Non-monotonic Syntax-Based Entailment: A Classification of Consequence Relations

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Abstract. The purpose of this paper is to provide a comparative study of non-monotonic syntax-based consequence relations, from different points of view. Starting from a (not necessarily consistent) belief base E and a pre-ordering on E, we first remind different mechanisms for selecting preferred consistent subbases in syntax-based approaches. Then, we present three entailment principles in order to cope with these multiple subbases. The crossing point of each generation mechanism and each principle defines a syntax-based consequence relation.
Pursuing previous work of the authors concerning the computational complexity point of view, we first provide a comparison from the cautiousness point of view. Our proposal restates previous results [18] in a single framework and provides new results. In the last part of this paper, we study the validity of deduction rules (such as those introduced by [15, 12]). Results are discussed in the conclusion.

1 Introduction

In this paper, we focus on syntactic approaches to non-monotonic inference. We assume that a set of formulae E (the belief base) is equipped with a complete pre-ordering structure (a priority relation) which, contrarily to [12], is not related to the semantical entailment ordering. Following [18], it is convenient to see non-monotonic syntax-based entailment as a two-step procedure which first generates and selects preferred consistent subbases (the "generation mechanism") and then manages these multiple subbases in order to conclude (the "conflict resolution principle"). For instance, the kind of inference "E infers Φ iff Φ is classically inferred in all the preferred consistent subbases of E" has been extensively considered in [2], with several meanings given to the term "preferred". A taxonomy of conflict resolution principles, according to cautiousness, can be found in [18]. The selection of preferred subbases relies upon the definition of aggregation modes which enable to extend the priority ordering (defined on the initial belief base) into a preference relation (between subbases). This problem has been already considered in [2, 9] from a more general point of view on preference-based reasoning.

In the framework described above, our purpose is to propose a comparative study of various syntax-based consequence relations, from different points of view. We have extensively studied the computational complexity of these relations (see [8]). Here, we first recall the three mechanisms for selecting preferred consistent subbases and the three conflict resolution principles, which were used to define the considered syntax-based consequence relations. Then, we provide a comparison from the cautiousness point of view. Our proposal restates previous results [18] in a single framework and provides new results. In the third part of this paper, we study the validity of deduction rules (such as those introduced by [15, 12]). Results are discussed in the conclusion.

2 Syntax-Based Consequence Relations

Throughout the paper, E denotes a non-empty finite set of propositional formulae and is referred to as the *belief base*. The belief base is considered syntactically, as in [17]: each belief is a distinct piece of information and only beliefs which are explicitely present in the base are taken into account. It departs from the logical point of view where a base is identified with the set of its models. Due to the belief status of its elements, E is not assumed to be consistent. Moreover, we assume that E is equipped with a complete pre-ordering $\leq$ (a priority relation), which modelizes an epistemic relevance ordering. It is equivalent to consider that E is stratified in a collection $(E_1, \ldots, E_n)$ of belief bases, where E_1 contains the formulae of highest priority (or relevance) and E_n those of lowest priority. The pair $(E, \leq)$ is called a *prioritized* (or *stratified*) belief base[1]. Each E_i is called a *stratum* of E. In the literature on non-monotonic inference, the most usual proposal for handling inconsistency is to work with maximal (w.r.t. set-inclusion) consistent subbases of E. Different approaches have been proposed to use the priority relation in order to select "preferred" consistent subbases (see [8] for a survey). For the purpose of our comparative study, we focus on three preference relations: the best-out preference which has been related to possibilistic inference, the inclusion-based preference which combines priorities and maximal consistent subbases and the lexicographic preference which combines priorities and subbases of maximal cardinality.

Definition 1 *Let $X = (X_1 \cup \ldots \cup X_n)$ and $Y = (Y_1 \cup \ldots \cup Y_n)$ be two consistent subbases of E (where $X_i = (X \cap E_i)$ and $Y_i = (Y \cap E_i)$), we define:*

