An axiomatic approach to support in argumentation

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Abstract

In the context of bipolar argumentation (argumentation with two kinds of interaction, attacks and supports), we present an axiomatic approach for taking into account a special interpretation of the support relation, the necessary support. We propose constraints that should be imposed to a bipolar argumentation system using this interpretation. Some of these constraints concern the new attack relations, others concern acceptability. We extend basic Dung's framework in different ways in order to propose frameworks suitable for encoding these constraints. By the way, we propose a formal study of properties of necessary support.

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1 Introduction

The main feature of argumentation framework is the ability to deal with incomplete and / or contradictory information, especially for reasoning [14; 2]. Moreover, argumentation can be used to formalize dialogues between several agents by modeling the exchange of arguments in, *e.g.*, negotiation between agents [4]. An argumentation system (AS) consists of a collection of arguments interacting with each other through a relation reflecting conflicts between them, called *attack*. The issue of argumentation is then to determine "acceptable" sets of arguments (*i.e.*, sets able to defend themselves collectively while avoiding internal attacks), called *"extensions"*, and thus to reach a coherent conclusion. Formal frameworks have greatly eased the modeling and study of AS. In particular, the framework of [14] allows for abstracting the "concrete" meaning of the arguments and relies only on binary interactions that may exist between them.

In this paper, we are interested in bipolar AS (BAS), which handle a second kind of interaction, the support relation. This relation represents a positive interaction between arguments and has been first introduced by [17; 26]. In [8], the support relation is left general so that the bipolar framework keeps a high level of abstraction. However there is no single interpretation of the support, and a number of researchers proposed specialized variants of the support relation: deductive support [5], necessary support [20; 21], evidential support [22; 23], backing support [12]. Each specialization can be associated with an appropriate modelling using an appropriate complex attack. These proposals have been developed quite independently, based on different intuitions and with different formalizations. [10] presents a comparative study in order to restate these proposals in a common setting, the bipolar argumentation framework (see also [12] for another survey). The idea is to keep the original arguments, to add complex attacks defined by the combination of the original attack and the support, and to modify the classical notions of acceptability. An important result of [10] is the highlight of a kind of duality between the deductive and the necessary specialization of support, which results in a duality in the modelling by complex attacks. In this context, new different papers have recently been written: some of them give a translation between necessary supports and evidential supports [24]; others propose a justification of the necessary support using the notion of subarguments [25]; an extension of the necessary support is presented in [19]. From all these works it seems interesting to focus on the necessary support. However, different interpretations remain possible, leading to different ways of introducing new attacks and different ways to define acceptability of sets of arguments.

Our purpose is to propose a kind of "axiomatic approach" for studying how necessary support should be taken into account. Indeed we propose requirements (or constraints) that should be imposed to a bipolar argumentation system as "axioms" describing a desired behaviour of this system. Some of these constraints concern the new attack relations, others concern acceptability. We extend basic Dung's framework in different ways in order to propose frameworks suitable for encoding these contraints. By the way, we propose a formal study of properties of necessary support.

Some background is given in Section 2 for AS and BAS, in particular the duality identified in [10]. Section 3 presents constraints that should be imposed for taking into account necessary support. Then different frameworks for handling these constraints are described in Section 4. Section 5 concludes and suggests perspectives of our work. The proofs are given in Appendix A.

2 Background on abstract bipolar argumentation systems

Bipolar abstract argumentation systems extend Dung's argumentation systems. So first we recall Dung's framework for abstract argumentation systems.

2.1 Dung's framework

Dung's abstract framework consists of a set of arguments and only one type of interaction between them, namely attack. The important point is the way arguments are in conflict.

Def. 1 (Dung AS) *A Dung's argumentation system (AS, for short) is a pair* $\langle \mathbf{A}, \mathbf{R} \rangle$ *where* $\mathbf{A}$ *is a finite and non-empty set of arguments and* $\mathbf{R}$ *is a binary relation over* $\mathbf{A}$ *(a subset of* $\mathbf{A} \times \mathbf{A}$*), called the* attack relation.

An argumentation system can be represented by a directed graph, called the *interaction graph*, in which nodes represent arguments and edges are defined by the attack relation: $\forall a, b \in \mathbf{A}$, $a\mathbf{R}b$ is represented by $a \not\to b$.

Def. 2 (Admissibility in AS) *Given* $\langle \mathbf{A}, \mathbf{R} \rangle$ *and* $S \subseteq \mathbf{A}$.

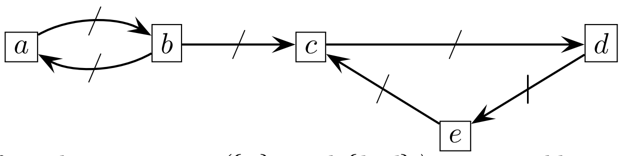
- S *is* conflict-free *in* $\langle \mathbf{A}, \mathbf{R} \rangle$ *iff*[1] *there are no arguments* $a, b \in S$, *s.t.*[2] $a\mathbf{R}b$.
- $a \in \mathbf{A}$ *is* acceptable *in* $\langle \mathbf{A}, \mathbf{R} \rangle$ *wrt*[3] S *iff* $\forall b \in \mathbf{A}$ *s.t.* $b\mathbf{R}a$, $\exists c \in S$ *s.t.* $c\mathbf{R}b$.
- S *is* admissible *in* $\langle \mathbf{A}, \mathbf{R} \rangle$ *iff* S *is conflict-free and each argument in* S *is acceptable wrt* S.

Standard semantics introduced by Dung (preferred, stable, grounded) enable to characterize admissible sets of arguments that satisfy some form of optimality.

Def. 3 (Extensions) *Given* $\langle \mathbf{A}, \mathbf{R} \rangle$ *and* $S \subseteq \mathbf{A}$.

- S *is* a preferred extension *of* $\langle \mathbf{A}, \mathbf{R} \rangle$ *iff it is a maximal (wrt* $\subseteq$*) admissible set.*
- S *is* a stable extension *of* $\langle \mathbf{A}, \mathbf{R} \rangle$ *iff it is conflict-free and for each* $a \notin S$, *there is* $b \in S$ *s.t.* $b\mathbf{R}a$.
- S *is* the grounded extension *of* $\langle \mathbf{A}, \mathbf{R} \rangle$ *iff it is the least (wrt* $\subseteq$*) admissible set* X *s.t. each argument acceptable wrt* X *belongs to* X.

Ex. 1 *Let* AS *be defined by* $\mathbf{A} = \{a, b, c, d, e\}$ *and* $\mathbf{R} = \{(a, b), (b, a), (b, c), (c, d), (d, e), (e, c)\}$. AS *is represented by the following graph.*



There are two preferred extensions ($\{a\}$ *and* $\{b, d\}$*), one stable extension (*$\{b, d\}$*) and the grounded extension is the empty set.*

2.2 Abstract bipolar argumentation systems

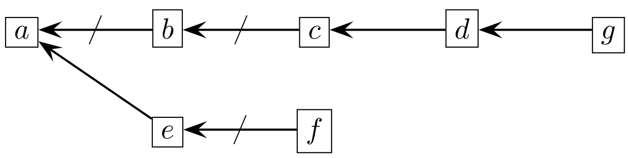
The abstract bipolar argumentation framework presented in [8; 9] extends Dung's framework in order to take into account both negative interactions expressed by the attack relation and positive interactions expressed by a support relation (see [3] for a more general survey about bipolarity in argumentation).

[1] if and only if
[2] such that
[3] with respect to

Def. 4 (BAS) *A bipolar argumentation system (BAS, for short) is a tuple* $\langle \mathbf{A}, \mathbf{R}_{\text{att}}, \mathbf{R}_{\text{sup}} \rangle$ *where* $\mathbf{A}$ *is a finite and non-empty set of arguments,* $\mathbf{R}_{\text{att}}$ *is a binary relation over* $\mathbf{A}$ *called the* attack relation *and* $\mathbf{R}_{\text{sup}}$ *is a binary relation over* $\mathbf{A}$ *called the* support relation.

A BAS can still be represented by a directed graph[4], called the *bipolar interaction graph*, with two kinds of edges. Let a_i and $a_j \in \mathbf{A}$, $a_i \mathbf{R}_{\text{att}} a_j$ (resp. $a_i \mathbf{R}_{\text{sup}} a_j$) means that a_i attacks a_j (resp. a_i supports a_j) and it is represented by $a \not\to b$ (resp. $a \to b$).

Ex. 2 *For instance, in the following graph representing a BAS, there is a support from* g *to* d *and an attack from* b *to* a



Handling support and attack at an abstract level has the advantage to keep genericity. An abstract bipolar framework is useful as an analytic tool for studying different notions of complex attacks, complex conflicts, and new semantics taking into account both kinds of interactions between arguments. However, the drawback is the lack of guidelines for choosing the appropriate definitions and semantics depending on the application. For solving this problem, some specializations of the support relation have been proposed and discussed recently. The distinction between deductive and necessary support has appeared first. Then, several interpretations have been given to the necessary support (sub-argument relation [25], evidential support [22; 23; 24], backing support [12]).

2.2.1 Deductive support

The deductive support has first appeared in [5]. This variant is intended to enforce the following constraint: If $b\mathbf{R}_{\text{sup}}c$ then "the acceptance of b implies the acceptance of c", and as a consequence "the non-acceptance of c implies the non-acceptance of b".

In relevant literature, this interpretation is usually taken into account by adding two kinds of complex attack. The idea is to produce a new AS, containing original and new attacks, and then to use standard semantics.

The first new attack, called mediated attack in [5], occurs when $b\mathbf{R}_{\text{sup}}c$ and $a\mathbf{R}_{\text{att}}c$: "the acceptance of a implies the non-acceptance of c" and so "the acceptance of a implies the non-acceptance of b".

Def. 5 ([5] Mediated attack)
Let $\mathsf{BAS} = \langle \mathbf{A}, \mathbf{R}_{\text{att}}, \mathbf{R}_{\text{sup}} \rangle$*. There is a* mediated attack *from* a *to* b *iff there is a sequence* $a_1 \mathbf{R}_{\text{sup}} \dots \mathbf{R}_{\text{sup}} a_{n-1}$*, and* $a_n \mathbf{R}_{\text{att}} a_{n-1}$*,* $n \geq 3$*, with* $a_1 = b$*,* $a_n = a$*.*

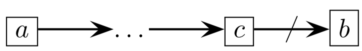
Another complex attack, called supported attacks in [9] occurs when $a\mathbf{R}_{\text{sup}}c$ and $c\mathbf{R}_{\text{att}}b$: "the acceptance of a implies the acceptance of c" and "the acceptance of c implies the non-acceptance of b"; so, "the acceptance of a implies the non-acceptance of b".

