1 et C_2 are two vs-defenders of A against B with* $\xrightarrow{j}\succ\xrightarrow{k}$ *(so C_1 is strictly better than C_2). $\{C_1, C_2\} \sqsupset \{C_2\}$, $\{C_1\} \sqsupset \{C_2\}$, but $\{C_1, C_2\}$ et $\{C_1\}$ are uncomparable.*
Indeed, although C_1 is strictly better than C_2, we cannot consider that $\{C_1\}$ offers a better defence than $\{C_1, C_2\}$ since C_1 already belongs to $\{C_1, C_2\}$.

[6] It is a weak alternative of the following definition given in [17]: $E_2 \sqsupset E_1$ iff there exist $C_1 \in E_1$ and $C_2 \in E_2 \setminus E_1$ such that C_2 `is-strictly-better-than` C_1 and there exist no $C_1 \in E_1$ and no $C_2 \in E_2$ such that C_1 `is-strictly-better-than` C_2.

Example 2 *Consider the* $\mathbf{AS_{vs}}$ *depicted by the following graph (note that* A *is not vs-acceptable wrt* $\{C_1\}$ *in this* $\mathbf{AS_{vs}}$*):*

$$\begin{tikzcd}
C_1 \arrow[r, "j"] & B_1 \arrow[dr, "i"] & \\
C_2 \arrow[ur, "k"] \arrow[dr, "m"] & & A \\
C_3 \arrow[r, "m"] & B_2 \arrow[ur, "m"] &
\end{tikzcd}$$

with $\xrightarrow{i} \not\succ \xrightarrow{k}$, $\xrightarrow{i} \not\succ \xrightarrow{j}$ *and* $\xrightarrow{j} \succ \xrightarrow{k}$

The following set-comparisons hold: $\{C_1, C_2\} \sqsupset \{C_2\}$, $\{C_1, C_3\} \sqsupset \{C_2\}$ *and* $\{C_1, C_3\} \sqsupset \{C_2, C_3\}$. *Note that these set-comparisons are all based on the only comparison concerning* C_1 *and* C_2 *as vs-defenders of* A *against* B.

Example 3

$$\begin{tikzcd}
C_1 \arrow[dr, "j", "k"'] & & \\
C_2 \arrow[r] & B \arrow[r, "i"] & A \\
C_3 \arrow[ur, "m"'] & &
\end{tikzcd}$$

Assume that C_1, C_2 *and* C_3 *are three vs-defenders of* A *against* B *with* C_3 *strictly better than* C_2 *et* C_2 *strictly better than* C_1.

We have $\{C_1, C_2, C_3\} \sqsupset \{C_1, C_3\}$ *since* C_2 *is strictly better than* C_1 *and* C_2 *does not belong to* $\{C_1, C_3\}$. *In contrast, according to the definition proposed in [17], these sets are uncomparable.*

Thus, S_1 and S_2 being two sets which collectively vs-defend A, we are able to determine whether S_1 offers a better defence for A than S_2, using the comparaison of $\mathbf{Def}(A, S_1)$ and $\mathbf{Def}(A, S_2)$ by the appropriate instantiation of Definition 8:

Definition 9 (Set-comparison wrt the defence of an argument) *Let* S_1 *and* S_2 *be two sets of arguments such that* A *is vs-acceptable wrt* S_1 *and wrt* S_2. S_2 strictly better than S_1 wrt the defence of A *will be denoted by* $S_2 \gg_A S_1$, *and is defined by:*

$$S_2 \gg_A S_1 \textit{ iff } \mathbf{Def}(A, S_2) \sqsupset \mathbf{Def}(A, S_1)$$

Note that, due to the definition of $\sqsupset$, the relation $\gg_A$ is irreflexive and asymmetric.

Definition 9 enables to compare two sets of vs-defenders of a given argument, wrt the defence of that argument. However, this comparaison is sometimes useless, as shown by the following example.

Example 4 *This example has been taken from [18].*

$$\begin{tikzcd}
 & C_1 \arrow[r, "j"] & B_1 \arrow[dr, "i"] & \\
D \arrow[r] & C_2 \arrow[r, "j"] & B_2 \arrow[r, "i"] & A \\
 & C_3 \arrow[ur, "k"'] & &
\end{tikzcd}$$

with $\xrightarrow{i} \not\succ \xrightarrow{j}$, $\xrightarrow{i} \not\succ \xrightarrow{k}$ *and* $\xrightarrow{j} \succ \xrightarrow{k}$

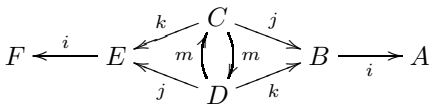
Let $S_1 = \{C_1, C_3\}$ *and* $S_2 = \{C_1, C_2\}$. A *is vs-acceptable wrt* S_1 *and wrt* S_2. *And* $S_2 \gg_A S_1$. *However, no vs-admissible set contains* S_2.

So, it seems important that the set-comparison wrt the defence of an argument is restricted to vs-admissible sets.

Given an argument A and using Definition 9, we are able to compare two vs-admissible sets which defend A. We will take advantage of this comparaison in the

search for a proof that A is credulously vs-accepted. Such a proof could simply be a vs-admissible set containing A. A proof exhibiting a best defence for A would be more informative. Moreover, from a computational point of view, shorter proofs are more satisfactory, since searching for defenders is expensive. Combining the two requirements will lead us to search for proofs proposing a minimal set of defenders and a best defence for the queried argument. This discussion is illustrated on the following example:

Example 5 *This example has been taken from [17].*



Assume that C and D are two vs-defenders of A and F.