- *best-out preference (see [2]): let X be a consistent subbase of E and $a(X) = \min \{i \mid \exists \Phi \in E_i \setminus X\}$. The best-out preference is the complete pre-ordering defined by $X \ll^{bo} Y$ iff $a(X) \leq a(Y)$[2] ; we say that Y is bo-preferred than X.*
- *inclusion-based preference (see [9] and [13] for equivalent definitions): this is the strict partial ordering defined by $X \ll^{incl} Y$ iff $\exists i \mid X_i \subset Y_i$ and $\forall j \mid 1 \leq j < i$, $X_j = Y_j$; we say that Y is incl-preferred than X.*
- *lexicographic preference (see [16, 2]): this is the strict ordering defined by $X \ll^{lex} Y$ iff $\exists i \mid |X_i| < |Y_i|$ and $\forall j \mid 1 \leq j < i$, $|X_j| = |Y_j|$ ($|Y|$ denotes the cardinality of Y) ; we say that Y is lex-preferred than X[3].*

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[1] In the following, the underlying pre-ordering will be omitted and a prioritized belief base will be denoted by E. The case of a flat belief base (*i.e.*, all the formulae are equally important) is a particular case.

[2] This ordering depends only on the most prioritary stratum where at least one formula has been removed in order to restore consistency.

[3] The lexicographic pre-ordering is complete. For any subbases X and Y of E, either $X \equiv^{lex} Y$ (*i.e.*, $|X_i| = |Y_i|$ for $i = 1 \ldots n$) or one of them is lex-preferred to the other one ($X \ll^{lex} Y$ or $Y \ll^{lex} X$).

The best-out preference is not very selective since some of the preferred subbases are not maximal for set-inclusion. Indeed, let *amax*$(E) = \max \{i \mid E_1 \cup \ldots \cup E_i$ is consistent$\}$. If *amax*$(E) = k$, then the best-out preferred consistent subbases of E are exactly the consistent subbases of E which contain $(E_1 \cup \ldots \cup E_k)$. The inclusion-based preference refines the set-inclusion in the sense that inclusion-based preferred consistent subbases are maximal consistent subbases. These incl-preferred subbases are of the form $(X_1 \cup \ldots \cup X_n)$ such that $(X_1 \cup \ldots \cup X_i)$ is a maximal-consistent subbase of $(E_1 \cup \ldots \cup E_i)$ for $i = 1 \ldots n$. It follows that incl-preferred subbases are also bo-preferred subbases. Note that incl-preferred consistent subbases are also called preferred sub-theories in [5], and exactly correspond to strongly maximal-consistent subbases in [10]. The lexicographic preference refines the inclusion-based preference. Any lex-preferred consistent subbase of E is an incl-preferred consistent subbase, but the converse is false.

As mentioned in the introduction of this paper, non-monotonic entailment from a given belief base can be viewed as a two-step procedure which first generates "preferred" belief states, and then manages these different belief states according to cautiousness principles. In the following, we call T (resp. INCL, LEX, BO) the mechanism which produces the set of maximal (resp. incl-preferred, lex-preferred, bo-preferred) consistent subbases of E. The two main entailment principles activated in presence of multiple conflicting belief states are the skeptical and credulous principles. A taxonomy of numerous entailment principles has been established by Pinkas and Loui [18] according to their cautiousness. Here, we focus on three of them.

Definition 2 *Let $m(E)$ denote a set of consistent subbases of E. For instance, $m(E)$ may be obtained with one of the mechanisms T, INCL, LEX or BO. Let Φ be a propositional formula. We define:*

- *UNI principle: Φ is inferred from $m(E)$ according to the skeptical (or universal) entailment principle iff Φ is classically inferred from each element of $m(E)$.*
- *EXI principle: Φ is inferred from $m(E)$ according to the credulous (or existential) entailment principle iff Φ is classically inferred from at least one element of $m(E)$[4].*
- *ARG principle: this intermediary principle consists in keeping only the credulous consequences whose negation cannot be inferred (see [3] for a discussion on the so-called argumentative inference). Φ is inferred from $m(E)$ according to the argumentative entailment principle iff Φ is classically inferred from at least one element of $m(E)$ and no element of $m(E)$ classically entails $\neg\Phi$.*

We are now ready to give a precise definition of syntax-based consequence relations generated by a prioritized belief base. Each one appears at the crossing point of a belief state generation mechanism m and an entailment principle p .

Definition 3 *Let E be a prioritized belief base and Φ a propositional formula. $E \mid\!\sim^{p,m} \Phi$ iff Φ is inferred from $m(E)$ according to the principle p ; $m(E)$ denotes the set of consistent subbases of E which are preferred for the mechanism m.*

[4] Obviously, each conclusion inferred from $m(E)$ by UNI principle is also obtained by EXI principle.