Def. 6 ([9] Supported attack)
Let $\mathsf{BAS} = \langle \mathbf{A}, \mathbf{R}_{\text{att}}, \mathbf{R}_{\text{sup}} \rangle$*. There is a* supported attack *from* a *to* b *iff there is a sequence* $a_1 \mathbf{R}_1 \dots \mathbf{R}_{n-1} a_n$*,* $n \geq 3$*, with* $a_1 = a$*,* $a_n = b$*,* $\forall i = 1 \dots n-2$*,* $\mathbf{R}_i = \mathbf{R}_{\text{sup}}$ *and* $\mathbf{R}_{n-1} = \mathbf{R}_{\text{att}}$*.*

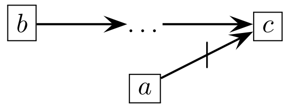
[4]This is an abuse of language since, stricly speaking, this is an edge-labeled graph (with two labels) rather than a directed graph.

So, with the deductive interpretation of the support, new kinds of attack, from a to b, can be considered in the following cases:

Supported attacks:



Mediated attacks:



2.2.2 Necessary support

The necessary support has been first proposed by [20; 21] with the following interpretation: If $c\mathbf{R}_{\text{sup}}b$ then "the acceptance of c is necessary to get the acceptance of b", or equivalently "the acceptance of b implies the acceptance of c". A example of this kind of support could be:

Ex. 3 *A dialog between three customers about the qualities of services of their hotel:*
- *"This hotel is very well managed." (Argument a)*
- *"Yes. In particular, the hotel staff is very competent." (Argument b)*
- *"They are not competent! The rooms are dirty." (Argument c)*

Here b necessarily supports a and c attacks b ($c \not\to b \to a$). The link between b and a is similar to the notion of subargument used in [25].

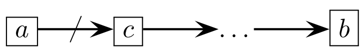
As for deductive support, the idea is to add complex attacks in order to use standard semantics on a new AS. The first added complex attack, called extended attack in [20] and secondary attack in [9] has been proposed in the following case: Suppose that $a\mathbf{R}_{\text{att}}c$ and $c\mathbf{R}_{\text{sup}}b$. "The acceptance of a implies the non-acceptance of c" and so "the acceptance of a implies the non-acceptance of b". Another kind of complex attack may be considered when $c\mathbf{R}_{\text{sup}}a$ and $c\mathbf{R}_{\text{att}}b$: "the acceptance of a implies the acceptance of c" and "the acceptance of c implies the non-acceptance of b". So, "the acceptance of a implies the non-acceptance of b". This new attack from a to b has been proposed in [21].

The formal definition of these two attacks is:

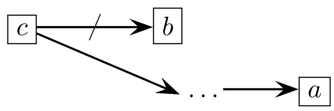
Def. 7 ([21] Extended attack) *Let* $\mathsf{BAS} = \langle \mathbf{A}, \mathbf{R}_{\text{att}}, \mathbf{R}_{\text{sup}} \rangle$*. There is an* extended attack *from a to b iff*
- *either $a\mathbf{R}_{\text{att}}b$ (direct attack),*
- *or there is a sequence $a_1\mathbf{R}_{\text{att}}a_2\mathbf{R}_{\text{sup}} \ldots \mathbf{R}_{\text{sup}}a_n$, $n \geq 3$, with $a_1 = a$, $a_n = b$ (Case 1),*
- *or there is a sequence $a_1\mathbf{R}_{\text{sup}} \ldots \mathbf{R}_{\text{sup}}a_n$, and $a_1\mathbf{R}_{\text{att}}a_p$, $n \geq 2$, with $a_n = a$, $a_p = b$ (Case 2).*

So, with the necessary interpretation of the support, new kinds of attack, from a to b, can be considered in the following cases:

Extended attacks – Case 1 (secondary attacks):



Extended attacks – Case 2:



2.2.3 Duality between deductive and necessary support

Deductive support and necessary support have been introduced independently. Nevertheless, they correspond to dual interpretations of the notion of support. Let us denote $a \overset{D}{\to} b$ (resp. $a \overset{N}{\to} b$) when there exists a deductive (resp. necessary) support from a to b. As $a \overset{D}{\to} b$ means that "the acceptance of a implies the acceptance of b", and $a \overset{N}{\to} b$ means that "the acceptance of a is necessary to get the acceptance of b", it follows that $a \overset{N}{\to} b$ is equivalent to $b \overset{D}{\to} a$.

Following this duality, it is easy to see that the mediated attack obtained by combining the attack relation $\mathbf{R}_{\text{att}}$ and the support relation $\mathbf{R}_{\text{sup}}$ exactly corresponds to the secondary attack obtained by combining the attack relation $\mathbf{R}_{\text{att}}$ and the support relation $\mathbf{R}^{-1}_{\text{sup}}$ which is the symmetric relation of $\mathbf{R}_{\text{sup}}$ ($\mathbf{R}^{-1}_{\text{sup}} = \{(b, a)|(a, b) \in \mathbf{R}_{\text{sup}}\}$). Similarly, the supported attack obtained by combining the attack relation $\mathbf{R}_{\text{att}}$ and the support relation $\mathbf{R}_{\text{sup}}$ exactly corresponds to the second case of extended attack obtained by combining the attack relation $\mathbf{R}_{\text{att}}$ and the support relation $\mathbf{R}^{-1}_{\text{sup}}$.

So in the following, we only focus on the necessary support since, taking advantage of the duality, all the results we obtain can be easily translated into results for deductive supports.

3 Axiomatic approach for handling necessary support

In relevant literature, as described in the previous section, taking into account support generally leads to add new attacks. It is the case for instance with the necessary support that leads to extended attacks. However, a deeper analysis of the original interpretation of necessary support suggests other ways to handle this support. In this section, we discuss several constraints induced by the intended meaning of necessary support, and we show that new frameworks must be proposed for encoding these constraints.

Let us come back to the original interpretation of necessary support: If $c\mathbf{R}_{\text{sup}}b$, "the acceptance of c is necessary to get the acceptance of b". Analysing this interpretation leads to at least four kinds of constraints.

Transitivity (TRA) This first requirement concerns the relation $\mathbf{R}_{\text{sup}}$ alone. It expresses transitivity[5] of the necessary support. It induces that a sequence of supports is considered as a support:

Def. 8 (Constraint TRA) *$\forall a, b \in \mathbf{A}$, if $\exists n > 1$ such that $a = a_1\mathbf{R}_{\text{sup}} \ldots \mathbf{R}_{\text{sup}}a_n = b$, then a* `supports` *b.*

Closure (CLO) A second constraint also concerns the relation $\mathbf{R}_{\text{sup}}$ alone and expresses the fact that if $c\mathbf{R}_{\text{sup}}b$, then "the acceptance of b implies the acceptance of c". So, if $c\mathbf{R}_{\text{sup}}b$, and there exists an extension S containing b, then S also contains c. This constraint can be expressed by the property of closure of an extension under $\mathbf{R}^{-1}_{\text{sup}}$:[6]

Def. 9 (Constraint CLO) *Let s be a semantics and E be an extension under s. $\forall a, b \in \mathbf{A}$, if $a\mathbf{R}_{\text{sup}}b$ and $b \in E$, then $a \in E$.*

Moreover, an interesting variant of this constraint could be induced by a slightly different reading of the original interpretation: "the acceptance of c is necessary to get the acceptance of b" because *c is the only attacker of a particular attacker of b*. This reading implies that there implicitly exists a special attack to b which can be only defeated by c. This interpretation will lead us to propose a framework with meta-arguments (see Section 4.2).

[5]Irreflexivity has also been considered for instance in [20; 21].

[6]Note that if $c\mathbf{R}_{\text{sup}}b$ and $c\mathbf{R}_{\text{att}}b$, as an extension must be conflict-free, there is no extension containing both c and b, so the constraint trivially holds. Some works, as for instance [10], exclude the case when $c\mathbf{R}_{\text{sup}}b$ and $c\mathbf{R}_{\text{att}}b$.

Conflicting sets (CFS) Now, we consider constraints induced by the presence of both attacks and supports in a BAS. Starting from the original interpretation, if $a\mathbf{R}_{\mathrm{att}}c$ and $c\mathbf{R}_{\mathrm{sup}}b$, "the acceptance of a implies the non-acceptance of c" and "the acceptance of b implies the acceptance of c". So, using contrapositives, "the acceptance of a implies the non-acceptance of b", and then "the acceptance of b implies the non-acceptance of a". Thus, we obtain a symmetric constraint involving a and b. However, the fact that "the acceptance of a implies the non-acceptance of b" is not equivalent to the fact that there is an attack from a to b. We have only the sufficient condition. So, the creation of a complex attack (here a secondary attack) from a to b can be viewed in some sense too strong. Hence, faced with the case when $a\mathbf{R}_{\mathrm{att}}c$ and $c\mathbf{R}_{\mathrm{sup}}b$, we propose to assert a conflict between a and b, or in other words that the set $\{a, b\}$ is a conflicting set. Similarly, if $c\mathbf{R}_{\mathrm{att}}b$ and $c\mathbf{R}_{\mathrm{sup}}a$, "the acceptance of a implies the acceptance of c" and so "the acceptance of a implies the non-acceptance of b".

Def. 10 (Constraint CFS) $\forall a, b, c \in \mathbf{A}$. *If* ($a\mathbf{R}_{\mathrm{att}}c$ *and* c `supports` b) *or* ($c\mathbf{R}_{\mathrm{att}}b$ *and* c `supports` a) *then* $\{a, b\}$ *is a conflicting set.*

Note that the Dung's abstract framework is not suitable for expressing such a constraint. So we will present in Section 4.1 a new framework for handling conflicting sets of arguments.