$\{A, C, F\}$ and $\{A, D, F\}$ are preferred vs-extensions containing A, so they are proofs for the credulous vs-acceptance of A. However, from a computational point of view, more interesting vs-admissible sets are $\{A, C\}$ and $\{A, D\}$ for proving that A is credulously vs-accepted. Now, assuming that $\xrightarrow{j}\succ\xrightarrow{k}$, we have $\{A, C\} \gg_A \{A, D\}$ and the best proof is $\{A, C\}$ (since C is a better vs-defender of A than D).

In the following, we formalize the above ideas, and first of all the concept of minimal vs-admissible set containing a given argument. As far as we know, very few works have adressed the credulous acceptance problem by the computation of minimal lines of defence. [23] has proposed an algorithm for computing minimally admissible defence sets, using the concept of a defence set around A. This is exactly what we call an A-min-admissible set (an admissible set which contains A and which is $\subseteq$-minimal among the admissible sets containing A). Then, taking into account vs-defence leads to:

Definition 10 (A-min-vs-admissible set) *Let $\mathbf{AS_{vs}}$ be an argumentation system with attacks of various strength. Let $A \in \mathbf{A}$, let $S \subseteq \mathbf{A}$, S is A-*min-vs-admissible *iff*

1. *S is vs-admissible,*
2. *$A \in S$ and*
3. *S is $\subseteq$-minimal among the sets satisfying the two previous conditions.*

Then, using Definition 9, we are able to compare vs-admissible sets which minimally defend A. The sets which are maximal for the relation $\gg_A$ among the min-vs-admissible sets, called A-min-best-defences, correspond to the best proofs we are looking for.

Definition 11 (A-min-best-defence) *$S \subseteq \mathbf{A}$ is a A-*min-best-defence *iff S is A-min-vs-admissible and $\nexists S' \subseteq \mathbf{A}$ A-min-vs-admissible, such that $S' \gg_A S$ (i.e. such that $\mathbf{Def}(A, S') \sqsupset \mathbf{Def}(A, S)$)*

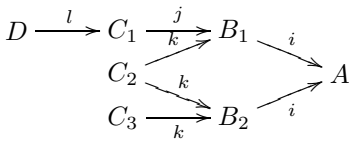
In other words, $S \subseteq \mathbf{A}$ is a A-min-best-defence iff

- S is A-min-vs-admissible and
- $\nexists S' \subseteq \mathbf{A}$ A-min-vs-admissible, such that ($\exists C \in S$ and $\exists C' \in S' \setminus S$ vs-defenders of A with C' is a strictly better defender of A than C) and ($\nexists B' \in S', B \in S \setminus S'$ vs-defenders of A with B a strictly better defender of A than B').

The above definitions are illustrated on the following examples.
In Example 1, if C_2 is *not* a vs-defender of A, then $\{C_2, A\}$ is A-min-admissible but not A-min-vs-admissible. If C_1 and C_2 are both vs-defenders of A, and C_1 is strictly better than C_2, then $\{C_1, A\}$ is the only A-min-best-defence, whereas, if C_1 and C_2 are equivalent or uncomparable, $\{C_1, A\}$ and $\{C_2, A\}$ are both A-min-best-defences.
In Example 5, $\{A, C\}$ and $\{A, D\}$ (resp. $\{F, C\}$ and $\{F, D\}$) are both A-min-vs-admissible (resp. F-min-vs-admissible). If $\xrightarrow{j}$ and $\xrightarrow{k}$ are equivalent or uncomparable, $\{A, C\}$ and $\{A, D\}$ are both A-min-best-defences. But, if $\xrightarrow{j}\succ\xrightarrow{k}$, $\{A, C\}$ is the only A-min-best-defence (and $\{F, D\}$ is the only F-min-best-defence). Note that, in that case, the A-min-vs-admissible set $\{A, D\}$ is not included in an A-min-best-defence.

Note that an A-min-best-defence does not always contain the best vs-defenders of A, since vs-admissibility is emphasized first. As shown on the following example, the best vs-defenders of A may not belong to a vs-admissible set.

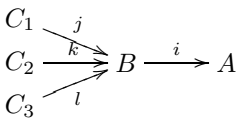
Example 6



Assume that all the C_i are vs-defenders of A, and $\xrightarrow{j}\succ\xrightarrow{k}$. $\{A, C_2\}$ is the only A-min-vs-admissible set and so it is also the only A-min-best-defence. And yet C_2 is not the best defender of A against B. However, choosing C_1 would lead to a set which is not vs-admissible.

The last remark concerns the existence of the A-min-best-defences, which is not guaranteed:[7]

Example 7



Assume that $\xrightarrow{j}\succ\xrightarrow{k}$, $\xrightarrow{k}\succ\xrightarrow{l}$ and $\xrightarrow{l}\succ\xrightarrow{j}$. $\{A, C_1\}$, $\{A, C_2\}$, $\{A, C_3\}$ are the three A-min-vs-admissible sets and one has the following preference between these sets: $\{A, C_1\} \gg_A \{A, C_2\}$, $\{A, C_2\} \gg_A \{A, C_3\}$ and $\{A, C_3\} \gg_A \{A, C_1\}$. So there is no maximal set for $\gg_A$ and, so no A-min-best-defence.

