# Additive refinements of qualitative decision criteria

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A numerical approach is classically advocated (see e.g. [12]) for encoding both the information pertaining to the state of nature and the preferences on $X$: uncertainty is represented by a probability distribution $p$ and preference is encoded by a utility function $u : X \to [0, 1]$ [1]. The pair $< p, u >$ will be called a *probabilistic utility model*, PU-model for short. Acts are then ranked according to their expected utility $EU_{p,u}$ (written here in the finite setting):

$$f \succeq_{EU,p,u} g \Leftrightarrow EU_{p,u}(f) \geq EU_{p,u}(g)$$

where, $\forall h \in \mathcal{A}\ EU_{p,u}(h) = \sum_{s \in S} p(s) \cdot u(h(s))$.

Information about preference and uncertainty in decision problems cannot always be quantified in a simple way, but only qualitative evaluations can sometimes be attained. As a consequence, the topic of qualitative decision theory is a natural one to consider: can we make efficient decision on the basis of qualitative information?

Giving up the quantification of utility and uncertainty has lead to give up expected utility (EU) criterion as well— the principe of qualitative decision [11, 6, 2, 4, 3, 8, 5]. making is to model uncertainty by an *ordinal* plausibility relation on events and preference by a *weak order* on consequences of decisions. In [6] two qualitative criteria based on possibility theory, an optimistic

[1]Since expected utility is not sensitive to linear transformations of $u$, the choice of $[0, 1]$ as the range for $u$ is made for convenience.

and a pessimistic one, whose definitions only require a (finite) completely ordered scale for utility and uncertainty are proposed. Let $S$ be a set of states, $X$ a set of consequence and $X^S$ the set of possible acts (in decision under uncertainty, an act is a function $f : S \mapsto X$):

**Definition 1 (Possibilistic utilities)** *Let $L = [0_L, 1_L]$ be a finite ordinal scale, $n : L \rightarrow L$ the order reversing function of $L$, $\pi : S \rightarrow L$ a possibility distribution on $S$ and $\mu : X \rightarrow L$ a utility function on $X$.*

- *$< S, X, L, \pi, \mu >$ will be called a qualitative possibisitic utility model (QPU-model)*
- *the optimistic possibilistic utility of $f$ is:* $U_{OPT,\pi,\mu}(f) = \max_{s \in S} \min(\pi(s), \mu(f(s)))$
- *pessimisitic utility of $f$ is:* $U_{PES,\pi,\mu}(f) = \min_{s \in S} \max(n(\pi(s)), \mu(f(s)))$
- *$\succeq_{OPT,\pi,\mu}$ and $\succeq_{PES,\pi,\mu}$ are classically defined from $U_{OPT,\pi,\mu}$ and $U_{PES,\pi,\mu}$*

These criteria proved to be not efficient enough, in the sense that they fail to satisfy the principle of strict Pareto dominance: $\forall s, \mu(f(s)) \geq \mu(g(s))$ and $\exists s^*, \pi(s^*) > 0$ and $\mu(f(s^*)) > \mu(g(s^*))$ does not imply $f \succ_{OPT,\pi,\mu} g$ nor $f \succ_{PES,\pi,\mu} g$

This drawback is not observed within expected utility theory since the following *Sure-Thing Principle* (STP) [12] insures that identical consequences do not influence the relative preference between two events.

$$\text{STP:} \forall f, g, h, h', fAh \succeq gAh \Leftrightarrow fAh' \succeq gAh'$$

So, the question is whether it is possible or not to reconciliate possibilistic criteria and efficiency. The answer seems to be no: in [7] it is shown that the possibilistic criteria cannot obey the STP, except in a very particular case: when the actual state of the world is known, i.e. when there is no uncertainty at all! So, we cannot both stay in the pure QPU framework and satisfy the Pareto principle. The idea is then to try to cope with this problem by proposing *refinements* of the possibilistic criteria that obey the Sure Thing Principle. Formally:

**Definition 2 (Refinement)** *$\succeq'$ refines $\succeq$ iff $\forall f, g \in X^S$, $f \succ g \Rightarrow f \succ' g$.*

Since we are looking for weak orders it is natural to think of refinements based on expected utility. Concerning the optimistic utility criterion, we obtain the following result:

**Theorem 1** *Let $< S, X, L, \pi, \mu >$ be a possibilistic model based on a scale $L = (\alpha_0 = 0_L < \alpha_1 < \ldots < \alpha_k = 1_L)$ . The function $\chi : L \to [0, 1]$ defined by:*
$\chi(0_L) = 0,\ \chi(\alpha_i) = \frac{v}{N^{2^{k-i}}}, i = 1, \ldots, k$ *where* $v = (\sum_{i=1,\ldots,k} \frac{n_i}{N^{2^{k-i}}})^{-1}$

*is such that:*

- $\chi \circ \pi$ *is a probability distribution*
- $\succeq_{EU,\chi\circ\pi,\chi\circ\mu}$ *refines* $\succeq_{OPT,\pi,\mu}$
- $\chi \circ \pi$ *(resp.* $\chi \circ \mu$ *) and* $\pi$ *(resp.* $\mu$*) are ordinaly equivalent*

So for any $< S, X, L, \pi, \mu >$ we are able to propose an EU model that refines the former. This model is thus perfectly compatible with the optimistic qualitative utility and more decisive than it. Moreover, since it is based on expected utility it satisfies the Sure Thing Principle as well as Pareto dominance and does not use other information than the original one - it is unbiased. Moreover, it can be shown that, if we do not accept to introduce a bias in the EU-refinement, it is unique, up to an isomorphism.

When considering the pessimistic qualitative model, the same kind of result can be obtained. First of all, notice that $\succeq_{PES,\pi,\mu}$ and $\succeq_{OPT,\pi,\mu}$ are dual relations:

**Proposition 1** *Let $< S, X, L, \pi, \mu >$ be a QPU model. It holds that:*
$\forall f, g \in X^S, f \succeq_{PES,\pi,\mu} g \Leftrightarrow g \succeq_{OPT,\pi,\mu'} f$*, where* $\mu' = n \circ \mu$

This gives rise to the following definition of pessimistic EU-refinement:

**Theorem 2** *Let $< S, X, L, \pi, \mu >$ be a QPU model and $\chi : L \to [0, 1]$ be the transformation of $L$ w.r.t. $\pi$ identified Theorem 1. Let $p = \chi \circ \pi$ and $u' = \chi(1_L) - \chi \circ n \circ \mu$; it holds that:*

- $\succeq_{EU,p,u'}$ *is a refinement of* $\succeq_{PES,\pi,\mu}$
- $p$ *(resp.* $u'$ *) and* $\pi$ *(resp.* $\mu$*) are ordinaly equivalent*
- *any unbiased EU-refinement of* $\succeq_{PES,\pi,\mu}$ *is ordinally equivalent to* $\succeq_{EU,p,u'}$

So, if $< S, X, L, \pi, \mu >$ a QPU model, it is always possible to build a probabilistic transformation $\chi$ using Theorem 1, and thus a probability $p = \chi \circ \pi$ and two utility functions $u = \chi \circ \mu$ and $u' = \chi(1) - \chi \circ n \circ \mu$ that define the unbiased EU-refinements of the optimistic and pessimistic utility criteria respectively.

This proves an important result for bridging qualitative possibilistic decision theory and expected utility theory: we have shown than any optimistic or pessimistic QPU model can be refined by a EU model. So, (i) possibilistic decision criteria are compatible with the classical expected utility criterion and (ii) choosing a EU model is advantageous, since it leads to a EU-refinement of the original rule (thus, a more decisive criterion) and it allows to satisfy the STP and the principle of Pareto.

But this does not mean that qualitativeness and ordinality are given up. For instance, in both cases, the probability measures are "big-stepped probabilities", i.e. satisfy [2]:

$$\forall s \in S, P(\{s\}) > P(\{s', P(\{s'\}) < P(\{s\})\})$$

States are clustered in ordinal classes and any state of one class is more plausible that any event built on the lower classes.

Although probabilistic and based on additive manipulations of utilities, these new criteria remain ordinal (it is actually possible to show they generalize well known ordinal weighted means, namely the leximin and leximax procedures.) And this is very natural: since we come from an ordinal model and do not accept any bias, we go to another (probabilistic but) ordinal model, in which the numbers only encode orders of magnitude.

Let us relate the previous EU criteria to the ordinal comparison of vectors. When $S$ is finite, the comparison of acts can indeed be seen as a comparison of vectors of pairs of elements of $L$:

**Definition 3** *The representative vector of any act $f \in \mathcal{A}$ is the vector:*

$$\vec{f} = ((\pi_1, \mu_1), \ldots, (\pi_i, \mu_i), \ldots (\pi_N, \mu_N))$$

*where $\pi_i$ stands for $\pi(s_i)$ and $\mu_i$ for $\mu(f(s_i))$.*

Comparing acts thus amounts to comparing elements of $(L^2)^N$. For instance, $\succeq_{OPT,\pi,\mu}$ is a restriction to the case $M = 2$ of the general $\succeq_{Maxmin}$ relation on $(L^M)^N$.