Addition of new attacks (nATT and n+ATT) Beyond these properties, according to the applications and the previous works presented in literature, we may impose stronger constraints corresponding to the addition of new attacks. Two cases may be considered:

Def. 11 (Constraint nATT) *If* $a\mathbf{R}_{\mathrm{att}}c$ *and* $c\mathbf{R}_{\mathrm{sup}}b$, *then there is a new attack from* a *to* b.

Def. 12 (Constraint n+ATT) *If* ($a\mathbf{R}_{\mathrm{att}}c$ *and* $c\mathbf{R}_{\mathrm{sup}}b$) *or* ($c\mathbf{R}_{\mathrm{att}}b$ *and* $c\mathbf{R}_{\mathrm{sup}}a$), *then there is a new attack from* a *to* b.

nATT (resp. **n+ATT**) corresponds to the addition of secondary (resp. extended) attacks. In Section 4.3 we present two frameworks for handling these constraints.

Continuing the discussion one step further, if the fact that "the acceptance of a implies the non-acceptance of b" is represented by an attack from a to b, due to contrapositive, this new attack must be symmetric. However, in that case, each attack should be turned into a symmetric one. Thus, we move towards symmetric argumentation frameworks which have been studied in [13]. We will not consider this case in the current paper. Some of the above constraints can he handled in a Dung's abstract framework (**CLO**, **TRA**, **nATT** and **n+ATT**) with the advantage of reusing all known Dung's results. However, as we noticed above, constraint **CFS** cannot be encoded in a Dung's framework. So in the next section we propose different variants of Dung's framework and of the bipolar framework in order to take into account these constraints.

4 New frameworks for handling necessary supports

Starting from the constraints discussed in Section 3, we propose several frameworks for handling necessary support. The first two are driven by Constraint **CLO** whereas the last two are driven by the constraints **nATT** and **n+ATT**. The section will end by a comparison of these frameworks.

4.1 Handling conflicting sets of arguments

We propose a generalized bipolar abstract argumentation framework consisting of a set of arguments, a binary relation representing an attack between arguments, a binary relation representing a support between arguments and a set of conflicting sets of arguments. Intuitively, knowing that a attacks b is stronger than knowing that $\{a, b\}$ is a conflicting set of arguments. Knowing that a set of arguments S is conflicting will only prevent any extension from containing S. Moreover, a conflicting set may contain more than two arguments.

Def. 13 (Generalized BAS, GBAS) *A generalized bipolar argumentation system is a tuple* $\langle \mathbf{A}, \mathbf{R}_{\text{att}}, \mathbf{R}_{\text{sup}}, \mathbf{C} \rangle$ *where*

- $\mathbf{A}$ *is a finite and non-empty set of arguments,*
- $\mathbf{R}_{\text{att}}$ *is a binary relation over* $\mathbf{A}$ *called the* attack relation*,*
- $\mathbf{R}_{\text{sup}}$ *is a binary relation over* $\mathbf{A}$ *called the* support relation *and*
- $\mathbf{C}$ *is a finite set of subsets of* $\mathbf{A}$ *such that* $\forall (a, b) \in \mathbf{R}_{\text{att}}, \{a, b\} \in \mathbf{C}$*.*

Conflict-freeness in a generalized bipolar argumentation system is defined as follows:

Def. 14 (Conflict-freeness in a GBAS) *Let* $\langle \mathbf{A}, \mathbf{R}_{\text{att}}, \mathbf{R}_{\text{sup}}, \mathbf{C} \rangle$ *be a GBAS and* $S \subseteq \mathbf{A}$*.* S *is conflict-free in the GBAS iff there does not exist* $C \in \mathbf{C}$ *such that* $C \subseteq S$*.*

However, the definition of semantics depends on the interpretation of the support and also on the constraints that have to be enforced. The generalized bipolar framework can be instantiated for encoding necessary support, due to the following definition:

Def. 15 *Let* $\mathsf{BAS} = \langle \mathbf{A}, \mathbf{R}_{\text{att}}, \mathbf{R}_{\text{sup}} \rangle$ *with* $\mathbf{R}_{\text{sup}}$ *being a set of necessary supports. The tuple* $\mathsf{GBAS} = \langle \mathbf{A}, \mathbf{R}_{\text{att}}, \mathbf{R}_{\text{sup}}, \mathbf{C} \rangle$ *with*

$$\begin{aligned} \mathbf{C} = \quad & \{\{a, b\} | (a, b) \in \mathbf{R}_{\text{att}}\} \\ \cup \quad & \{\{a, b\} | a\mathbf{R}_{\text{att}}c \textit{ and } c \texttt{ supports } b\} \\ \cup \quad & \{\{a, b\} | c\mathbf{R}_{\text{att}}b \textit{ and } c \texttt{ supports } a\} \end{aligned}$$

is the generalized argumentation system associated with BAS*.*

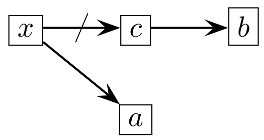
It is easy to see that the generalized argumentation system associated with BAS enables to enforce the constraints **TRA** and **CFS**, whereas it satisfies neither Constraint **nATT**, nor Constraint **n+ATT**.

The next step is the study of acceptability in a GBAS in order to check whether Contraint **CLO** is taken into account. For that purpose, the first proposal is to use conflict-freeness as defined in Def. 14 and admissible, preferred and stable extensions as defined in Dung's systems. In this case, it can be proved that every stable extension is closed under $\mathbf{R}_{\text{sup}}^{-1}$.

Prop. 1 *Let* $\mathsf{BAS} = \langle \mathbf{A}, \mathbf{R}_{\text{att}}, \mathbf{R}_{\text{sup}} \rangle$ *and its associated* GBAS*. Let* $S \subseteq \mathbf{A}$*. If* S *is conflict-free in* GBAS*, and for each* $a \notin S$*, there is* $b \in S$ *s.t.* $b\mathbf{R}_{\text{att}}a$*, then* S *is closed under* $\mathbf{R}_{\text{sup}}^{-1}$*.*

However, this approach produces many conflicts, without adding any attacks. So in many cases, there will be no stable extension. Moreover, Constraint **CLO** is generally not satisfied with the preferred semantics. The following example illustrates these two drawbacks.

Ex. 4 *Consider* BAS *represented by the following graph.*



$\mathbf{C} = \{\{x,c\},\{x,b\},\{a,c\}\}$. *Using the classical definition of semantics with conflict-freeness as defined in Def. 14, the preferred extensions of the associated* GBAS *are* $\{a,x\}$ *and* $\{a,b\}$*, and there is no stable extension. Moreover, the preferred extension* $\{a,b\}$ *is not closed under* $\mathbf{R}_{\text{sup}}^{-1}$.

The preferred semantics has to be redefined in order to enforce Constraint **CLO**. So, our second proposal is to enforce a notion of coherence by combining conflict-freeness and closure under $\mathbf{R}_{\text{sup}}^{-1}$. Moreover it can be proven that:

Prop. 2 *Let* $\langle \mathbf{A}, \mathbf{R}_{\text{att}}, \mathbf{R}_{\text{sup}} \rangle$ *and its associated* GBAS*. Let* $S \subseteq \mathbf{A}$*. If* S *is closed under* $\mathbf{R}_{\text{sup}}^{-1}$ *then (*S *is conflict-free in* GBAS *iff* S *is conflict-free in* $\langle \mathbf{A}, \mathbf{R}_{\text{att}} \rangle$*).*

Def. 16 (Coherence in a GBAS) *Let* $\langle \mathbf{A}, \mathbf{R}_{\text{att}}, \mathbf{R}_{\text{sup}}, \mathbf{C} \rangle$ *be a GBAS and* $S \subseteq \mathbf{A}$*.* S *is coherent in the GBAS iff* S *is conflict-free in* $\langle \mathbf{A}, \mathbf{R}_{\text{att}} \rangle$ *and* S *is closed under* $\mathbf{R}_{\text{sup}}^{-1}$.

Using coherence in place of conflict-freeness leads to new definitions:

Def. 17 (Admissibility in a GBAS) *Let* $\langle \mathbf{A}, \mathbf{R}_{\text{att}}, \mathbf{R}_{\text{sup}}, \mathbf{C} \rangle$ *be a GBAS and* $S \subseteq \mathbf{A}$*.*
- S *is* admissible in the GBAS *iff* S *is coherent in the GBAS and* $\forall a \in S$*,* $\forall b \in \mathbf{A}$ *s.t.* $b\mathbf{R}_{\text{att}}a$*,* $\exists c \in S$ *s.t.* $c\mathbf{R}_{\text{att}}b$*.*
- S *is* a preferred extension of the GBAS *iff it is a maximal (wrt* $\subseteq$*) admissible set.*
- S *is* a stable extension of the GBAS *iff* S *is coherent*[7] *in the GBAS and for each* $a \notin S$*, there is* $b \in S$ *s.t.* $b\mathbf{R}_{\text{att}}a$*.*

Ex. 4 (cont'd) *Taking into account coherence, as in Def.17,* $\{a,x\}$ *is the unique preferred extension of the associated* GBAS*, and it is closed under* $\mathbf{R}_{\text{sup}}^{-1}$.

So, using Def.17 and 16, the associated GBAS enables to enforce Constraint **CLO**.[8] Moreover, as in Dung's framework, stable extensions are also preferred.

Prop. 3 *Let* $\langle \mathbf{A}, \mathbf{R}_{\text{att}}, \mathbf{R}_{\text{sup}}, \mathbf{C} \rangle$ *be a GBAS and* $S \subseteq \mathbf{A}$*. If* S *is a stable extension of the GBAS then* S *is also a preferred extension of the GBAS.*

A thorough study of the generalized bipolar abstract argumentation framework would demand to define other semantics such as grounded one. However, this is not our purpose in this paper. We focus on the way to enforce different kinds of constraints related to necessary support.

4.2 A meta-framework encoding necessary support

The fact that "the acceptance of c is necessary to get the acceptance of b" can be encoded in another way. As explained in Section 3, the idea is to assume the existence of a special argument attacking b for which c is the *only* attacker. More precisely, if $c\mathbf{R}_{\text{sup}}b$, we create a new argument N_{cb} and two attacks $c\mathbf{R}_{\text{att}}N_{cb}$ and $N_{cb}\mathbf{R}_{\text{att}}b$. As c is the unique attacker of N_{cb}, "the acceptance of b implies the acceptance of c". The meaning of N_{cb} could be that the support from c to b is not active. A similar idea can be found in [27; 11] for the more general purpose of representing recursive and defeasible attacks and supports.