[7]Note that this existence is not guaranteed even if the relation $\succeq$ between the attack relations is transitive (see the example given in Appendix B).

4 Dialectical proofs

4.1 Dialectical framework

A proof of credulous acceptance for an argument A can be presented under a dialectical form (see for instance [14, 10, 6]) where two players exchange arguments. The proponent (**PRO**) proposes the initial argument A and the arguments which directly or indirectly defend A in the proof. The opponent (**OPP**) advances attackers of A or of its defenders. The precise rules of the game depend on the problem to be solved.

[6] has proposed dialectical proof theories for the credulous acceptance problem under the preferred semantics. Following the formal approach of [6], we propose a dialectical framework which will enable us to provide, in the following section, proof theories for the credulous vs-acceptance problem.

The game played by **OPP** and **PRO** takes the form of a dialogue governed by rules expressed by what is usually called a legal-move function. In the following, given a set E, $E^\star$ denotes the set of finite sequences of elements from E.

Definition 12 (ϕ-dialogue)

- *A* dialogue type *is a tuple* $(\mathbf{A}, \mathbf{ATT}, \succeq, \phi)$ *where* $\langle \mathbf{A}, \mathbf{ATT}, \succeq \rangle$ *is an* $\mathbf{AS_{vs}}$ *and* $\phi : (\mathbf{A}X\mathbf{A})^\star \longrightarrow 2^{(\mathbf{A}X\mathbf{A})}$ *is a function called* legal-move function.
- *A* move in $\mathbf{AS_{vs}}$ *is a pair [*P*,* (B, A)*] where* $P \in \{\mathbf{PRO}, \mathbf{OPP}\}$ *and* $(B, A) \in (\mathbf{A}X\mathbf{A})$*.* B *is the argument advanced by* P*, in response to the argument* A*. For a move* $\mu = [P, (B, A)]$*, we use*
  - **Pl**(μ) *to denote* P*,*
  - **Att**(μ) *to denote the pair* (B, A)*,*
  - **Arg**(μ) *to denote* B *and*
  - **Pre**(μ) *to denote* A*.*
- *A* dialogue d *in* $(\mathbf{A}, \mathbf{ATT}, \succeq, \phi)$ *(or* ϕ*-dialogue) is a countable sequence* $\mu_0\mu_1\ldots$ *of moves in* $\mathbf{A}$ *such that:*
  1. **Pl**$(\mu_0) =$ **PRO***,*
  2. **Pl**$(\mu_i) \neq$ **Pl**(μ_{i+1}) *and*
  3. **Att**$(\mu_{i+1}) \in \phi($**Att**$(\mu_0)\ldots$**Att**$(\mu_i))$*.*

  We say d is about **Arg**(μ_0)*.*

Each player plays in turn and **PRO** plays first. The legal-move function defines, at each step, what moves can be used to continue the dialogue. Note that, in contrast with classical dialectical frameworks, playing a move consists in playing an attack (B, A). A is the predecessor of B in this attack. This modification will enable us to take into account vs-defence. In order to be sure that **PRO** proposes a vs-defender C of A against B, we must be able to compare the attack from B to A (the pair (B, A)) with

the attack from C to B (the pair (C, B)). However, wo do not require that $\mathbf{Arg}(\mu_{i+1})$ attacks $\mathbf{Arg}(\mu_i)$.

We assume the existence of a special imaginary argument (used for initializing dialogues), denoted by $\_$ and usually called the "empty argument", such that $\mathbf{Pre}(\mu_0) = \_$.

Let us give some useful notations.

Notation 1 *Let $d = \mu_0\mu_1 \ldots \mu_i$ be a finite ϕ-dialogue:*

- *μ_i is denoted by* $\mathbf{Last}(d)$*;*
- *$\phi(\mathbf{Att}(\mu_0) \ldots \mathbf{Att}(\mu_i))$ is denoted by $\phi(d)$;*
- $\mathbf{PRO}(d)$ *denotes the set of arguments advanced by* **PRO** *during d;*
- *The extension of d with a move μ in $\mathbf{AS_{vs}}$ such that $\mu_o \ldots \mu_i\mu$ is a ϕ-dialogue is denoted by the juxtaposition $d.\mu$.*

A dialogue type gives a formal framework for the definition of rules, using the legal-move function. Then, it is necessary to define formal conditions under which a given ϕ-dialogue is successful. These conditions are usually called winning criteria. We consider the first criterion given in [14, 6].

Definition 13 (winning criterion) *Let $(\mathbf{A}, \mathbf{ATT}, \succeq, \phi)$ be a dialogue type.*
A ϕ-dialogue d is won by **PRO** *if and only if d is finite, cannot be continued (i.e. $\phi(d) = \varnothing$), and* $\mathbf{Pl}(\mathbf{Last}(d)) = \mathbf{PRO}$.

Combining a dialogue type $(\mathbf{A}, \mathbf{ATT}, \succeq, \phi)$ and a winning criterion, we obtain a ϕ-dialectical proof theory and ϕ-proofs. So, according to Definition 13, we define:

Definition 14 (ϕ-proof) *A* ϕ-proof *is a ϕ-dialogue won by* **PRO**.