For our comparative study, m belongs to {T, INCL, LEX, BO} and p belongs to {UNI, EXI, ARG}. More generally, we will consider non-monotonic consequence relations of the form "Φ is inferred from Ψ with respect to E".

Following previous works on the relationship between non-monotonic inference and belief revision [17, 12], we define:

Definition 4 *$\Psi \mid\!\sim_E^{p,m} \Phi$ iff $\Psi \oplus E \mid\!\sim^{p,m} \Phi$, where $\Psi \oplus E$ denotes the prioritized belief base obtained from E by adding $\{\Psi\}$ as first stratum. More precisely, if E is stratified into $(E_1, \ldots, E_n)$, then $\Psi \oplus E = (E_0 = \{\Psi\}, E_1, \ldots, E_n)$ and if E is a flat belief base, then $\Psi \oplus E = (E_0 = \{\Psi\}, E)$.*

3 The Cautiousness Ordering

In this section, we compare syntax-based consequence relations generated by a belief base, from the point of view of cautiousness [18]. Given two consequence relations R_1 and R_2, R_1 is said more cautious than R_2 iff every conclusion obtained by R_1 is also obtained by R_2. Results have already been obtained for some of the relations considered in section 2. Our proposal restates these previous results in a single framework and provides new results (proofs are in [7]). We obtain the taxonomy depicted in Figure 1.

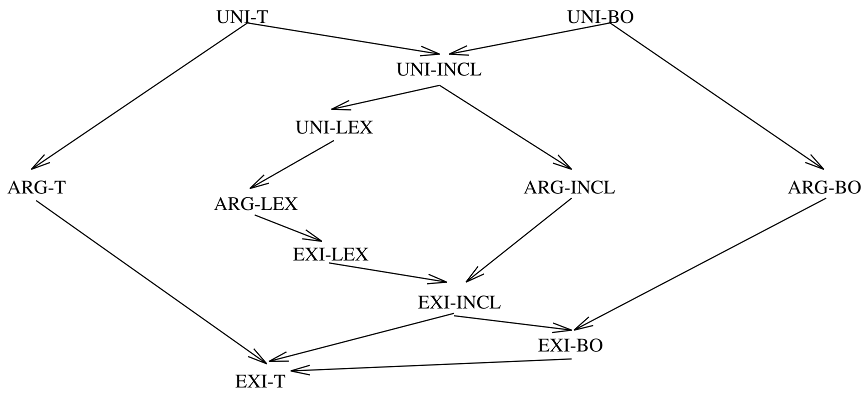

Fig. 1. Cautiousness ordering. A syntax-based consequence relation is represented by the associated selection principle p and generation mechanism m and is simply denoted by the pair p-m. The arrow means "more cautious than".

Remarks about previous works: The taxonomy proposed in [18] contains other selection principles (than EXI, ARG and UNI) but does not specify the generation mechanism. We propose a refinement since the relationships we establish concern four specified generation mechanisms. Brass [4] compared four selection principles (including EXI and UNI) on the particular case of the INCL mechanism and a belief base stratified in two levels, the first one being consistent. A comparison of $\mid\!\sim^{\text{UNI,T}}$, $\mid\!\sim^{\text{EXI,T}}$, $\mid\!\sim^{\text{ARG,T}}$

can be found in [3]. The relations $\mid\!\sim^{\text{UNI,BO}}$, $\mid\!\sim^{\text{UNI,INCL}}$, and $\mid\!\sim^{\text{UNI,LEX}}$ are thoroughly discussed in [2] but with a different definition. However, both presentations are equivalent due to the following property:

Proposition 1 *Let E be a prioritized belief base and Φ a propositional formula. For each m in $\{$BO, INCL, LEX$\}$: the m-preferred consistent subbases of $\Psi \oplus E$ are of the form $\{\Psi\} \cup S$, where S is a subbase of E which is m-preferred among the Ψ-consistent subbases of E.*

4 Classification According to Deduction Rules

Much work has been done concerning the characterization of classes of non-monotonic consequence relations by means of logical properties, or deduction rules. The best known classes are:

- the preferential relations defined by the rules of the System P (see [15]): Supraclassicality (SCL), Left Logical Equivalence (LLE), Right Weakening (RW), Cut, Cautious Monotony (CM), Or.
- the rational relations defined by the system P and the Rational Monotony (RM).
- the non-monotonic relations defined by the Basic Postulates (see [12]): Supraclassicality (SCL), Left Logical Equivalence (LLE), Right Weakening (RW), And, Weak Rational Monotony (WRM), Weak Conditionalization (WC), Consistency Preservation (CP).
- the comparative relations defined by the set of Extended Postulates (see [12]): Basic postulates with Cumulativity (CUM)[5], Rational Monotony (RM), Or.