[7]Due to Proposition 1, coherent may be replaced by conflict-free.
[8]Note that enforcing coherence makes the set C useless due to Prop.2.

Def. 18 *Let* $\mathsf{BAS} = \langle \mathbf{A}, \mathbf{R}_{\text{att}}, \mathbf{R}_{\text{sup}} \rangle$ *with* $\mathbf{R}_{\text{sup}}$ *being a set of necessary supports. Let* $\mathbf{A}_n = \{N_{cb} | (c,b) \in \mathbf{R}_{\text{sup}}\}$ *and* $\mathbf{R}_n = \{(c, N_{cb}) | (c,b) \in \mathbf{R}_{\text{sup}}\} \cup \{(N_{cb}, b) | (c,b) \in \mathbf{R}_{\text{sup}}\}$. *The tuple* $\mathsf{MAS} = \langle \mathbf{A} \cup \mathbf{A}_n, \mathbf{R}_{\text{att}} \cup \mathbf{R}_n \rangle$ *is the meta-argumentation system* [9] *associated with* BAS.

Let us check whether the minimal requirements are satisfied. Let us first consider constraint **TRA**. From $a\mathbf{R}_{\text{sup}}b$ and $b\mathbf{R}_{\text{sup}}c$, we obtain the sequence of attacks $a\mathbf{R}_{\text{att}}N_{ab}\mathbf{R}_{\text{att}}b\mathbf{R}_{\text{att}}N_{bc}\mathbf{R}_{\text{att}}c$. So, the acceptance of c implies the acceptance of b, which in turn implies the acceptance of a, as if we had directly encoded $a\mathbf{R}_{\text{sup}}c$. So, **TRA** is taken into account. The same result holds for **CLO**:

Prop. 4 *Let* $\mathsf{BAS} = \langle \mathbf{A}, \mathbf{R}_{\text{att}}, \mathbf{R}_{\text{sup}} \rangle$ *and its associated* MAS. *Let* $S \subseteq \mathbf{A} \cup \mathbf{A}_n$. *If* S *is admissible in* MAS, *then* $S \cap \mathbf{A}$ *is closed under* $\mathbf{R}_{\text{sup}}^{-1}$ *in* BAS.

Constraint **CFS** is not enforced. We only have the following property:

Prop. 5 *Let* $\mathsf{BAS} = \langle \mathbf{A}, \mathbf{R}_{\text{att}}, \mathbf{R}_{\text{sup}} \rangle$ *and its associated* MAS. *Let* a, b, c *be arguments of* $\mathbf{A}$. *If* ($a\mathbf{R}_{\text{att}}c$ *and* c `supports` b) *or* ($c\mathbf{R}_{\text{att}}b$ *and* c `supports` a) *then no admissible set in* MAS *contains* $\{a, b\}$.

Note that this result is weaker than **CFS** since it does not imply that $\{a, b\}$ is a conflicting set.

Obviously, stronger constraints such as **nATT** or **n+ATT** are not directly enforced. If $a\mathbf{R}_{\text{att}}c$ and $c\mathbf{R}_{\text{sup}}b$, we obtain the sequence $a\mathbf{R}_{\text{att}}c\mathbf{R}_{\text{att}}N_{cb}\mathbf{R}_{\text{att}}b$. No attack from a to b is added. However, we will see in Section 4.4 that the meta-argumentation framework associated with BAS enables to recover the extensions obtained when enforcing Constraint **nATT**.

4.3 A framework with complex attacks

In this subsection we discuss two frameworks enabling to handle necessary support through the addition of complex attacks. According to the various interpretations of the necessary support, all the complex attacks are not justified. For instance, if the necessary support models a subargument relation as in [25], only the secondary attack makes sense. Other works [21] have considered both cases of extended attack. However, to the best of our knowledge, there has been no formal study of the properties of these extended attacks, and of the consequences of these attacks on the acceptable sets of arguments.

From Def. 7, new attacks called `n+-attacks` can be generated inductively as follows:

Def. 19 (`n+-attacks`) *Let* $\mathsf{BAS} = \langle \mathbf{A}, \mathbf{R}_{\text{att}}, \mathbf{R}_{\text{sup}} \rangle$ *with* $\mathbf{R}_{\text{sup}}$ *being a set of necessary supports. There exists a* `n+-attack` *from* a *to* b *iff*

- *either* $a\mathbf{R}_{\text{att}}b$, *or there is a (case 1 or case 2) extended attack from* a *to* b,
- *or there exists an argument* c *s.t.* a `n+-attacks` c *and* c `supports` b,
- *or there exists an argument* c *s.t.* c `supports` a *and* c `n+-attacks` b.

$\mathbf{N+}^{\mathbf{R}_{\text{sup}}}_{\mathbf{R}_{\text{att}}}$ *denoted the set of* `n+-attacks` *generated by* $\mathbf{R}_{\text{sup}}$ *on* $\mathbf{R}_{\text{att}}$. *The AS defined by* $\langle \mathbf{A}, \mathbf{N+}^{\mathbf{R}_{\text{sup}}}_{\mathbf{R}_{\text{att}}} \rangle$ *is denoted by* AS^{N^+}.

Obviously Constraints **TRA**, **nATT** and **n+ATT** are enforced in AS^{N^+}.

Let us now consider the case when the extended attacks are restricted to secondary attacks (Case 1 of extended attacks). Following the above definition, our purpose is to define a `n-attack` from a to b when either $a\mathbf{R}_{\text{att}}b$, or there exists a secondary attack from a to b, or there exists

[9] Note that it is an argumentation system in Dung's sense.

an argument c s.t. a `n-attacks` c and c `supports` b. Indeed, it is easy to prove that the formal definition of this `n-attack` can be simplified as follows:

Def. 20 (n-attack) *Let* $\mathsf{BAS} = \langle \mathbf{A}, \mathbf{R}_{\mathrm{att}}, \mathbf{R}_{\mathrm{sup}} \rangle$. *There is* `n-attack` *from a to b iff*
- *either* $a\mathbf{R}_{\mathrm{att}}b$,
- *or there is a secondary attack from a to b.*

$\mathbf{N}_{\mathbf{R}_{\mathrm{att}}}^{\mathbf{R}_{\mathrm{sup}}}$ *denoted the set of* `n-attacks` *generated by* $\mathbf{R}_{\mathrm{sup}}$ *on* $\mathbf{R}_{\mathrm{att}}$. *The AS defined by* $\langle \mathbf{A}, \mathbf{N}_{\mathbf{R}_{\mathrm{att}}}^{\mathbf{R}_{\mathrm{sup}}} \rangle$ *is denoted by* AS^N.

Note that both AS^N and AS^{N^+} are Dung's argumentation systems; so the classical notions given in Def. 2 and 3 can be applied without restriction, nor redefinition.

Obviously Constraints **TRA** and **nATT** are enforced in AS^N, whereas Constraint **n+ATT** is not.

Def. 19 looks complex. However the following proposition enables to rewrite `n+-attacks` and `n-attacks` in a form which will be much easier to handle for studying their properties.

Prop. 6 *Let* $\mathsf{BAS} = \langle \mathbf{A}, \mathbf{R}_{\mathrm{att}}, \mathbf{R}_{\mathrm{sup}} \rangle$. *There is an* `n+-attack` *from a to b iff there is a sequence* $a_1\mathbf{R}_{\mathrm{att}}b_1\mathbf{R}_{\mathrm{sup}} \ldots \mathbf{R}_{\mathrm{sup}}b_m$, *with* $b_m = b$ *and* $m \geq 1$, *and a sequence* $a_1\mathbf{R}_{\mathrm{sup}} \ldots \mathbf{R}_{\mathrm{sup}}a_n$ *with* $a_n = a$ *and* $n \geq 1$.

`n+-attacks` corresponding to this proposition can be illustrated by the following figure:

$$\begin{tikzcd}
\boxed{a_1} \arrow[r, "/" marking] \arrow[dr] & \boxed{b_1} \arrow[r] & \ldots \arrow[r] & \boxed{b_m = b} & m \geq 1 \\
& \boxed{a_2} \arrow[r] & \ldots \arrow[r] & \boxed{a_n = a} & n \geq 1
\end{tikzcd}$$

Moreover, Prop. 6 can be used for identifying the following particular cases:
- The case when $m = n = 1$ corresponds to a direct attack from a to b.
- The case when $n = 1$ and $m \geq 1$ corresponds to a `n-attack` from a to b (direct or secondary attacks, see Def. 20).
- The case when $n = 1$ and $m > 1$ corresponds to an extended attack - Case 1 (secondary attack) from a to b (see Def. 7).
- The case when $n > 1$ and $m = 1$ corresponds to an extended attack - Case 2 from a to b (see Def. 7).

An obvious consequence of this proposition is:

Corol. 1 *Let* $\mathsf{BAS} = \langle \mathbf{A}, \mathbf{R}_{\mathrm{att}}, \mathbf{R}_{\mathrm{sup}} \rangle$ *and its associated* AS^N *and* AS^{N^+}. *Let* $S \subseteq \mathbf{A}$. *If S is conflict-free in* AS^{N^+}, *then S is conflict-free in* AS^N.

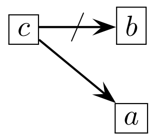
As said above, in some works necessary support can be handled by only considering `n-attacks`, that is by adding secondary attacks. However, although both cases of extended attacks are independent, we show that taking into account only `n-attacks` is already enough for inducing constraints on AS^{N^+}.

Prop. 7 *Let* $\mathsf{BAS} = \langle \mathbf{A}, \mathbf{R}_{\mathrm{att}}, \mathbf{R}_{\mathrm{sup}} \rangle$ *and its associated* AS^N. *If a* `n+-attack` *from a to b can be built from* BAS, *there exists no admissible set in* AS^N *containing* $\{a, b\}$.

As an immediate consequence (contrapositive of Prop. 7), we have:

Corol. 2 *Let* $\mathsf{BAS} = \langle \mathbf{A}, \mathbf{R}_{\mathrm{att}}, \mathbf{R}_{\mathrm{sup}} \rangle$ *and the associated* AS^N *and* AS^{N^+}. *Let* $S \subseteq \mathbf{A}$. *If S is admissible in* AS^N, *then S is conflict-free in* AS^{N^+}.