4.2 Proofs of credulous vs-acceptance: from "basic proofs" to "best proofs"

The credulous vs-acceptance problem comes to decide if a given argument X belongs to (at least) one vs-admissible set. So, a proof that X is credulously vs-accepted must exhibit a vs-admissible set containing X. Thus, we can start from the dialectical proof theory proposed in [6] and add constraints to the legal-move function, in order to ensure that **PRO** always advances vs-defenders of arguments she previously advanced. Thus, we can define ϕ_1^{vs}-proofs for credulous vs-acceptance (called "basic proofs").

The second step is the production of proofs for credulous vs-acceptance which are "less expensive" than ϕ_1^{vs}-proofs; in this case, we want to exhibit a X-min-vs-admissible set, so a more constrained legal-move function is proposed corresponding to ϕ_2^{vs}-proofs (called "minimal proofs").

The last step concerns the production of "best proofs" (*i.e.* proofs exhibiting a X-min-best-defence). However, for that purpose, the dialectical framework defined above is not appropriate. Indeed, there is no simple way to obtain such best proofs by adding

new constraints on the legal-move function, as shown by Example 6, where **PRO** does not advance the best vs-defender of A when she builds the best proof for A. So, in this paper, we propose to determine best proofs for a given argument X by comparing the ϕ_2^{vs}-proofs for X.

Before the basic definition for ϕ_i^{vs}-legal move functions ($i = 1, 2$), let us give some useful notations.

Notation 2 *Let* $\mathbf{AS}$ *denote the classical system* $\langle \mathbf{A}, \mathbf{R} = \bigcup_i \xrightarrow{i} \rangle$ *associated with the argumentation system with attacks of various strength* $\mathbf{AS_{vs}} = \langle \mathbf{A}, \mathbf{ATT}, \succeq \rangle$.

- *Let* $A \in \mathbf{A}$, $\mathbf{R}^+(A)$ *(resp.* $\mathbf{R}^-(A)$*) denotes the set of arguments which are attacked by* A *(resp. which attack* A*) in the sense of* $\mathbf{R}$*;* $\mathbf{R}^{\pm}(A) = \mathbf{R}^+(A) \cup \mathbf{R}^-(A)$*;*
- *Let* $S \subseteq \mathbf{A}$, $\mathbf{R}^+(S)$ *denotes the set of arguments which are attacked by an element of* S*;* $\mathbf{R}^-(S)$ *and* $\mathbf{R}^{\pm}(S)$ *are defined in an analogous way;*
- **Refl** *denotes the set of arguments which are self-attacking;*
- $\mathbf{vsDef}(A, B)$ *will denote the set of the vs-defenders of* A *against* B.

Following [6], we state the following constraints which must be verify for each ϕ_1^{vs}-dialogue. Every move [**OPP**, (B, A)] must reply to a preceding move [**PRO**, (A, F)] in the dialogue, that is **OPP** advances B which attacks an argument A previoulsy advanced by **PRO**. Every move [**PRO**, (C, B)], except the first one, must be *immediately* preceded in the dialogue by a move [**OPP**, (B, A)] such that C is a *vs-defender* of A against B.
Moreover, **PRO**(d) must be vs-admissible, so **PRO** cannot choose any argument in $\mathbf{R}^{\pm}$(**PRO**(d)), nor any self-attacking argument. And it is useless for **PRO** to advance an argument previously advanced by **PRO**. It is also useless for **OPP** to play an attack (B, A) if **PRO**(d) contains a vs-defender of A against B.

Then, in order to produce "minimal proofs" for the credulous vs-acceptance problem, the idea is to enforce the restriction on moves by **PRO** as follows : if **PRO** plays an attack (C, B) with C authorized by ϕ_1^{vs}, **PRO** will not be permitted to advance any argument from $\mathbf{vsDef}(A, B)$ in the continuation of the dialogue.

So, we propose the following definition for legal-move functions:

Definition 15 (ϕ_i^{vs}-dialogues ($i = 1, 2$)) $\phi_i^{vs} : (\mathbf{A}X\mathbf{A})^\star \longrightarrow 2^{(\mathbf{A}X\mathbf{A})}$ *are defined by:*

- *if* d *is a dialogue about the argument* X *of odd length (next move is by* **OPP***),*

$$\phi_i^{vs}(d) = \{(B, A) \in \mathbf{A} \times \mathbf{A} \mid A \in \mathbf{PRO}(d) \text{ and } B \in \mathbf{R}^-(A) \text{ such that } (\mathbf{vsDef}(A, B) \cap \mathbf{PRO}(d)) \text{ is empty}\}$$

- *if* d *is a dialogue about the argument* X *of even length (next move is by* **PRO***) with* **Att**(**Last**(d)) $= (B, A)$*,*

$$\phi_i^{vs}(d) = \{(C, B) \in \mathbf{A} \times \mathbf{A} \mid C \in \mathbf{vsDef}(A, B) \cap \mathbf{POSS}_i(d)\}$$

with

– $\mathbf{POSS}_1(d) = \mathbf{A} \setminus (\mathbf{Refl} \cup \mathbf{PRO}(d) \cup \mathbf{R}^{\pm}(\mathbf{PRO}(d)))$
– $\mathbf{POSS}_2(d) = \mathbf{POSS}_1(d) \setminus \{X \in \mathbf{vsDef}(Y, Z) \mid \mathbf{vsDef}(Y, Z) \cap \mathbf{PRO}(d) \neq \varnothing\}$

Note that each ϕ_2^{vs}-dialogue is also a ϕ_1^{vs}-dialogue. Moreover, the restriction on moves by **OPP** is the same in both dialogue types.

