Egalitarian Collective Decision Making under Qualitative Possibilistic Uncertainty: Principles and Characterization

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Abstract

This paper raises the question of collective decision making under possibilistic uncertainty; We study four egalitarian decision rules and show that in the context of a possibilistic representation of uncertainty, the use of an egalitarian collective utility function allows to get rid of the Timing Effect. Making a step further, we prove that if both the agents' preferences and the collective ranking of the decisions satisfy Dubois and Prade's axioms (1995), and particularly risk aversion, and Pareto Unanimity, then the egalitarian collective aggregation is compulsory. This result can be seen as an ordinal counterpart of Harsanyi's theorem (1955).

Keywords: Decision under Uncertainty, Possibility Theory, Collective Choice, Egalitarianism, Timing Effect.

Introduction

The handling of collective decision problem under uncertainty resorts on (i) the identification of a theory of decision making under uncertainty (DMU) that captures the decision makers' behaviour with respect to uncertainty and (ii) the specification of a collective utility function (CUF) as it may be used when the problem is not pervaded with uncertainty. One also needs to precise when the utility of the agents is to be evaluated: before (*ex-ante*) or after (*ex-post*) the realisation of the uncertain events. In the first case, the global utility function is a function of the DMU utilities of the different agents; in the second case it is an aggregation, w.r.t. the likelihood of the final states, of the collective utilities of the states. For instance, in the probabilistic framework, the comparison of decisions is based on the expected utility model axiomatized by von Neumann and Morgenstern (1944). When several agents are involved, utilitarianism prescribes the maximization of the sum of the expected utilities (*ex-ante*) or the expected utility of the sum of the agents' utilities (*ex-post*) - both actually coincide. An egalitarian approach can be based either on the min of the expected utilities (*ex-ante*), or on the expected utility of the least satisfied of the agents (*ex-post*) - but the two approaches can lead to divergent rankings. This phenomenon has been called the "Timing Effect" by Myerson (1981) in the early eighties.

Following Fleming (1952), Harsanyi (1955) showed that if the collective preference satisfies von Neumann and Morgenstern's axioms (i.e. when the *ex-ante* preference is an expected utility) and the preference relations of the agents also satisfy these axioms (i.e. when the *ex-post* preferences also follow the EU model) then the sole possible collective decision making approach (satisfying Pareto unanimity) is the utilitarian one. Making a step further Myerson (1981) proved that only the choice of an utilitarian social welfare function can reconciliate the *ex-ante* and *ex-post* approaches.

These results rely on the assumption that the knowledge of the agents about the consequences of their decisions is rich enough to be modelled by probabilistic lotteries. When the information about uncertainty cannot be quantified in a probabilistic way the topic of possibilistic decision theory is often a natural one to consider (Dubois and Prade 1995; Dubois et al. 1998; Giang and Shenoy 2000; Dubois, Prade, and Sabbadin 2001; Dubois et al. 2002; Dubois, Fargier, and Perny 2003). The present paper raises the question of collective decision making under possibilistic uncertainty. The next Section recalls the basic notions on which our work relies (decision under possibilistic uncertainty, collective utility functions, etc.). We then present four egalitarian possibilistic utilities and show that if both the collective preference and the individual preferences do satisfy Dubois and Prade's axioms (1995), and in particular risk aversion, then an egalitarian CUF is mandatory. This theorem can be considered as an ordinal counterpart to Harsanyi's theorem. For space reasons, proofs are omitted; they can be found online at url ftp://ftp.irit.fr/IRIT/ADRIA/PapersFargier/aaai15.pdf.

Background

Collective utility functions

Let us consider a multi-agent decision problem defined by a set $\mathcal{A} = \{1, \ldots, p\}$ of agents, each agent $i \in \mathcal{A}$ being supposed to express her preferences on a set of alternatives (say, a set X), by a ranking function or a utility function u_i that associates to each element of X a value in a subset of $\mathbb{R}^+$ (typically in the interval $[0, 1]$). The problem is then to determine, for each $x \in X$, a collective utility degree that reflects the collective preference.