Ex. 5 *Consider* BAS *represented by the following graph:*



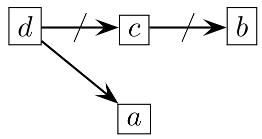
The associated AS^N *only contains the original attack from* c *to* b *(there is no secondary attack). If we consider only* `n-attacks`*, there is no conflict between* a *and* b*. However, it can be proved that no admissible set in* AS^N *contains* $\{a, b\}$*.*

The following results establish links between extensions in AS^N and AS^{N^+}.

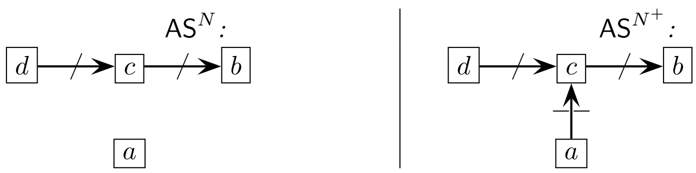
Prop. 8 *Let* $\mathsf{BAS} = \langle \mathbf{A}, \mathbf{R}_{\text{att}}, \mathbf{R}_{\text{sup}} \rangle$ *and the associated* AS^N *and* AS^{N^+}*. Let* $S \subseteq \mathbf{A}$*. If* S *is admissible in* AS^N*, then* S *is also admissible in* AS^{N^+}*.*

The converse of Prop 8 generally does not hold as shown by the following example.

Ex. 6 *Consider* BAS *and its associated* AS^N *and* AS^{N^+} *represented by the following graphs:*



The associated AS^N *and* AS^{N^+} *are represented by the following graphs:*



The set $\{a, b\}$ *is admissible in* AS^{N^+} *but is not admissible in* AS^N *(since* a *does not attack* c *in* AS^N*).*

However, the converse of Prop. 8 holds for maximal admissible sets:

Prop. 9 *Let* $\mathsf{BAS} = \langle \mathbf{A}, \mathbf{R}_{\text{att}}, \mathbf{R}_{\text{sup}} \rangle$ *and its associated* AS^N *and* AS^{N^+}*. Let* $S \subseteq \mathbf{A}$*.* S *is maximal admissible in* AS^{N^+} *iff* S *is maximal admissible in* AS^N*.*

The same holds for stable semantics:

Prop. 10 *Let* $\mathsf{BAS} = \langle \mathbf{A}, \mathbf{R}_{\text{att}}, \mathbf{R}_{\text{sup}} \rangle$ *and its associated* AS^N *and* AS^{N^+}*. Let* $S \subseteq \mathbf{A}$*.* S *is stable in* AS^{N^+} *iff* S *is stable in* AS^N*.*

We conclude this section by providing results about the property of closure under the relation $\mathbf{R}_{\text{sup}}^{-1}$.

Prop. 11 *Let* $\mathsf{BAS} = \langle \mathbf{A}, \mathbf{R}_{\text{att}}, \mathbf{R}_{\text{sup}} \rangle$ *and its associated* AS^{N^+}*. Let* $S \subseteq \mathbf{A}$ *and* $a, b \in \mathbf{A}$*.*

- *If* S *is conflict-free in* AS^{N^+}*,* $a \in S$ *and* $b\mathbf{R}_{\text{sup}}a$*, then* $S \cup \{b\}$ *is conflict-free in* AS^{N^+}*.*
- *If* S *is maximal (wrt* $\subseteq$*) conflict-free in* AS^{N^+}*, then* S *is closed for the relation* $\mathbf{R}_{\text{sup}}^{-1}$*.*

Prop. 11 does not hold when considering AS^N instead of AS^{N^+}, as shown by the following example.

Ex. 5 (cont'd) *$S = \{a, b\}$ is maximal conflict-free in AS^N but it is not closed under $\mathbf{R}_{\text{sup}}^{-1}$. We have $c\mathbf{R}_{\text{sup}}a$ but $S \cup \{c\}$ is not conflict-free in AS^N.*

However, the property of closure under $\mathbf{R}_{\text{sup}}^{-1}$ is recovered in AS^N, if preferred (resp. stable) extensions are considered.

Prop. 12 *Let $\mathsf{BAS} = \langle \mathbf{A}, \mathbf{R}_{\text{att}}, \mathbf{R}_{\text{sup}} \rangle$ and the associated AS^N and AS^{N^+}. Let $S \subseteq \mathbf{A}$.*

- *If S is a preferred extension in AS^N (resp. AS^{N^+}), then S is closed for the relation $\mathbf{R}_{\text{sup}}^{-1}$.*
- *If S is stable in AS^N (resp. AS^{N^+}), then S is closed for the relation $\mathbf{R}_{\text{sup}}^{-1}$.*

Due to Prop. 12, each stable (resp. preferred) extension of AS^N (resp. AS^{N^+}) is closed under $\mathbf{R}_{\text{sup}}^{-1}$. In that sense, Constraint **CLO** is enforced in AS^N (resp. AS^{N^+}).

It remains to consider Constraint **CFS**. This constraint is obviously satisfied by AS^{N^+} since a new attack is built for each conflict in the sense of **CFS**, whereas the Dung's argumentation system AS^N does not capture all the conflicts induced by **CFS**, as illustrated by the following example.

Ex. 4 (cont'd) *In the associated AS^N, there is one `n-attacks` from x to c and one from x to b. $\{a, x\}$ is the unique preferred extension of AS^N. It is also stable. Note that $\{a, c\}$ is conflict-free in AS^N. Nevertheless $\{a, c\}$ is a conflicting set in the sense of* **CFS**.

4.4 Comparison between the different frameworks

In the previous sections, starting from a set of constraints, several frameworks (GBAS, MAS, AS^N and AS^{N^+}) have been proposed for handling necessary support. In this section, we compare these frameworks wrt two different points of view: the satisfaction of the constraints and the extensions that are produced.

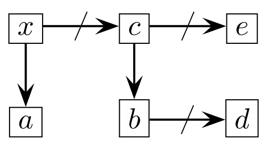
First, the following table synthesizes the previous results:

	GBAS	MAS	AS^N	AS^{N^+}
TRA	X	X	X	X
CLO	X	X	X	X
CFS	X	$-$	$-$	X
nATT	$-$	$-$	X	X
n+ATT	$-$	$-$	$-$	X

X (resp. $-$) means that the corresponding property is (resp. not) satisfied in the corresponding framework.

Now, let us consider AS^N and GBAS. We know that AS^N does not satisfy **CFS** whereas GBAS does. However, due to Prop. 7, if S is a conflicting set of GBAS, it is conflicting in AS^{N^+} and then there is no admissible set of AS^N containing S. Moreover, it can be proved that each preferred extension of GBAS is (generally strictly) included in a preferred extension of AS^N. This is illustrated by the following example.

Ex. 7 *Consider* BAS *represented by:*



In the associated GBAS*, we have* $\mathbf{C} = \{\{x,c\},\{x,b\},\{c,e\},\{b,d\},\{a,c\},\{b,e\}\}$. *The unique preferred extension of* GBAS *is* $\{a,x,e\}$.
In the associated AS^N*, the* `n-attacks` *from* x *to* b *is used for ensuring the acceptability of* d *wrt* $\{a,x,e\}$. *So, the unique preferred extension is* $\{a,d,x,e\}$.

Prop. 13 *Let* $\mathsf{BAS} = \langle \mathbf{A}, \mathbf{R}_{\text{att}}, \mathbf{R}_{\text{sup}} \rangle$ *and its associated* GBAS *and* AS^N. *Let* $S \subseteq \mathbf{A}$.
- *If* S *is admissible in* GBAS*, then* S *is also admissible in* AS^N.
- *If* S *is a preferred extension in* GBAS*, then* S *is included in a preferred extension of* AS^N.
- *If* S *is a stable extension in* GBAS*, then* S *is also a stable extension of* AS^N.

Note that Prop. 13 holds when considering AS^{N+} instead of AS^N, due to Prop. 8, 9 and 10.

The next issue concerns the comparison between AS^N and the associated MAS of BAS. It seems that encoding a necessary support $c\mathbf{R}_{\text{sup}}b$ by a meta-argument N_{cb} and the sequence $a\mathbf{R}_{\text{att}}c\mathbf{R}_{\text{att}}N_{cb}\mathbf{R}_{\text{att}}b$ is less strong than encoding `n-attacks`. However, there is a correspondence between the extensions which are obtained in each framework.

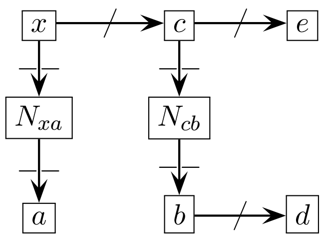
Prop. 14 *Let* $\mathsf{BAS} = \langle \mathbf{A}, \mathbf{R}_{\text{att}}, \mathbf{R}_{\text{sup}} \rangle$ *and its associated* MAS *and* AS^N.
- *Let* $S \subseteq \mathbf{A} \cup \mathbf{A}_n$. *If* S *is admissible in* MAS*, then* $S \cap \mathbf{A}$ *is also admissible in* AS^N.
- *Let* $S \subseteq \mathbf{A}$. *If* S *is a preferred extension in* AS^N*, there exists* S' *admissible in* MAS *such that* $S = S' \cap \mathbf{A}$.
- *Let* $S \subseteq \mathbf{A} \cup \mathbf{A}_n$. *If* S *is stable in* MAS*, then* $S \cap \mathbf{A}$ *is also stable in* AS^N.
- *Let* $S \subseteq \mathbf{A}$. *If* S *is a stable extension in* AS^N*, then there exists* S' *stable in* MAS *such that* $S = S' \cap \mathbf{A}$.

From Prop. 13 and 14, the following comparison between GBAS and MAS can be easily established.