Here, we study the validity of the above deduction rules for the relations considered in definition 4. The mechanism T will not be considered when studying relations of the form $\Psi \mid\!\sim_E^{p,m} \Phi$, since the base $\Psi \oplus E$ is stratified. Let us first recall the previous results on the subject. Brass [4] has proved characterization theorems for $\mid\!\sim^{\text{UNI,INCL}}$ and $\mid\!\sim^{\text{EXI,INCL}}$ in the particular case of a belief base stratified in two levels, the first one being consistent. For instance, $\mid\!\sim^{\text{UNI,INCL}}$ is the more cautious "semantics" for the set of rules REF, RW, AND, OR, LLE, CWA[6]. Two important characterizations can be found in [2]: The set of $\mid\!\sim^{\text{UNI,LEX}}$ relations is exactly the set of comparative inference relations (defined by the extended postulates). The set of $\mid\!\sim^{\text{UNI,INCL}}$ relations strictly contains the set of comparative relations and is strictly included in the set of preferential relations. [1] discusses $\mid\!\sim^{\text{UNI,BO}}$ and the relationship with possibilistic inference. Due to space limitations, we summarize our results in the Table 1. All the proofs are in [7].

5 Conclusions

First of all, we did not consider the semantical point of view but rather an operational point of view. However, the $\mid\!\sim^{\text{UNI},m}$ relations can be given a preferential-models seman-

[5] CUM is a consequence of CUT, CM, LLE, SCL.

[6] A consequence relation R satisfies the rule CWA iff for each Ψ, if the union of all the INCL-preferred subbases of $\Psi \oplus E$ is consistent and entails Φ, then $(\Psi \oplus \text{E}) R\ \Phi$.

				UNI-BO	UNI-INCL	UNI-LEX	EXI-BO	EXI-INCL	EXI-LEX	ARG-BO	ARG-INCL	ARG-LEX
Kraus, Lehmann and Magidor's systems												
	System P											
			REF	♦	♦	♦	♦	♦	♦	♦	♦	♦
			LLE	♦	♦	♦	♦	♦	♦	♦	♦	♦
			RW	♦	♦	♦	♦	♦	♦	♦	♦	♦
			CUT	♦	♦	♦	–	–	–	–	–	–
			CM	♦	♦	♦	–	–	–	–	–	–
			OR	♦	♦	♦	–	–	–	–	–	–
	RM			♦	–	♦	♦	♦	♦	–	–	–
Gärdenfors and Makinson's postulates												
	Extended Postulates											
		Basis Postulates										
			SCL	♦	♦	♦	♦	♦	♦	♦	♦	♦
			LLE	♦	♦	♦	♦	♦	♦	♦	♦	♦
			RW	♦	♦	♦	♦	♦	♦	♦	♦	♦
			AND	♦	♦	♦	–	–	–	–	–	–
			WRM	♦	–	♦	♦	♦	♦	–	–	–
			WC	♦	♦	♦	♦	♦	♦	♦	♦	♦
			CP	♦	♦	♦	♦	♦	♦	♦	♦	♦
		CUM		♦	♦	♦	–	–	–	–	–	–
		RM		♦	–	♦	♦	♦	♦	–	–	–
		OR		♦	♦	♦	–	–	–	–	–	–

Table 1. Deduction rules – ♦ (resp. $-$) $=$ means that the rule is (resp. is not) satisfied

tics (see [2, 15]). More generally, the preference relations defined on consistent subbases of E induce pre-ordering on models (see [2]).