The following results enable us to establish the soundness and the completeness of these two ϕ_i^{vs}-proof theories.

Lemma 1 *If d is a ϕ_1^{vs}-dialogue,* $\mathbf{PRO}(d)$ *is conflict-free.*

Theorem 1 (Soundness of ϕ_i^{vs}-proofs) *Let X be an argument of* $\mathbf{A}$.

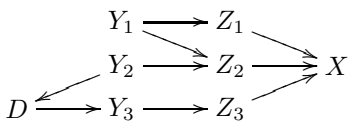
1. *If d is a ϕ_1^{vs}-proof for X, then* $\mathbf{PRO}(d)$ *is a vs-admissible set containing X.*
2. *If d is a ϕ_2^{vs}-proof for X, then* $\mathbf{PRO}(d)$ *is a X-min-vs-admissible set.*

Theorem 2 (Completeness of ϕ_i^{vs}-proofs) *Let X be an argument of* $\mathbf{A}$.

1. *If X is in a vs-admissible set, then there exists a ϕ_1^{vs}-proof for X.*
2. *If X belongs to S, a X-min-vs-admissible set, then there exists a ϕ_2^{vs}-proof d for X such that* $\mathbf{PRO}(d) = S$.

The following example illustrates the above theorem by showing the ϕ_2^{vs}-proof which is built.

Example 8



Assume that Y_1, Y_2, Y_3 are vs-defenders of X, and that Y_2 is a vs-defender of Y_3.

Let $S = \{Y_1, Y_2, Y_3, X\}$. S is a X-min-vs-admissible set. So, there is a ϕ_2^{vs}-proof d for X such that $\mathbf{PRO}(d) = S$. *Here is d:*

$\mu_0 = [\mathbf{PRO}, (X, \_\_)]$ *with* $\phi_2^{vs}(\mu_0) = \{(Z_1, X), (Z_2, X), (Z_3, X)\}$
$\mu_1 = [\mathbf{OPP}, (Z_1, X)]$ *with* $\phi_2^{vs}(\mu_0\mu_1) = \{(Y_1, Z_1)\}$
$\mu_2 = [\mathbf{PRO}, (Y_1, Z_1)]$ *with* $\phi_2^{vs}(\mu_0\mu_1\mu_2) = \{(Z_3, X)\}$ *(***OPP** *cannot advance Z_2 since $Y_1 \in \mathbf{vsDef}(X, Z_2)$)*
$\mu_3 = [\mathbf{OPP}, (Z_3, X)]$ *with* $\phi_2^{vs}(\mu_0\mu_1\mu_2\mu_3) = \{(Y_3, Z_3)\}$
$\mu_4 = [\mathbf{PRO}, (Y_3, Z_3)]$ *with* $\phi_2^{vs}(\mu_0\mu_1\mu_2\mu_3\mu_4) = \{(D, Y_3)\}$
$\mu_5 = [\mathbf{OPP}, (D, Y_3)]$ *with* $\phi_2^{vs}(\mu_0\mu_1\mu_2\mu_3\mu_4\mu_5) = \{(Y_2, D)\}$
$\mu_6 = [\mathbf{PRO}, (Y_2, D)]$ *with* $\phi_2^{vs}(\mu_0\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6) = \varnothing$

Concerning the production of "best proofs" for X, we propose the following methodology. Due to the above results, we know that if there exists S, a X-min-best-defence, S is X-min-vs-admissible. So, there exists a ϕ_2^{vs}-proof d for X such that **PRO**$(d) = S$. Thus, we just have to compute all the ϕ_2^{vs}-proofs d_i for X, and to compare the sets **PRO**(d_i) with the relation $\gg_X$.

As said above, the dialectical framework proposed in this paper is not appropriate for designing a proof theory able to produce X-min-best-defences, when they exist. The main reason is that **PRO** should not be permitted to advance a vs-defender for which there exists no ϕ_1^{vs}-proof. Indeed, due to Definition 13, a ϕ-proof is a sequential proof. And it is not possible to backtrack on the choice of a vs-defender advanced by **PRO**. We are currently investigating another dialectical framework, using tree-like proofs.

5 Conclusion and related works

Our proposal in this paper is a further contribution to the development of argumentation with various attacks of different strength, based on the abstract framework introduced by [17]. The basic idea is to use the relative strength of the attacks for refining the concept of reinstatement: we define a new notion of defence, the vs-defence, requiring that the counter-attack is not weaker than the attack. This enables us to revisit Dung's classical semantics with the definition of vs-admissible sets, preferred vs-extensions, and the credulous vs-acceptance problem.
Another issue is to compare the defence offered by sets of arguments. We propose comparisons at different levels: between vs-defenders, between sets of vs-defenders and between vs-admissible sets.
This enables us to define two dialectical proof theories for credulous vs-acceptance. The first one improves the ϕ_1-proof theory proposed in [6] by taking into account the new notion of defence. So, a ϕ_1^{vs}-proof for a given argument A produces a vs-admissible set containing A. Then, we propose the ϕ_2^{vs}-proof theory, a more constrained proof theory which produces $\subseteq$-minimal sets among the vs-admissible sets containing A. For the ϕ_1^{vs} and ϕ_2^{vs}-proofs, theorems of soundness and completeness are given.
The last issue concerns the production of A-min-best-defences, which represent the best proofs for A. These proofs cannot be easily obtained by a simple restriction of the ϕ_2^{vs}-dialogues but can be computed by comparing the ϕ_2^{vs}-proofs for A.