Prop. 15 *Let* $\mathsf{BAS} = \langle \mathbf{A}, \mathbf{R}_{\text{att}}, \mathbf{R}_{\text{sup}} \rangle$ *and its associated* MAS *and* GBAS. *Let* $S \subseteq \mathbf{A}$.
- *If* S *is a preferred extension of* GBAS*, then there exists* S' *preferred in* MAS *such that* $S \subseteq S' \cap \mathbf{A}$.
- *If* S *is a stable extension of* GBAS*, then there exists* S' *stable in* MAS *such that* $S = S' \cap \mathbf{A}$.

The following example illustrates the above propositions.

Ex. 7 (cont'd) *Consider the associated* MAS *represented by:*



In GBAS*, the unique preferred (and also stable) extension is the set* $\{a,x,e\}$. *In* AS^N*, the unique preferred (and also stable) extension is the set* $\{a,x,e,d\}$. *In* MAS*, the unique preferred (and also stable) extension is the set* $\{a,x,e,N_{cb},d\}$.

5 Conclusion and future works

Recent studies in argumentation have addressed the notion of support, with several interpretations (such as deductive, evidential, necessary, backing) and several approaches developed independently. In this paper we focus on necessary support and show that the intended meaning of necessary support can induce different ways to handle it. Our main contribution is to propose an axiomatic approach that is helpful for understanding and comparing the different existing proposals for handling support. First, we have proposed different kinds of constraints that should be imposed to a bipolar argumentation system using necessary supports. Then we have studied different frameworks suitable for encoding these contraints.

This paper reports a preliminary work that could be pursued along different lines. First, our study must be deepened in order to give a more high-level analysis and comparison of all these frameworks. Then the axiomatic approach could be enriched by considering other constraints, such as for instance the strong requirement leading to the addition of symmetric attacks in the case of a necessary support. Moreover, it would be interesting to define such an axiomatic for other interpretations of support, or to consider other frameworks which do not explicitely define a notion of support, such as Abstract Dialectical Frameworks [6]. Another direction for further research would be to study how to encode necessary (or other variants) support by the addition of attacks of various strengths (see for instance [18; 7; 15; 16]). Moreover it would be interesting to see the link between our approaches and the ranking semantics proposed by [1].

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A Proofs

Proof of Prop. 1: Let $S \subseteq \mathbf{A}$ be conflict-free in GBAS. Let $b \in S$ and $c \in \mathbf{A}$ such that $c\mathbf{R}_{\text{sup}}b$. If $c \notin S$, there exists $a \in S$ such that $a\mathbf{R}_{\text{att}}c$. So $\{a, b\}$ is a conflicting set in GBAS, which contradicts the fact that S is conflict-free in GBAS. Hence, $c \in S$ and S is closed under $\mathbf{R}_{\text{sup}}^{-1}$. □

Proof of Prop. 2: Let $S \subseteq \mathbf{A}$ closed under $\mathbf{R}_{\text{sup}}^{-1}$. Let $\mathsf{AS} = \langle \mathbf{A}, \mathbf{R}_{\text{att}} \rangle$ be the associated AS of GBAS.
$\Rightarrow$ Following Def.13 and 14, it is obvious that S conflict-free in GBAS implies that S conflict-free in AS.
$\Leftarrow$ Consider that S is conflict-free in AS. Assume that S is not conflict-free in GBAS. So there exist $a \in S$ and $b \in S$ such that:

- either $a\mathbf{R}_{\text{att}}c$ and c `supports` b: since $b \in S$ and S closed under $\mathbf{R}_{\text{sup}}^{-1}$, then $c \in S$; since a attacks c, S is not conflict-free in AS; contradiction;
- or $c\mathbf{R}_{\text{att}}b$ and c `supports` a: since $a \in S$ and S closed under $\mathbf{R}_{\text{sup}}^{-1}$, then $c \in S$; since c attacks b, S is not conflict-free in AS; contradiction.

So if S is closed for $\mathbf{R}_{\text{sup}}^{-1}$ then (S conflict-free in AS implies that S conflict-free in GBAS). □

Proof of Prop. 3: Let $S \subseteq \mathbf{A}$ be a stable extension of the GBAS. By definition, S is coherent. Consider $z \in S$. There are two cases:

- Either z is unattacked and so z is acceptable wrt S.
- Or z is attacked by x. Since S is stable, $x \notin S$ and there $y \in S$ such that y attacks x; so z is also acceptable wrt S.

Thus S is admissible. Moreover, assume that S is not maximal for set-inclusion among the admissible sets of the GBAS; then there exists S' admissible in the GBAS such that $S \subset S'$; consider $a \in S' \setminus S$; since S is stable then S attacks a and so S' is not conflict-free; contradiction. So S is maximal for set-inclusion among the admissible sets of the GBAS and thus S is a preferred extension of the GBAS. □

Proof of Prop. 4: Let $S \subseteq \mathbf{A} \cup \mathbf{A}_n$ be admissible in MAS. Let $a \in S \cap \mathbf{A}$ and $b \in \mathbf{A}$ such that $b\mathbf{R}_{\text{sup}}a$. In MAS we have the sequence $b\mathbf{R}_{\text{att}}N_{ba}\mathbf{R}_{\text{att}}a$. As S is admissible, there exists $c \in S$ such that c attacks N_{ba} in MAS. Due to Def. 18, the only argument that attacks N_{ba} in MAS is b. Hence $b = c \in S$. We have proved that $S \cap \mathbf{A}$ is closed under $\mathbf{R}_{\text{sup}}^{-1}$ in BAS.

□

Proof of Prop. 5: Let a, b, c be arguments of $\mathbf{A}$.

- Let $a\mathbf{R}_{\text{att}}c$ and $c\mathbf{R}_{\text{sup}}b$ in BAS. In MAS we have the sequence $a\mathbf{R}_{\text{att}}c\mathbf{R}_{\text{att}}N_{cb}\mathbf{R}_{\text{att}}b$. Assume that there exists E admissible in MAS containing $\{a, b\}$. As c is the unique attacker of N_{cb}, E must contain c. The fact that E contains a contradicts the fact that E is conflict-free. Hence E does not exist.
  If $a\mathbf{R}_{\text{att}}c$ and c `supports` b, we prove the result by induction on the length of the sequence `supports`.
- Let $c\mathbf{R}_{\text{att}}b$ and $c\mathbf{R}_{\text{sup}}a$ in BAS. In MAS we have the sequences $c\mathbf{R}_{\text{att}}b$ and $c\mathbf{R}_{\text{att}}N_{ca}\mathbf{R}_{\text{att}}a$. Assume that there exists E admissible in MAS containing $\{a, b\}$. As c is the unique attacker of N_{ca}, E must contain c, which attacks b. That contradicts the fact that E is conflict-free.
  If $c\mathbf{R}_{\text{att}}b$ and c `supports` a, we prove the result by induction on the length of the sequence `supports`.

□

Proof of Prop. 6: Let us denote by $C(a, b)$ the fact that there is a sequence $a_1\mathbf{R}_{\text{att}}b_1\mathbf{R}_{\text{sup}} \dots \mathbf{R}_{\text{sup}}b_m$, with $b_m = b$ and $m \geq 1$, and a sequence $a_1\mathbf{R}_{\text{sup}} \dots \mathbf{R}_{\text{sup}}a_n$ with $a_n = a$ and $n \geq 1$.
$\Rightarrow$) We proceed by induction, following Def. 19.
Basic case : If $a\mathbf{R}_{\text{att}}b$, $C(a, b)$ trivially holds with $n = m = 1$. If there is an extended attack-Case 1 (resp. Case 2) from a to b, $C(a, b)$ holds with $n = 1$ (resp. $m = 1$).
General case : First, let us assume that $C(a, c)$ holds and c supports b. So we have a new sequence $a_1\mathbf{R}_{\text{att}}b_1\mathbf{R}_{\text{sup}} \dots \mathbf{R}_{\text{sup}}b_m = c\mathbf{R}_{\text{sup}}b$, and the sequence $a_1\mathbf{R}_{\text{sup}} \dots \mathbf{R}_{\text{sup}}a_n = a$. Hence $C(a, b)$ holds. Now, let us assume that $C(c, b)$ holds and c supports a. So we have a new sequence $a_1\mathbf{R}_{\text{sup}} \dots \mathbf{R}_{\text{sup}}a_n = c\mathbf{R}_{\text{sup}}a$ and the sequence $a_1\mathbf{R}_{\text{att}}b_1\mathbf{R}_{\text{sup}} \dots \mathbf{R}_{\text{sup}}b_m = b$. Hence $C(a, b)$ holds.

$\Leftarrow$) We proceed by induction on the length of the sequences.
Basic case $n = 1$: $C(a, b)$ means that there is an extended attack-Case 1 from a to b.
Basic case $m = 1$: $C(a, b)$ means that there is an extended attack-Case 2 from a to b.
General case: Assume that the result holds for $n \geq 1, m \geq 1$. Let us first consider $C(a, b)$ with $(n+1, m)$. We have a sequence $a_1\mathbf{R}_{\text{att}}b_1\mathbf{R}_{\text{sup}}\ldots\mathbf{R}_{\text{sup}}b_m = b$ and a sequence $a_1\mathbf{R}_{\text{sup}}\ldots\mathbf{R}_{\text{sup}}a_n\mathbf{R}_{\text{sup}}a_{n+1} = a$. By induction hypothesis, there is a `n+-attack` from a_n to b and $a_n\mathbf{R}_{\text{sup}}a$. By the third item of Def. 19, we conclude that there is a `n+-attack` from a to b.
Now let us consider $C(a, b)$ with $(n, m+1)$. We have a sequence $a_1\mathbf{R}_{\text{att}}b_1\mathbf{R}_{\text{sup}}\ldots\mathbf{R}_{\text{sup}}b_m\mathbf{R}_{\text{sup}}b_{m+1} = b$ and a sequence $a_1\mathbf{R}_{\text{sup}}\ldots\mathbf{R}_{\text{sup}}a_n = a$. By induction hypothesis, there is a `n+-attack` from a to b_m and $b_m\mathbf{R}_{\text{sup}}b$. By the second item of Def. 19, we conclude that there is a `n+-attack` from a to b.