Following [12, 15] approaches to non-monotonic inference leads to prefer the class of $|\!\sim^{\mathrm{UNI},m}$ relations, which are the more cautious relations. Hence, it is not surprising that the $|\!\sim^{\mathrm{EXI},m}$ and $|\!\sim^{\mathrm{ARG},m}$ relations obtain bad results in the Table 1 (because these rules are basically defined for the $|\!\sim^{\mathrm{UNI},m}$ relations). The Rational Monotony has been justified by the ability to cope with the problem of irrelevance: If Φ is a plausible consequence of Ψ_1 and if Ψ_2 has nothing to do with Ψ_1, then Φ should be also a plausible consequence of $(\Psi_1 \wedge \Psi_2)$. However, the relation $|\!\sim^{\mathrm{UNI,BO}}$ (which satisfies the property of Rational Monotony) suffers from the so-called "drowning effect" [2]: the beliefs which appear below *amax*(E) are completely ignored by the inference process. Contrastedly, the problem of "drowning effect" is solved with the relations $|\!\sim^{\mathrm{UNI,LEX}}$ and $|\!\sim^{\mathrm{UNI,INCL}}$ (which rely upon a selection of maximal consistent subbases). However, with $|\!\sim^{\mathrm{UNI,INCL}}$ Rational Monotony is lost, while it is recovered with $|\!\sim^{\mathrm{UNI,LEX}}$. Besides, the LEX-based approaches are very sensitive to redundance of information, since they rely upon cardinality. They should be probably reserved to specific applications (such as for instance consistency-based approaches to model-based diagnosis).

At the other extremity, the $\mid\!\sim^{\text{EXI},m}$ relations are too permissive. In particular, they may lead to pairs of mutually exclusive conclusions. It seems reasonable to require the property of safety for a relation. A relation is safe if it does not produce pairwise contradictory conclusions. Requiring safety naturally leads to the third class of relations, the $\mid\!\sim^{\text{ARG},m}$ relations.

The $\mid\!\sim^{\text{ARG},m}$ relations are more productive than the $\mid\!\sim^{\text{UNI},m}$ relations, while respecting safety[7]. They are well-suited to applications where the inconsistency of the belief base may be generated by the fusion of different belief bases. In that case, it seems that with the ARG principle, the inference process remains faithful to the contents of the base. That point is illustrated by the following example: $E = E_1 \cup E_2 \cup E_3$ with $E_1 = \{a, b, c, a \to \neg b\}$, $E_2 = \{a \to \neg c\}$ and $E_3 = \{a \to d\}$. The incl-preferred consistent subbases are: $S_1 = \{a, c, a \to \neg b, a \to d\}$, $S_2 = \{a, b, c, a \to d\}$, $S_3 = \{b, c, a \to \neg b, a \to \neg c, a \to d\}$. Only one subbase is lex-preferred: S_3. The conclusion d is inferred by $\mid\!\sim^{\text{ARG,INCL}}$ (but not by $\mid\!\sim^{\text{UNI,INCL}}$). Besides, note that d is not inferred by $\mid\!\sim^{\text{ARG,LEX}}$. Indeed, due to cardinality considerations, the LEX mechanism hides the argument in favor of d. Another example shows that the ARG principle can solve the "drowning effect", even when the mechanism BO is chosen. $E = E_1 \cup E_2 \cup E_3 \cup E_4$ with $E_1 = \{a \to \neg b\}$, $E_2 = \{a\}$, $E_3 = \{b\}$ and $E_4 = \{c\}$. c is not inferred by $\mid\!\sim^{\text{UNI,BO}}$ but is inferred by $\mid\!\sim^{\text{ARG,BO}}$.

Some directions for further research may be drawn from the above discussion. First, we think that syntax-based inference based on the ARG principle deserves deeper investigation. Promising approaches to defeasible reasoning have been recently proposed in the framework of argumentation (for instance in [19, 11, 14]). Roughly speaking, a conclusion is inferred if the arguments supporting it can be successfully defended against the arguments supporting the opposite statement. In the case of a flat belief base, most of the argument-based inference relations have been restated in the framework of syntax-based entailment described in this paper (see [6] for a report on that topic). The comparison of both methodologies will be extended to stratified belief bases. Another interesting problem is the complete characterization of the so-called irrelevance principle (which is not achieved by the Rational Monotony). More generally, other logical properties should be outlined in order to characterize the $\mid\!\sim^{\text{EXI},m}$ and $\mid\!\sim^{\text{ARG},m}$ relations in counterbalancing the preference for the $\mid\!\sim^{\text{UNI},m}$ relations due to the logical properties of [15, 12].

Our main contribution is a unifying framework which enables to provide:

- a comparative study according to different points of view: computational complexity (previous work), cautiousness, and validity of deduction rules,
- a better understanding of underlying mechanisms and a help for choosing one of the specified mechanisms/principles depending on the pragmatics of the application or on complexity considerations.

[7] The price paid for these advantages is an expensive computational complexity.

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