To our best knowledge, this is the first attempt to construct best proofs for the credulous acceptance problem. [23] has proposed a procedure for computing minimal lines of defence around a given argument. But this procedure does not use a dialectical method, and does not always produce minimal admissible sets. For many years, there has been substantial research on dialectical proof procedures for classical abstract argumentation. Recently, [9, 21] have proposed a unifying framework able to capture dialectical proof procedures for handling both credulous and skeptical acceptance in abstract argumentation. This framework is based on the notions of dispute derivation and base derivation. Dispute derivation provides a way to define proofs, and is very

close to the approach of [6], which has inspired our work. Base derivation enables to represent backtracking in the search for a proof, and is used for proving skeptical acceptance. As a future work, we plan to consider this powerful unifying framework for obtaining proofs accounting for strength of attacks. Another future work will concern both the study of worst-case complexity and the development of algorithms dedicated to the proofs theories presented in this paper.

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A Proofs

Proof of Property 3: Point 1 follows directly from Definition 4 since $\mathbf{A}$ is finite. Point 2 follows directly from the previous point, since the empty set is vs-admissible. And for the third point, we have: let E be a preferred vs-extension; E is vs-admissible, so E is admissible for $\mathbf{AS}$ (due to Property 2); so, using a result of [8], E is included in a preferred extension of $\mathbf{AS}$. ■

Proof of Lemma 1: We prove the result by induction on the number of elements of **PRO**(d). If **PRO**(d) only contains one element, it is trivially conflict-free.

Suppose the property holds for any ϕ_1^{vs}-dialogue such that **PRO**(d) contains at most n elements, $n > 0$. Let d be a ϕ_1^{vs}-dialogue such that **PRO**(d) contains $n+1$ elements.

Suppose first that the last move of d is played by **PRO**. Then d has the form $d'.[\textbf{OPP}, (B, A)].[\textbf{PRO}, (C, B)]$ where d' is a ϕ_1^{vs}-dialogue, $(B, A) \in \phi_1^{vs}(d')$ and $(C, B) \in \phi_1^{vs}(d'.[\textbf{OPP}, (B, A)])$. Then **PRO**$(d) =$ **PRO**$(d') \cup \{C\}$. From Definition 15, we know that $C \notin$ **PRO**(d'), so **PRO**(d') contains strictly less than $n+1$ elements. By induction hypothesis, **PRO**(d') is conflict-free. Since $C \notin \mathbf{R}^{\pm}(\textbf{PRO}(d'))$, **PRO**$(d)$ is conflict-free too.

Suppose now that the last move of d is played by **OPP**. Then d has the form $d'.[\textbf{OPP}, (B, A)]$ where d' is a ϕ_1^{vs}-dialogue. Obviously, **PRO**$(d) =$ **PRO**(d') so **PRO**(d) is conflict-free. ■

A.1 Proofs for soundness

Proof of Theorem 1.1: If d is a ϕ_1^{vs}-proof for the argument X, $\phi_1^{vs}(d) = \varnothing$ and **Pl**(**Last**(d)) $=$ **PRO**. So, it means that $\nexists (B, A) \in \mathbf{A} \times \mathbf{A}$ such that $A \in$ **PRO**(d), $B \in \mathbf{R}^{-}(A)$ and $(\mathbf{vsDef}(A, B) \cap$ **PRO**$(d))$ is empty. Or equivalently, for each B attacking A, **PRO**(d) vs-defends A against B. So, **PRO**(d) is vs-admissible. ■

Proof of Theorem 1.2: Let d be a ϕ_2^{vs}-proof for the argument X. Any ϕ_2^{vs}-dialogue is also a ϕ_1^{vs}-dialogue. Moreover, the restriction on moves by **OPP** is the same in both dialogue types. So, using Lemma 1 and Theorem 1.1, we conclude that **PRO**(d) is vs-admissible. So, there remains to prove minimality. Assume that S is a X-min-vs-admissible set strictly included in **PRO**(d). Let N be the number of elements of S. From the proof of Theorem 2.1, we can build a ϕ_1^{vs}-proof for X, denoted by d_N, such that **PRO**$(d_N) = S$.

We are going to show that d_N is not a ϕ_2^{vs}-dialogue. As $S \subset$ **PRO**(d), there exists an argument $Z_1 \in$ **PRO**(d) and $Z_1 \notin S$, so $Z_1 \neq X$. Therefore, Z_1 has been advanced by **PRO** in d for vs-defending an argument $W_1 \in$ **PRO**(d) against an argument Y_1. Either $W_1 \in S$ or $W_1 \in$ **PRO**$(d) \setminus S$. Obviously, due to the rules governing ϕ_1^{vs}-dialogues, $W_1 \neq Z_1$. If $W_1 \in$ **PRO**$(d) \setminus S$, we can iterate the construction and find $W_2 \in$ **PRO**(d) such that W_2 is a vs-defender of W_1. However, **PRO**$(d) \setminus S$ is finite, so the iteration will stop with $W_k \in S$. Thus, we have found an argument

$Z \in (\mathbf{PRO}(d) \setminus S)$ ($Z = W_{(k-1)}$) and $W \in S$ ($W = W_k$) such that Z vs-defends W against an argument Y.