□

Proof of Prop. 7: Assume that $S \subseteq \mathbf{A}$ is an admissible set in AS^N containing $\{a, b\}$.
As S is conflict-free in AS^N, there is no `n-attack` between a and b. So the `n+-attack` from a to b has the following form: $a_1\mathbf{R}_{\text{att}}b_1\mathbf{R}_{\text{sup}}\ldots\mathbf{R}_{\text{sup}}b_m = b$ and $a_1\mathbf{R}_{\text{sup}}\ldots\mathbf{R}_{\text{sup}}a_n = a$ with $n > 1, m \geq 1$. Hence there is a `n-attack` from a_1 to b.
As S is admissible in AS^N and $b \in S$, there is a `n-attack` from S to a_1. There exists $x \in S$ such that x `n-attacks` a_1. Either $x\mathbf{R}_{\text{att}}a_1$ or there is a secondary attack from x to a_1.
In the first case, we have $x\mathbf{R}_{\text{att}}a_1\mathbf{R}_{\text{sup}}\ldots\mathbf{R}_{\text{sup}}a_n = a$, which shows a `n-attack` from x to a.
In the second case, we have $x\mathbf{R}_{\text{att}}y\mathbf{R}_{\text{sup}}\ldots\mathbf{R}_{\text{sup}}a_1\mathbf{R}_{\text{sup}}\ldots\mathbf{R}_{\text{sup}}a_n = a$, which also shows a `n-attack` from x to a.
As S is conflict-free in AS^N, there cannot exist any `n-attack` from $x(\in S)$ and $a(\in S)$. Hence the initial hypothesis does not hold. □

Proof of Prop. 8: Assume that $S \subseteq \mathbf{A}$ is an admissible set in AS^N. Due to Corollary 2, S is conflict-free in AS^{N^+}. It remains to prove that each element of S is acceptable wrt S in AS^{N^+}.
Let $b \in S$ and a such that a `n+-attacks` b. Due to Prop. 6 we have the two following sequences: $a_1\mathbf{R}_{\text{att}}b_1\mathbf{R}_{\text{sup}}\ldots\mathbf{R}_{\text{sup}}b_m = b$, and $a_1\mathbf{R}_{\text{sup}}\ldots\mathbf{R}_{\text{sup}}a_n = a$ with $m \geq 1$, $n \geq 1$. Hence there is a `n-attack` from a_1 to b.
As S is admissible in AS^N, there exists $x \in S$ such that x `n-attacks` a_1, or equivalently such that there is a direct or secondary attack from x to a_1.
In the first case, we have $x\mathbf{R}_{\text{att}}a_1\mathbf{R}_{\text{sup}}\ldots\mathbf{R}_{\text{sup}}a_n = a$, which shows a `n-attack` from x to a.
In the second case, we have $x\mathbf{R}_{\text{att}}y\mathbf{R}_{\text{sup}}\ldots\mathbf{R}_{\text{sup}}a_1\mathbf{R}_{\text{sup}}\ldots\mathbf{R}_{\text{sup}}a_n = a$, which also shows a `n-attack` from x to a.
Hence there is a `n-attack` from x to a and so also a `n+-attack` from x to a. That proves that b is acceptable wrt S in AS^{N^+}.

□

Proof of Prop. 9:
$\Rightarrow$) Assume that S is maximal admissible in AS^{N^+}. Obviously, S is conflict-free in AS^N. We have to prove that S is admissible in AS^N and then that S is maximal admissible.

- We prove that S is admissible in AS^N. If S is not admissible in AS^N, there exists $x \in S$ such that x is not acceptable wrt S in AS^N. So there is a `n-attack` from b to x such that S does not `n-attack` b. As S is admissible in AS^{N^+} there exists $a \in S$ such that a `n+-attacks` b. Hence we have the following sequences: $a_1\mathbf{R}_{\text{att}}b_1\mathbf{R}_{\text{sup}}\ldots\mathbf{R}_{\text{sup}}b_m = b$, and $a_1\mathbf{R}_{\text{sup}}\ldots\mathbf{R}_{\text{sup}}a_n = a$ with $m \geq 1$, $n \geq 1$. Moreover, as S does not `n-attack` b we are sure that $n > 1$ and $a_1 \notin S$. As S is maximal admissible in AS^{N^+}, $S \cup \{a_1\}$ is not admissible in AS^{N^+}. Hence, either $S \cup \{a_1\}$ is not conflict-free in AS^{N^+}, or a_1 is not acceptable wrt S in AS^{N^+}.
  In the first case, as $a_1\mathbf{R}_{\text{sup}}\ldots\mathbf{R}_{\text{sup}}a_n = a$ and $a \in S$, if there is a conflict between S and a_1 in AS^{N^+}, following Def. 19, there is also a conflict between S and a in AS^{N^+}. That is impossible since $a \in S$ and

S is conflict-free in AS^{N^+}.

In the second case, there exists a `n+-attack` from an argument d to a_1 such that there is no `n+-attack` from S to d. Following again Def. 19, we obtain a `n+-attack` from d to a. The fact that there is no `n+-attack` from S to d contradicts the fact that S is admissible in AS^{N^+}. Hence the initial hypothesis does not hold and we have proved that S is admissible in AS^N.

- It remains to prove that S is maximal admissible in AS^N. If it is not the case, there exists $S' \subseteq \mathbf{A}$ such that S' is admissible in AS^N and $S \subset S'$. Due to Prop. 8, S' is also admissible in AS^{N^+}, which contradicts the fact that S is maximal admissible in AS^{N^+}.

$\Leftarrow$) Assume that S is maximal admissible in AS^N. Due to Prop. 8, S is admissible in AS^{N^+}. If S is not maximal admissible in AS^{N^+}, there exists $S' \subseteq \mathbf{A}$ such that S' is maximal admissible in AS^{N^+} and $S \subset S'$. Due to the first item S' is maximal admissible in AS^N, which contradicts the fact that S is maximal admissible in AS^N.

□

Proof of Prop. 10:

$\Rightarrow$) Assume that S is stable in AS^{N^+}. Obviously, S is conflict-free in AS^N. Let $b \notin S$. There exists $a \in S$ such that a `n+-attacks` b. If we do not have a `n-attacks` b, we have the following sequences: $a_1\mathbf{R}_{\text{att}}b_1\mathbf{R}_{\text{sup}}\ldots\mathbf{R}_{\text{sup}}b_m = b$, and $a_1\mathbf{R}_{\text{sup}}\ldots\mathbf{R}_{\text{sup}}a_n = a$ with $m \geq 1$, $n > 1$.

If $a_1 \notin S$, as S is stable in AS^{N^+} there exists $c \in S$ such that c `n+-attacks` a_1. Following Def. 19, we obtain a `n+-attack` from c to a, which contradicts the fact that S is conflict-free in AS^{N^+}. So $a_1 \in S$ and S `n-attacks` b.

$\Leftarrow$) Assume that S is stable in AS^N. Then S is admissible in AS^N. Due to Corollary 2 S is conflict-free in AS^{N^+}. Let $b \notin S$, there exists $a \in S$ such that a `n-attacks` b and so a `n+-attacks` b. Hence S is stable in AS^{N^+}. □

Proof of Prop. 11: $a, b \in \mathbf{A}$.

- Let S be conflict-free in AS^{N^+}, $a \in S$ and $b\mathbf{R}_{\text{sup}}a$. Assume that $S \cup \{b\}$ is not conflict-free in AS^{N^+}. As S is conflict-free, there exists a `n+-attack` between b and an element c of S. Either c `n+-attacks` b or b `n+-attacks` c. Following Def. 19, we obtain a `n+-attack` from c to a in the first case and a `n+-attack` from a to c in the second case. That contradicts the fact that S is conflict-free in AS^{N^+}.
- Let S be maximal (wrt $\subseteq$) conflict-free in AS^{N^+}, $a \in S$ and $b\mathbf{R}_{\text{sup}}a$. Due to the first item, $S \cup \{b\}$ is conflict-free in AS^{N^+}. As S is maximal conflict-free, b must belong to S. Hence S is closed under $\mathbf{R}_{\text{sup}}^{-1}$.

□

Proof of Prop. 12:

- Let S be a preferred extension in AS^{N^+}. If S is not closed for the relation $\mathbf{R}_{\text{sup}}^{-1}$, there exists $a \in S$ and $b \notin S$ such that $b\mathbf{R}_{\text{sup}}a$. Due to Prop. 11, $S \cup \{b\}$ is conflict-free in AS^{N^+}. As S is maximal admissible, it follows that b is not acceptable wrt S in AS^{N^+}. Hence there exists c such that c `n+-attacks` b and S does not `n+-attack` c. Following Def. 19, we obtain a `n+-attack` from c to a. The fact that S does not `n+-attack` c contradicts the fact that S is admissible. Hence the initial hypothesis does not hold and S is closed for the relation $\mathbf{R}_{\text{sup}}^{-1}$.

  The same holds if S is a preferred extension in AS^N due to Prop. 9.
- Let S be a stable extension in AS^{N^+}. S is also a preferred extension of AS^{N^+}. Due to the first item, we conclude that S is closed for the relation $\mathbf{R}_{\text{sup}}^{-1}$.

□

Proof of Prop. 13:

- Let S be admissible in GBAS. S is coherent in GBAS and each element of S is acceptable wrt S in GBAS. Following Def.s 14, 15 and 16, as S is coherent in GBAS it is clear that S is conflict-free in AS^N.

  It remains to prove that each element of S is acceptable wrt S in AS^N. Let $b \in S$ and assume there exists a `n-attack` from an argument a to b. Either $a\mathbf{R}_{\text{att}}b$ or there is a sequence $a\mathbf{R}_{\text{att}}b_1\mathbf{R}_{\text{sup}}\ldots\mathbf{R}_{\text{sup}}b_m = b$. In the second case, as S is closed for the relation $\mathbf{R}_{\text{sup}}^{-1}$, $b_1 \in S$. As we have $a\mathbf{R}_{\text{att}}b_1$, $b_1 \in S$ and S admissible in GBAS, there exists $c \in S$ such that $c\mathbf{R}_{\text{att}}a$. The same holds in the first case. Hence we have proved that b is acceptable wrt S in AS^N.