When this collective preference depends only on the individual utilities of the agents, the collective utility can be obtained by a collective utility function (CUF; for more details about collective utility functions see (Moulin 1988)) of the form $u(x) = f(u_1(x), \ldots, u_p(x))$. Classical utility theory prescribes that the best decisions are those that maximize the sum of the individual utilities, i.e.:

$$u(x) = \sum_{i \in \mathcal{A}} u_i(x)$$

This function possesses several good properties but fails to ensure equity between agents. The egalitarian approach on the contrary proposes to maximize the satisfaction of the least satisfied agent, i.e. the CUF:

$$u(x) = \min_{i \in \mathcal{A}} u_i(x)$$

When the agents are not equally important (e.g. in an administration board, or when the aim is more to aggregate criteria than to satisfy a group), a weight w_i can be associated to each i; this yields the use of a weighted sum in the utilitarian case:

$$u(x) = \sum_{i \in \mathcal{A}} w_i \cdot u_i(x)$$

or of a weighted minimum in the egalitarian case:

$$u(x) = \min_{i \in \mathcal{A}} \max((1 - w_i), u_i(x))$$

Multi-agent decision making under risk

In presence of risk, i.e. when the information about the consequences of decisions is probabilistic, a popular criterion to compare decisions is the expected utility model axiomatized by von Neumann and Morgenstern (1944): an elementary decision is modelled by a probabilistic lottery over the set X of its possible outcomes. The preferences of a single decision maker are supposed to be captured by a utility function assigning a numerical value to each outcome. The evaluation of a lottery is performed through the computation of its expected utility (the greater, the better)[1]. When several agents are involved, two approaches are possible, depending on when the utility of the agents is to be evaluated: after or before the realization of the uncertain events. The *ex-post* approach comes down to a problem of mono-agent decision making under uncertainty (this agent being "the collectivity") by defining the utility function u as a CUF. On the contrary, the *ex-ante* approach combines the DMU utilities of the different agents with the collective utility function.

In the probabilistic context, utilitarianism comes down to calculate either the expected collective utility (*ex-post*), or the aggregation of the individual expected utilities (*ex-ante*). Egalitarianism prescribes to maximize either the expectation of the minimum of the satisfaction degrees or the minimum of the mathematical expectations. The two approaches do not always coincide: this is the so called *Timing Effect*.

Counter-example 1. *Consider two agents* 1 *and* 2*, two consequences* x_1 *and* x_2 *and the probabilistic lotteries* L_1 *and* L_2 *given in Figure 1. The expected value of the minimum of the utilities are:*
$$0.7 * \min(0.3, 0.5) + 0.3 * \min(1, 0.4) = 0.33 \quad for\ L_1$$
$$0.2 * \min(0.3, 0.5) + 0.8 * \min(1, 0.4) = 0.38 \quad for\ L_2$$
So, ex-post, $L_2 \succ L_1$*. On the contrary, ex-ante, computing the minimum of the expected utilities leads to* $L_1 \succ L_2$*. Indeed:*
$$\min(0.7 * 0.3 + 0.3 * 1, 0.7 * 0.5 + 0.3 * 0.4) = 0.47 \quad for\ L_1$$
$$\min(0.2 * 0.3 + 0.8 * 1, 0.2 * 0.5 + 0.8 * 0.4) = 0.42 \quad for\ L_2.$$

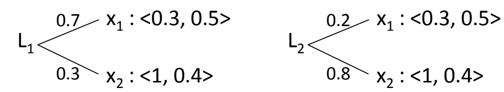

Figure 1: Two probabilistic lotteries in a bi-agent context. The probability of x_i according to a lottery labels the corresponding edge; each x_i is labelled by its vector of utilities.

In 1955, Harsanyi provided a theorem that is often interpreted as a justification of utilitarianism; he showed that if (i) the collective preference satisfies von Neumann and Morgenstern's axioms (1944), (ii) the preferences of each agent also satisfy these axioms, and (iii) if two lotteries are indifferent for each agent they are considered as collectively indifferent (Pareto indifference axiom), then the only appropriate collective CUF is the classical utilitarian one. Myerson (1981) proved that, in the probabilistic context, only the use of an affine collective aggregation function overcomes the Timing Effect, and conversely, that any attempt to introduce equity causes a divergence between the *ex-post* and *ex-ante* approaches.

(Mono agent) Decision Making under Possibilistic Uncertainty

Harsanyi's and Myerson's results are strongly related to the assumption of a probabilistic uncertainty and are valid only in such a rich and sophisticated context. When the information about uncertainty cannot be quantified in a probabilistic way, the topic of possibilistic decision theory is often a natural one to consider.