Let us first consider the degenerate case of total ignorance, where $\forall s \in S, \pi(s) = 1_L$. In this case, the comparison of acts comes down to the comparison of utility degrees: $\vec{f} = ((1_L, \mu_1), \ldots, (1_L, \mu_N))$ becomes $\vec{f} = (\mu_1, \ldots, \mu_N)$. So, $f \succeq_{OPT,\pi,\mu} g$ iff $\vec{f} \succeq_{Max} \vec{g}$ and $f \succeq_{PES,\pi,\mu} g$ iff $\vec{f} \succeq_{Min} \vec{g}$. In decision making, the comparison of vectors by the max and min operators is well known, as it is known that it suffers from a lack of decisive power. That is why refinements of $\succeq_{min}$ and $\succeq_{max}$ have been proposed [10]:

[2]This notion of big-stepped probability generalizes the one of [13, 1], where each cluster is a singleton

**Definition 4 (Leximax, Leximin)** *Let $\vec{u}, \vec{v} \in L^N$. Then*

- $\vec{u} \succeq_{lmax} \vec{v} \Leftrightarrow (\forall j, u_{(j)} = v_{(j)}$ *or* $\exists i, \forall j < i, u_{(j)} = v_{(j)}$ *and* $u_{(i)} > v_{(i)})$
- $\vec{u} \succeq_{lmin} \vec{v} \Leftrightarrow (\forall j, u_{(j)} = v_{(j)}$ *or* $\exists i, \forall j > i, u_{(j)} = v_{(j)}$ *and* $u_{(i)} > v_{(i)})$

*where, for any $\vec{w} \in L^N$, $w_{(k)}$ is the k-th* biggest *element of $\vec{w}$ (i.e. $w_{(1)} \geq \ldots \geq w_{(N)}$).*

In practice, the leximin (resp. leximax) comparison consists in ordering both vectors in increasing (resp. decreasing) order and then lexicographically comparing them. It is obvious that $\succeq_{lmax}$ refines $\succeq_{max}$ and $\succeq_{lmin}$ refines $\succeq_{min}$, that both relations escape the drowning effect and are very efficient.

Since the leximax and leximin comparisons are good candidates in a particular case, we have imagined an extension of these procedures to the case of 2 dimensions ( $(L^M)^N$ instead of $L^N$). The only thing that we need is to use any complete preorder $\trianglerighteq$ on vectors of $L^M$ instead of the classical relation $\geq$ on $L$. It then possible to order the sub-vectors of any $\vec{w}$ according to $\trianglerighteq$ and to apply any of the two previous procedures ($\triangleright$ is the strict part of $\trianglerighteq$):

**Definition 5 (Leximax($\trianglerighteq$), Leximin($\trianglerighteq$))** *Let $\trianglerighteq$ be a complete preorder on $L^M$, $\cong$ the associated equivalence relation ($\vec{a} \cong \vec{b} \Leftrightarrow \vec{a} \trianglerighteq \vec{b}$ and $\vec{b} \trianglerighteq \vec{a}$). Let $\vec{u}, \vec{v} \in (L^M)^N$. Then*

- $\vec{u} \succeq_{lmax(\trianglerighteq)} \vec{v} \Leftrightarrow (\forall j, u_{(\trianglerighteq,j)} \cong v_{(\trianglerighteq,j)}$ *or* $\exists i$ *s.t.* $\forall j < i, u_{(\trianglerighteq,j)} \cong v_{(\trianglerighteq,j)}$ *and* $u_{(\trianglerighteq,i)} \triangleright v_{(\trianglerighteq,i)})$.
- $\vec{u} \succeq_{lmin(\trianglerighteq)} \vec{v} \Leftrightarrow (\forall j, u_{(\trianglerighteq,j)} \cong v_{(\trianglerighteq,j)}$ *or* $\exists i$ *s.t.* $\forall j > i, u_{(\trianglerighteq,j)} \cong v_{(\trianglerighteq,j)}$ *and* $u_{(\trianglerighteq,i)} \triangleright v_{(\trianglerighteq,i)})$.