S is vs-admissible, Yattacks W and $W \in S$, so there exists $A \in S$ such that A vs-defends W against Y. Since $Z \notin S$, we are sure that A and Z are different. Thus, $\mathbf{PRO}(d)$ contains two elements of $\mathbf{vsDef}(Y, W)$, and so d is not a ϕ_2^{vs}-dialogue.

■

A.2 Proofs for completeness

Lemma 2 *Let d be a ϕ_1^{vs}-dialogue such that* $\mathbf{Pl}(\mathbf{Last}(d)) = \mathbf{PRO}$. *Let $S \subseteq \mathbf{A}$ be a minimal vs-admissible set containing* $\mathbf{PRO}(d)$. *If $S \neq \mathbf{PRO}(d)$, there exist $Y, Z, W \in \mathbf{A}$ such that $d' = d.[\mathbf{OPP}, (Y, W)].[\mathbf{PRO}, (Z, Y)]$ is a ϕ_1^{vs}-dialogue, and S is a minimal vs-admissible set containing* $\mathbf{PRO}(d')$.

Proof of Lemma 2: If $\mathbf{PRO}(d)$ is strictly included in S, then the minimality of S implies that $\mathbf{PRO}(d)$ is not vs-admissible. From Lemma 1, we know that $\mathbf{PRO}(d)$ is conflict-free. So, there exist $W \in \mathbf{PRO}(d)$ and $Y \notin \mathbf{PRO}(d)$ such that $Y \in \mathbf{R}^-(W)$ and $(\mathbf{vsDef}(W, Y) \cap \mathbf{PRO}(d)) = \varnothing$. Since $\mathbf{PRO}(d)$ is included in S and S is vs-admissible, there exists $Z \in (\mathbf{vsDef}(W, Y) \cap S)$. So, $Z \in \mathbf{R}^-(Y)$ and $Z \notin \mathbf{PRO}(d)$. Moreover, S is conflict-free, so $Z \notin$ **Refl** and $Z \notin \mathbf{R}^{\pm}(\mathbf{PRO}(d))$. Therefore, the move (Z, Y) belongs to $\phi_1^{vs}(d.[\mathbf{OPP}, (Y, W)])$, and $d' = d.[\mathbf{OPP}, (Y, W)].[\mathbf{PRO}, (Z, Y)]$ is a ϕ_1^{vs}-dialogue.

$\mathbf{PRO}(d') = \mathbf{PRO}(d) \cup \{Z\}$ which is included in S since $Z \in S$. There remains to prove that no vs-admissible set S' strictly included in S contains $\mathbf{PRO}(d')$. Suppose that S' is a strict subset of S such that $\mathbf{PRO}(d') \subseteq S'$. $\mathbf{PRO}(d) = \mathbf{PRO}(d') \setminus \{Z\}$ so $\mathbf{PRO}(d)$ is strictly included in S'. By definition of S, that implies that S' is not vs-admissible. ■

Proof of Theorem 2.1: Assume that X belongs to a vs-admissible set S. Without loss of generality, we can assume that S is X-min-vs-admissible. Let N be the number of elements of S (remember that $\mathbf{A}$ is finite). We build a sequence of ϕ_1^{vs}-dialogues as follows:

Let $d_1 = [\mathbf{PRO}, (X, \_\_)]$, $\ldots$, let $d_n = d_{n-1}.[\mathbf{OPP}, (Y, W)].[\mathbf{PRO}, (Z, Y)]$ for $2 \leq n \leq N$, where $W \in \mathbf{PRO}(d_{n-1}), Z \in (\mathbf{R}^-(Y) \cap S \cap \mathbf{vsDef}(W, Y))$ and $(\mathbf{vsDef}(W, Y) \cap \mathbf{PRO}(d_{n-1})) = \varnothing$. Lemma 2 proves that the sequence is well-defined and that d_N is a ϕ_1^{vs}-dialogue won by **PRO** (indeed $\phi_1^{vs}(d_N) = \varnothing$ since $\mathbf{PRO}(d_N) = S$ and S is vs-admissible). ■

Proof of Theorem 2.2: Assume that X belongs to S a X-min-vs-admissible set. We are going to build d a ϕ_2^{vs}-proof for X, such that $\mathbf{PRO}(d) = S$, in different steps. Each step will correspond to a level of vs-defence for X, and for each level, we will choose the attaker which justifies the presence of the defender belonging to S: for the first step, we will consider vs-defenders of X which are in some sense essential; then, for each vs-defender of X which has to be defended, we consider its defenders, ...