The basic building block in possibility theory is the notion of *possibility distribution*. Let S be a variable whose value is ill-known and Ω its domain. The knowledge about the value of S is encoded by a possibility distribution $\pi : \Omega \to [0, 1]$; given $\omega \in \Omega$, $\pi(\omega) = 1$ means that realization of ω is totally possible and $\pi(\omega) = 0$ means that ω is impossible. It is assumed that π is *normalized*, i.e. that there exist at least one ω which is totally possible.

From π, one can compute the possibility $\Pi(A)$ and the necessity $N(A)$ of an event $A \subseteq \Omega$: $\Pi(A) = \sup_{\omega \in A} \pi(\omega)$ evaluates to which extent A is *consistent* with the knowledge represented by π, while $N(A) = 1 - \Pi(\bar{A}) = 1 - \sup_{\omega \notin A} \pi(\omega)$ corresponds to the extent to which $\bar{A}$ is impossible and thus evaluates at which level A is certain.

[1]Notice that this kind of modelling is orthogonal to the one more recently investigated in Computational Social Choice (Konczak and Lang 2005; Mattei 2011; Lang et al. 2012); in the lottery framework, the uncertainty pertains to the consequences of the decisions and the agents have well established preferences on these consequences; in the latter works, the preferences of the agents are ill-known and can be represented or elicited in sophisticated ways. Besides, the purpose of these works is generally more related to complexity theory (determine the complexity of sophisticated decision rules or the one of manipulating the issue of the vote, etc.).

Giving up the probabilistic quantification of uncertainty yielded to give up the EU criterion as well. The development of possibilistic decision theory has lead to the proposition and the characterization of a series of possibilistic counterparts of the EU criterion (Dubois and Prade 1995; Dubois et al. 1998; Giang and Shenoy 2000; Dubois, Prade, and Sabbadin 2001; Dubois, Fargier, and Perny 2003). Following (Dubois and Prade 1995) a one stage decision is modelled by a (simple) possibilistic lottery, i.e. a normalized possibility distribution over a finite set of outcomes X. In a finite setting, a possibilistic lottery L can be written $L = \langle \lambda_1/x_1, \ldots, \lambda_n/x_n \rangle$ where $\lambda_j = \pi_L(x_j)$ is the possibility of getting outcome x_j when choosing decision L; this degree are also denoted by $L[x_j]$.

A *compound* possibilistic lottery is a normalized possibility distribution over a set of (simple or compound) lotteries. We shall denote such a lottery $L = \langle \lambda_1/L_1, \ldots, \lambda_m/L_m \rangle$, λ_i being the possibility of getting lottery L_i according to L.

The possibility $\pi_{i,j}$ of getting consequence x_j from L_i depends on the possibility λ_i of getting L_i and on the possibility λ_i^j of getting x_j from L_i (for the sake of simplicity, we assume that the L_i's are simple lotteries; the principle extends to the general case); in other words, $\pi_{i,j} = \min(\lambda_i, \lambda_i^j)$. The possibility of getting x_j from $L = \langle \lambda_1/L_1, \ldots, \lambda_m/L_m \rangle$ is simply the $\max$, over all the L_i's, of the $\pi_{i,j}$. In decision theory, a compound lottery is generally assumed to be indifferent (according to the DM's preference) to the simple lottery defined by:

$$Reduction(L) = \langle \max_{i=1,m} \min(\lambda_i^1, \lambda_i)/x_1, \ldots, \max_{i=1,m} \min(\lambda_i^n, \lambda_i)/x_n \rangle$$

Example 1. *The following figure provides an example of a possibilistic compound lottery with its reduction:*

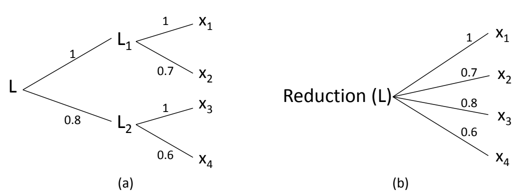

Figure 2: A possibilistic compound lottery and its reduction.

(Dubois and Prade 1995) then proposed two global utilities for evaluating any simple lottery:

$$U^-(L) = \min_{x \in X} \max(n(L[x]), u(x))$$

$$U^+(L) = \max_{x \in X} \min(L[x], u(x))$$

where n is an order reversing function (e.g. $n(x) = (1 - x)$). Pessimistic utility $U^-(L)$ estimates to what extent it is certain (i.e. necessary according to a measure N) that L is good. Its optimistic counterpart, $U^+(L)$ estimates to what extent it is possible that L is good. They extend to any kind of possibilistic lottery, by considering that the utility of a compound lottery is simply the one of its reduction.