*where, for any $\vec{w} \in (L^M)^N$, $w_{(\trianglerighteq,i)}$ the $i^{th}$ biggest sub-vector of $\vec{w}$ according to $\trianglerighteq$.*

The leximax procedure can in particular be applied to the preorder $\trianglerighteq = \succeq_{lmin}$. In practice, this comparison consists in first ordering the elements of each sub-vector in increasing order w.r.t $\succeq$, then in ordering the sub-vectors in decreasing order (w.r.t. $\succeq_{lmin}$). It is then enough to lexicographically compare the two new vectors of vectors.

$\succeq_{lmax(\succeq lmin)}$ is the refinement of $\succeq_{maxmin}$ we desire. Let us now compare representative vectors of acts using this relation (letting $M = 2$) – we obtain a refinement of $\succeq_{OPT,\pi,\mu}$:

**Proposition 2** *The following relation $\succeq_{lmax(\succeq lmin)}$ on $\mathcal{A}$ refines $\succeq_{OPT,\pi,\mu}$ :*

$$f \succeq_{lmax(\succeq lmin)} g \Leftrightarrow \vec{f} \succeq_{lmax(\succeq lmin)} \vec{g}$$

The same kind of reasoning can be followed to refine $\succeq_{PES,\pi,\mu}$. The pessimistic utility of act $f$ is $\min_{s\in S}\max(n(\pi(s)),\mu(f(s)))$. We need to refine a minimax procedure, and this can be done using $\succeq_{lmin(\succeq lmax)}$. Since operator *max* does not apply to $(\pi(s),\mu(f(s)))$ but to $(n(\pi(s)),\mu(f(s)))$, we use the $\pi$-reverse vectors of acts:

**Definition 6** *The $\pi$-reverse vector of an act $f \in \mathcal{A}$ is:*

$$n(\vec{f}) = ((n(\pi(s_1)),\mu(f(s_1)),\dots(n(\pi(s_N)),\mu(f(s_N))).$$

**Proposition 3** *The following relation $\succeq_{lmin(\succeq lmax,n)}$ refines $\succeq_{PES,\pi,\mu}$ :*

$$f \succeq_{lmin(\succeq lmax,n)} g \Leftrightarrow n(\vec{f}) \succeq_{lmin(\succeq lmax)} n(\vec{g})$$

Refining $\succeq_{PES,\pi,\mu}$ leads to the application of the leximin(leximax) comparison to the $\pi$-reverse vectors, while refining $\succeq_{OPT,\pi,\mu}$ applies the leximax(leximin) comparison directly to the representative vectors. Both procedures *are purely ordinal*: the degrees in $L$ are only compared using min, max and reverse operators — only their relative orders matter. Our final result is that these refinements are equivalent to the EU-refinements identified previously.

**Theorem 3** *Let $< S, X, L, \pi, \mu >$ be a QPU model and $\chi$ a unbiased probabilistic transformation of $L$ w.r.t. $\pi$, $p = \chi\circ\pi$, $u = \chi\circ\mu$, $u' = \chi(1_L)-\chi\circ n\circ\mu$:*

*i)* $\succeq_{EU,p,u}$ *refines* $\succeq_{OPT,\pi,\mu} \Leftrightarrow \succeq_{EU,p,u} \equiv \succeq_{lmax(\succeq lmin)}$.

*ii)* $\succeq_{EU,p,u'}$ *refines* $\succeq_{PES,\pi,\mu} \Leftrightarrow \succeq_{EU,p,u'} \equiv \succeq_{lmin(\succeq lmax,n)}$.

So, the probabilistic refinements of possibilistic utilities are equivalent to purely comparative procedures, this is why we can say that efficient QPU refinements are probabilistic but remain qualitative. Reciprocally, we can prove that the $\succeq_{lmax(\succeq lmin)}$ and $\succeq_{lmin(\succeq lmax,n)}$ preference relations over vectors of vectors always admit a representation by a sum (on N) of products (on M), provided that $L$ is discrete – but this is beyond of the scope of this paper, that focuses on decision under uncertainty (where M=2).

The result of the present research can be viewed in a more general perspective: the optimistic and pessimistic utilities are not limited to decision under uncertainty and can be view as general maximin and minimax procedures (used for instance in multi criteria decision making, voting theory, etc): we have shown that they can be refined by a classical weighted sum, when the strict Pareto principle is required. This raises a new question: can we extend this principle to any other instance of Sugeno integral ?

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