Let N be the number of elements of S. Let $d_1 = [\mathbf{PRO}, (X, \_\_)]$. If $N = 1$, obvioulsly, d_1 is a ϕ_2^{vs}-proof for X. Otherwise, X is attacked and is not vs-defended by $\mathbf{PRO}(d_1)$. As S is vs-admissible and contains X, S vs-defends X. Let $\mathbf{Def}(X, S)$ = $\{Y_1, \ldots, Y_k\}$. As S is X-min-vs-admissible, each Y_i is essential for the vs-defence of X, or one of its vs-defenders[8] (direct or indirect). More precisely, some of them are not necessary as vs-defenders of X. Let us formalize that. For each Y_i, let Y_i^+ denote the set of arguments which attack X and against which Y_i vs-defends X. Formally, $Y_i^+ = \{Z \in \mathbf{R}^-(X) \mid Y_i \in \mathbf{vsDef}(X, Z)\}$. Let us consider a minimal (for set-inclusion) subset E of $\mathbf{Def}(X, S)$, such that $\bigcup\{Y_i^+, Y_i \in E\} = \mathbf{R}^-(X)$. Such a subset will be called an essential subset of vs-defenders. Due to minimality, we are sure that in such a subset, each Y_i^+ is not included in $\bigcup\{Y_j^+, Y_j \in E, j \neq i\}$. So, for each Y_i^+ in an essential subset of vs-defenders of X, there is at least one attacker of X against which Y_i is the only vs-defender. Such an attacker of X justifies the presence of Y_i in S. In the sequel of this proof, if Y_i belongs to E an essential subset of vs-defenders of X, $J(Y_i)$ will denote $Y_i^+ \setminus \bigcup\{Y_j^+, Y_j \in E, j \neq i\}$.

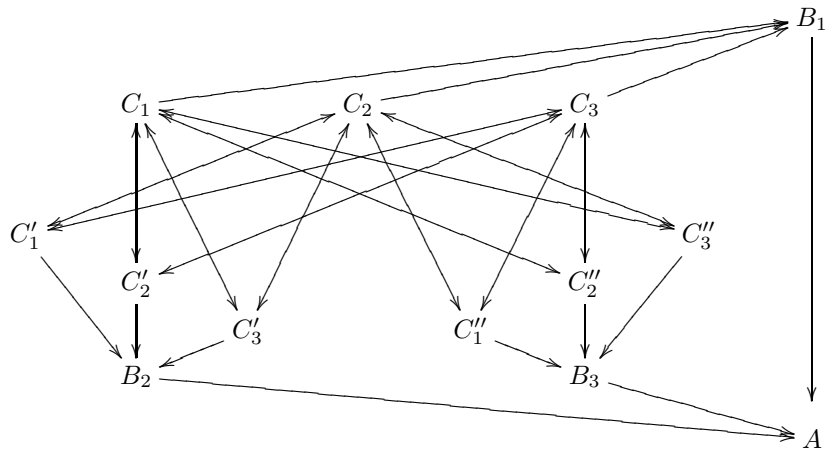
The idea is to extend d_1 by a move where **OPP** chooses one element of $J(Y_i)$ in order to attack X. So, let us take E an essential subset of vs-defenders of X, and $Y_i \in E$. For simplicity, we assume it is Y_1.
Let $Z_1 \in J(Y_1)$ and $d_2 = d_1.[\mathbf{OPP}, (Z_1, X)].[\mathbf{PRO}, (Y_1, Z_1)]$. We iterate the construction, by choosing $Y_2 \in E$ and $Z_2 \in J(Y_2)$. The iteration will stop when each element of E has been advanced. So, we obtain d_{l+1} the last move of which is of the form $[\mathbf{PRO}, (Y_l, Z_l)]$. Due to the choice of each $Z_i \in J(Y_i)$, d_{l+1} is a ϕ_2^{vs}-dialogue.

Now, if $\mathbf{PRO}(d_{l+1})$ is strictly included in S, there exists $Y_i \in E$, (and so advanced in d_{l+1}) which is not vs-defended by $\mathbf{PRO}(d_{l+1})$, but is vs-defended by S. We use the same construction as done before, just replacing X by Y_i. This iteration will stop when the dialogue d which is obtained satisfies $\mathbf{PRO}(d) = S$. As S is vs-admissible, $\phi_2^{vs}(d) = \varnothing$. And due to the choice of moves by **OPP**, d is a ϕ_2^{vs}-dialogue. So, d is a ϕ_2^{vs}-proof for X. ■

B Existence of a A-min-best-defence?

This existence is not guarantee even if the relation $\succeq$ between the attack relations is transitive:

[8] Even if Y_i is a vs-defender of X, it can be also a vs-defender of another argument.



Assume that all the C_i, C_i' and C_i'' are vs-defenders of A and consider that the relation $\succeq$ gives the following preferences between vs-defenders ($\succeq$ is transitive):

- C_1 is stricly better than C_2
- C_2' is stricly better than C_3'
- C_3'' is stricly better than C_1''
- C_3 is uncomparable with C_1 and C_2
- C_1' is uncomparable with C_2' and C_3'
- C_2'' is uncomparable with C_1'' and C_3''

$\{A, C_1, C_1', C_1''\}$, $\{A, C_2, C_2', C_2''\}$, $\{A, C_3, C_3', C_3''\}$ are the three A-min-vs-admissible sets and one has the following preference between these sets:

- $\{A, C_1, C_1', C_1''\} \gg_A \{A, C_2, C_2', C_2''\}$,
- $\{A, C_2, C_2', C_2''\} \gg_A \{A, C_3, C_3', C_3''\}$ and
- $\{A, C_3, C_3', C_3''\} \gg_A \{A, C_1, C_1', C_1''\}$.

So there is no maximal set for $\gg_A$ and, so no A-min-best-defence.