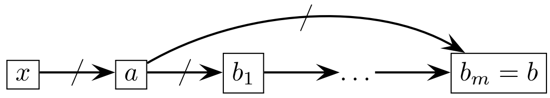
- As a consequence of the result proved by the first item, if S is a preferred extension in GBAS, then S is admissible in AS^N and then is included in a preferred extension of AS^N.
- Let S be a stable extension of GBAS. Following Prop. 3, S is a preferred extension of GBAS, so S is admissible in GBAS. Moreover, following the first item of the current proposition, S is admissible in AS^N and then S is conflict-free in AS^N. Consider $a \notin S$. Since S is stable in GBAS, there exists $x \in S$ such that $x\mathbf{R}_{\text{att}}a$. Since $\mathbf{R}_{\text{att}}$ is included in the `n-attacks`, $x\mathbf{R}_{\text{att}}a$ in AS^N and S is stable in AS^N.

□

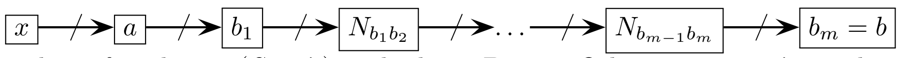
Proof of Prop. 14:

- Let $S \subseteq \mathbf{A} \cup \mathbf{A}_n$, S admissible in MAS. Due to Prop. 5, we know that if there is a `n-attack` between two arguments a and b of $\mathbf{A}$, no admissible set in MAS contains $\{a, b\}$. Hence $S \cap \mathbf{A}$ is conflict-free in AS^N.

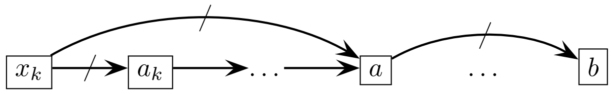
  Let $b \in S \cap \mathbf{A}$ such that there is a `n-attack` from an argument a to b in $\mathbf{A}$. Either there is a direct or a secondary attack from a to b. That is we have the sequence $a\mathbf{R}_{\text{att}}b_1\mathbf{R}_{\text{sup}}\ldots\mathbf{R}_{\text{sup}}b_m = b$, with $m \geq 1$. If $m > 1$, we obtain the following sequence in MAS: $a\mathbf{R}_{\text{att}}b_1\mathbf{R}_{\text{att}}N_{b_1b_2}\mathbf{R}_{\text{att}}b_2\ldots\mathbf{R}_{\text{att}}N_{b_{m-1}b_m}\mathbf{R}_{\text{att}}b_m = b$. As S is admissible in MAS and $b \in S$, we have $b_{m-1} \in S, \ldots, b_2 \in S, b_1 \in S$ and there exists $x \in S$ such that x attacks a in MAS. Note that if $m = 1$ the sequence in MAS is reduced to $a\mathbf{R}_{\text{att}}b$ which also implies that there exists $x \in S$ such that x attacks a in MAS. So we have in AS^N:



  and in MAS:



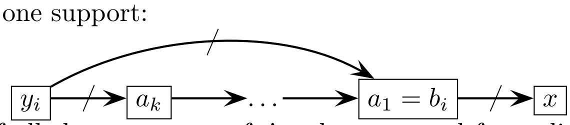
  If $x \in \mathbf{A}$ we have found $x \in (S \cap \mathbf{A})$ such that $x\mathbf{R}_{\text{att}}a$. Otherwise, $x \in \mathbf{A}_n$ and $x = N_{a_1a}$ with $a_1\mathbf{R}_{\text{att}}N_{a_1a}\mathbf{R}_{\text{att}}a$. That means that there is a support $a_1\mathbf{R}_{\text{sup}}a$ in BAS. As $x = N_{a_1a}$ belongs to S, a_1 is attacked in MAS. By iterating the process, we build a sequence $a_1, a_2, \ldots$. Let us consider the longest sequence that can be built. It has the form: $a_k\mathbf{R}_{\text{sup}}\ldots\mathbf{R}_{\text{sup}}a_1\mathbf{R}_{\text{sup}}a$ `n-attacks` b with a_k being attacked in MAS by $x_k \in S \cap \mathbf{A}$. So we have the sequence $x_k\mathbf{R}_{\text{att}}a_k\mathbf{R}_{\text{sup}}\ldots\mathbf{R}_{\text{sup}}a_1\mathbf{R}_{\text{sup}}a$ `n-attacks` b which proves that x_k `n-attacks` a:



  Hence there is `n-attack` from $S \cap \mathbf{A}$ to a, and we have proved that $S \cap \mathbf{A}$ is admissible in AS^N.
- Let $S \subseteq \mathbf{A}$, S preferred extension in AS^N. Obviously, S is conflict-free in MAS, since the attacks in MAS between elements of $\mathbf{A}$ are direct attacks coming from BAS. Assume that S is not admissible in MAS. Let $x \in S$ which is not acceptable wrt S in MAS. There exists b in MAS such that x is attacked by b and no argument of S attacks b in MAS.

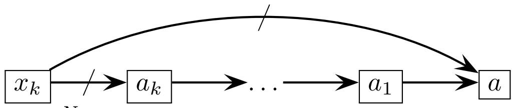
  If $b \in \mathbf{A}_n$, b has the form N_{yx} and there is a sequence $y\mathbf{R}_{\text{att}}b\mathbf{R}_{\text{att}}x$ in MAS, which corresponds to a support $y\mathbf{R}_{\text{sup}}x$ in BAS. Due to Prop. 12, S is closed under $\mathbf{R}_{\text{sup}}^{-1}$, so $y \in S$. However that contradicts the fact that no argument of S attacks b in MAS.

  Hence, if x is attacked by b such that no argument of S attacks b in MAS, $b \in A$. Let us consider all the arguments b_i of that form (*i.e.* such that x is attacked by b_i and no argument of S attacks b_i in MAS). Each b_i belongs to A. As S is admissible in AS^N, for each b_i there exists $y_i \in S$ and a `n-attack` from y_i to b_i. This `n-attack` cannot be a direct attack due to the assumption on b_i. It is a secondary attack involving at least one support:



  Let N_x denote the set of all the arguments of $\mathbf{A}_n$ that are used for coding all the secondary attacks from y_i to b_i for each b_i. It is easy to see that all the arguments of N_x are acceptable wrt S in MAS, and that $S \cup N_x$ is admissible in MAS. Hence, if we add to S all the N_x such that $x \in S$ is not acceptable wrt S in MAS, we still obtain an admissible set of MAS whose restriction to $\mathbf{A}$ is S. We call it the extension of S to MAS in the following.

- Let $S \subseteq \mathbf{A} \cup \mathbf{A}_n$, S stable in MAS. As S is stable in MAS, S is also admissible in MAS. From the first item, we know that $S \cap \mathbf{A}$ is conflict-free in AS^N. Let $a \in \mathbf{A}$ such that $a \notin S$. As S is stable in MAS, there exists $x \in S$ such that x attacks a in MAS. If $x \in S \cap \mathbf{A}$, the proof is done. Otherwise, $x \in \mathbf{A}_n$ and x has the form $N_{a_1 a}$ and there is a sequence $a_1 \mathbf{R}_{\text{att}} x \mathbf{R}_{\text{att}} a$ in MAS, which corresponds to a support $a_1 \mathbf{R}_{\text{sup}} a$ in BAS. Since S is admissible in MAS, there exists $x_1 \in S$ such that x_1 attacks a_1 in MAS. If $x_1 \in S \cap \mathbf{A}$ we obtain a `n-attack` from x_1 to a in BAS and the proof is done. Otherwise, x_1 is of the form $N_{a_2 a_1}$. By iterating the process, we build a sequence $a_1, a_2, \ldots$. Let us consider the longest sequence that can be built. It has the form: $a_k \mathbf{R}_{\text{sup}} \ldots \mathbf{R}_{\text{sup}} a_1 \mathbf{R}_{\text{sup}} a$ with a_k being attacked in MAS by $x_k \in S \cap \mathbf{A}$. So we have the sequence $x_k \mathbf{R}_{\text{att}} a_k \mathbf{R}_{\text{sup}} \ldots \mathbf{R}_{\text{sup}} a_1 \mathbf{R}_{\text{sup}} a$ which proves that x_k `n-attacks` a.



Hence $S \cap \mathbf{A}$ is also stable in AS^N.

- Let $S \subseteq \mathbf{A}$, S stable in AS^N. Obviously, S is conflict-free in MAS. Let $b \in A$ such that $b \notin S$ and there is no direct attack from S to b. As S is stable in AS^N, there exists $a_i \in S$ such that a_i `n-attacks` b. That is we have a sequence $a_i \mathbf{R}_{\text{att}} b_1 \mathbf{R}_{\text{sup}} \ldots \mathbf{R}_{\text{sup}} b_m = b$, with $m > 1$. So we have the following sequence in MAS: $a_i \mathbf{R}_{\text{att}} b_1 \mathbf{R}_{\text{att}} N_{b_1 b_2} \mathbf{R}_{\text{att}} b_2 \ldots \mathbf{R}_{\text{att}} N_{b_{m-1} b_m} \mathbf{R}_{\text{att}} b_m = b$.
Let N_b denote the set of all the arguments of $\mathbf{A}_n$ that are used for coding all the secondary attacks from a_i to b for each $a_i \in S$. Let us add to S all the N_b such that $b \in \mathbf{A} \setminus S$. We obtain a set denoted by S_1.
Then we add to S_1 all the arguments $b \in A_n$ such that there is no attack from S to b in MAS. We obtain a set denoted by S_2.
By construction, S_2 attacks each argument of $\mathbf{A} \cup \mathbf{A}_n \setminus S$ in MAS and there is no attack between $S_2 \cap A_n$ and S. Moreover, S is conflict-free and there is no attack between arguments of A_n. So S_2 is conflict-free in MAS. That proves that S_2 is stable in MAS.

□

Proof of Prop. 15: Let $\mathsf{BAS} = \langle \mathbf{A}, \mathbf{R}_{\text{att}}, \mathbf{R}_{\text{sup}} \rangle$ and its associated AS^N, GBAS, and MAS. Let $S \subseteq \mathbf{A}$.

- If S is a preferred extension of GBAS then, following Prop. 13, there exists S'' preferred in AS^N such that $S \subseteq S''$; and then, following Prop. 14, there exists S' preferred in MAS such that $S'' \subseteq S' \cap \mathbf{A}$. So $S \subseteq S' \cap \mathbf{A}$.
- If S is a stable extension of GBAS then, following Prop. 13, S is a stable extension of AS^N; and then, following Prop. 14, there exists S' stable in MAS such that $S = S' \cap \mathbf{A}$.

□