To the best of our knowledge, the question of multi-agent decision making under possibilistic uncertainty has never been studied. Beyond the proposition of egalitarian utilities that suits possibilistic knowledge, we show in the present paper that they do not necessarily suffer from the Timing Effect, and we provide a representation theorem that can be viewed as an ordinal counterpart of Harsanyi's one.

Egalitarian collective decision making under possibilistic uncertainty

In the more qualitative, ordinal, case of possibilistic lotteries, four egalitarian utilities can be proposed (Ben Amor, Essghaier, and Fargier 2014) - two pessimistic utilities and two optimistic ones[2]:

2in

Definition 1.

$$\begin{aligned}
U^{-\min}_{ante}(L) &= \min_{i=1,p} \max(1 - w_i, \min_{x \in X} \max(u_i(x), 1 - L[x])) \\
U^{-\min}_{post}(L) &= \min_{x \in X} \max(1 - L[x], \min_{i=1,p} \max(1 - w_i, u_i(x))) \\
U^{+\min}_{ante}(L) &= \min_{i=1,p} \max(1 - w_i, \max_{x \in X} \min(u_i(x), L[x])) \\
U^{+\min}_{post}(L) &= \max_{x \in X} \min(L[x], \min_{i=1,p} \max(1 - w_i, u_i(x)))
\end{aligned}$$

Example 2. *As a matter of fact, consider two agents, the first agent being less important than the second one ($w_1 = 0.6$, $w_2 = 1$), and the simple lotteries L_1, L_2 on $X = \{x_1, x_2, x_3\}$ depicted[3] in Figure 3. We have:*

$$\begin{aligned}
U^{-\min}_{ante}(L_1) &= \min(&& \max(1-0.6, \min\max(1-1, 0.8), \max(1-0.9, 0.1)), \\
& && \max(1-1, \min\max(1-1, 0.1), \max(1-0.9, 0.8))) \\
&= 0.1 \\
U^{-\min}_{post}(L_1) &= \min(&& \max(1-1, \min(\max(1-0.6, 0.8), \max(1-1, 0.1))), \\
& && \max(1-0.9, \min(\max(1-0.6, 0.1), \max(1-1, 0.8)))) \\
&= 0.1 \\
U^{+\min}_{ante}(L_1) &= \min(&& \max(1-0.6, \max(\min(1, 0.8), \min(0.9, 0.1))), \\
& && \max(1-1, \max(\min(1, 0.1), \min(0.9, 0.8)))) \\
&= 0.8 \\
U^{+\min}_{post}(L_1) &= \max(&& \min(1, \min(\max(1-0.6, 0.8), \max(1-1, 0.1))), \\
& && \min(0.9, \min(\max(1-0.6, 0.1), \max(1-1, 0.8)))) \\
&= 0.4 \\
U^{-\min}_{ante}(L_2) &= U^{-\min}_{post}(L_2) = U^{+\min}_{ante}(L_2) = U^{+\min}_{post}(L_2) = 0.8.
\end{aligned}$$

Figure 3: Two bi-agent possibilisitic lotteries.

It is obviously possible to define in the same way a series of utilitarian possibilistic utilities (U^{-sum}_{ante}, U^{+sum}_{ante}, etc.) and a series of max-oriented ones ($U^{-\max}_{post}$, $U^{+\max}_{post}$, etc.). The present paper prefers to focus in details on the egalitarian ones, that look more coherent, and also more appealing from an ethical point of view. First of all, it appears that the coincidence between *ex-post* and *ex-ante* approaches does not imply utilitarianism. It is indeed easy to show that:

Proposition 1. $U^{-\min}_{ante}(L) = U^{-\min}_{post}(L)$.

We shall thus simply use the notation $U^{-\min}$. Such a coincidence does not happen in the "optimistic" case; with

[2] In our notation system the first exponent indicates the attitude with respect to uncertainty: optimistic (+) or pessimistic(-); the second one indicates the type of CUF used (min, max, sum, etc.).

[3] The consequences with a possibility degree of 0 are not represented in the drawings.

$U^{+\min}_{post}$ a lottery is good as soon as there exists a possible outcome satisfying all the agents; with $U^{+\min}_{ante}$, a lottery is good when each agent forecasts an outcome that is good for her (but it is not necessarily the same one for all): it may happen that $U^{+\min}_{post}(L) < U^{+\min}_{ante}(L)$, as shown by Counter-example 2. It holds that:

Proposition 2. $U^{+\min}_{post}(L) \leq U^{+\min}_{ante}(L)$.

Axioms for collective qualitative decision making

Let us now propose an axiomatization of the pessimistic egalitarian utility (we denote it $U^{-\min} = U^{-\min}_{post} = U^{-\min}_{ante}$). Consider a set $\mathcal{A}$ of p agents, a finite set of consequences X, a possibilistic scale V, the set of possibilistic lotteries $\mathcal{L}$ obtained from V and X. The preference profile $\langle \succeq_1, \ldots, \succeq_p \rangle$ gathers the preference relations $\succeq_i$ of each agent i on $\mathcal{L}$. $\succeq$ denotes the collective preference on $\mathcal{L}$.

Let us denote x the "constant" lottery leading to consequence x for sure (s.t. $L[y] = 0$ for each $y \neq x$; e.g. L_2 in Figure 3): constant lotteries and elements of X are identified. In the same way, let Y be the lottery that represents a subset Y of X (it provides the possibility degree 1 to each $y \in Y$, and 0 otherwise). First of all, we formulate a continuity axiom on the consequences:

Axiom C (Continuity on X): $\forall x, y \in X, \forall B \subseteq \mathcal{A}, \exists z \in X$ such that: $z \sim_i x$ if $i \in B$ and $z \sim_j y$ if $i \notin B$.

This axiom requires that there exists a z in X that is indifferent to x for the agents in B and indifferent to y for the others. When two agents are involved, Axiom **C** says that if x and y are two elements of X, then X contains a z corresponding to the vector of satisfaction $\langle x_i, y_j \rangle$. More generally, this axiom requires the set of lotteries to be rich enough to contain all the constant acts corresponding to all the vectors of satisfaction (in a sense, **C** deals more with $\mathcal{L}$ than with $\succeq$). This implies in particular that X contains a consequence x^* that is ideal for all the agents, and a consequence x_* anti-ideal for all the agents. When the set of consequences X is too small, it is harmless to extend and enrich it in order to obtain all the z that we need: in the following, Axiom **C** is supposed by construction (in Harsanyi's paper it is implicit: X is identified with the set of utility vectors).

We now introduce the axiom of Pareto unanimity, that is essential for collective choice:

Axiom P (Pareto Unanimity): If $\forall i \in \mathcal{A}$, $L \succeq_i L'$, then $L \succeq L'$.

Because we are rather interested in a cautious way of decision making than in an adventurous one, the next axioms are those proposed by (Dubois and Prade 1995) (their interpretation is detailed in the literature, hence we refrain to comment them further) . We write them below for any relation $\trianglerighteq$; in the following, these axioms will apply to $\succeq$ and to the $\succeq_i$'s:

Axiom 1: $\trianglerighteq$ on $\mathcal{L}$ is an equivalence relation (i.e. is complete and transitive).

Axiom 2 (Certainty equivalence): $\forall Y \subseteq X$, $\exists x \in Y$ s.t. x and Y are equivalent for $\trianglerighteq$.

Axiom 3 (Risk aversion): If $\forall x \in X, L[x] \leq L'[x]$ (L is more specific than L'), then $L \trianglerighteq L'$.

Axiom 4 (Weak independence): If L and L' are equivalents, then $\langle \lambda/L, \mu/L'' \rangle$ and $\langle \lambda/L', \mu/L'' \rangle$ are also equivalents, for any λ, μ s.t. $\max(\lambda, \mu) = 1$.

Axiom 5 (Lottery reduction): For any (compound lottery) $L, L \sim Reduction(L)$.

Axiom 6 (Continuity of $\mathcal{L}$): If $\forall x \in X$, $L'[x] \leq L[x]$ then $\exists \lambda$ s.t. $L' \sim \langle 1/L, \lambda/X \rangle$.

Dubois and Prade (1995) show that the preference relation on $\mathcal{L}$ defined by U^- do satisfy Axioms 1-6 and that, reciprocally, if Axioms 1-6 are satisfied by some relation $\trianglerighteq$, then the simple lotteries are ranked as if they where be evaluated by their pessimistic utilities. Technically, the satisfaction of these axioms allows the definition of an ordered scale U, an utility function $u : X \mapsto U$, an order reversing function $n : V \mapsto U$ such that $L \trianglerighteq L'$ iff $U^-(L) \geq U^-(L')$. This axiomatization is a qualitative counterpart to von Neumann and Morgenstern's characterization of expected utility.

Properties of possibilistic collective utility functions

We now study the four decision rules in light of the axioms and show that they are consistent, namely obeyed by the pessimistic egalitarian collective utility ($U^{-\min}$). Consider: a set $\mathcal{L}$ of possibilistic lotteries built from a set X and a scale V; a set u_i, $i \in \mathcal{A}$ of utility functions on X taking their values in $[0, 1]$; and a weight vector $\vec{w} \in [0, 1]^p$ (where w_i is the weight of agent i). It holds that:

Proposition 3. *The relations $\succeq$ and $\succeq_i$ defined by:*

$L \succeq L'$ *iff* $U^{-\min}(L) \geq U^{-\min}(L')$,
$L \succeq_i L'$ *iff* $U^-_i(L) \geq U^-_i(L')$

satisfy Axioms 1-6, as well as the Pareto unanimity axiom.

The satisfaction of Axioms 1-6 by the $\succeq_i$'s is obvious (U^-_i is by definition a pessimistic utility). Their satisfaction by $\succeq$ is also straightforward: $U^{-\min}$ is clearly a pessimistic DMU utility based on the utility function $u(x) = \min_{i=1,p} \max(u_i(x), (1 - w_i))$.

The satisfaction of Pareto Unanimity is also easy to prove. Suppose that $L \succeq_i L'$, for all i. By definition, $L \succeq_i L'$ iff $U^-_i(L) \geq U^-_i(L')$; $L \succeq_i L'$ for each i implies that the aggregation by the weighted minimum of the U^-_i's for L is greater or equal to the one given to L' (this aggregation is non decreasing); then $U^{-\min}(L) \geq U^{-\min}(L')$. In other words, $L \succeq L'$. Axiom **P** is thus satisfied.

Generally, we believe that all the *ex-ante* possibilistic aggregations, and in particular $U^{+\min}_{ante}$, satisfy **P**, be they egaliriarian or not (e.g. so do U^{-sum}_{ante}, $U^{-\max}_{ante}$, etc.) - simply because the CUFs are non decreasing. In our egalitarian context:

Proposition 4. *The relations $\succeq$ and $\succeq_i$ defined by:*

$L \succeq L'$ *iff* $U^{+\min}_{ante}(L) \geq U^{+\min}_{ante}(L')$,
$L \succeq_i L'$ *iff* $U^+_i(L) \geq U^+_i(L')$

satisfy Pareto Unanimity.

The problem is that $U^{+\min}_{ante}$ may fail to satisisfy weak independence, as shown by the following counter-example:

Counter-example 2. *Consider two equally important agents i.e. ($w_1 = w_2 = 1$), and the three lotteries depicted in Figure 4.*

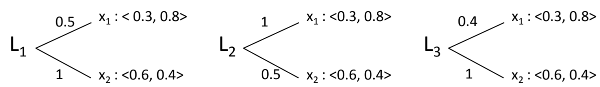

Figure 4: A counter-example to weak independance.

Let L and L' be the lotteries defined by:
$L = \langle 1/L_1, 0.9/L_3\rangle$, $L' = \langle 1/L_2, 0.9/L_3\rangle$.
$U^{+\min}_{ante}(L_1) = U^{+\min}_{ante}(L_2) = 0.5$*: L_1 and L_2 are indifferent*

$$
\begin{aligned}
U^{+min}_{ante}(L) &= U^{+min}_{ante}(Reduction(L)) \\
&= U^{+min}_{ante}(\langle 0.5/x_1, 1/x_2\rangle) = 0.5 \\
U^{+m\min}_{ante}(L') &= U^{+min}_{ante}(Reduction(L')) \\
&= U^{+min}_{ante}(\langle 1/x_1, 0.9/x_2\rangle) = 0.6.
\end{aligned}
$$

Then, $U^{+\min}_{ante}(L') > U^{+\min}_{ante}(L)$, which contradicts the axiom of weak independence.

Concerning $U^{+\min}_{post}$, news are very bad, since it even fails to satisfy Pareto Unanimity:

Counter-example 3. *Consider the two lotteries of Figure 3 on $X = \{x_1, x_2, x_3\}$ and suppose now that the two agents are equally important i.e. ($w_1 = w_2 = 1$). We get for agent 1:*

$$
\begin{aligned}
U^+(L_1) &= \max(\min(1, 0.8), \min(0.9, 0.1)) = 0.8 \\
U^+(L_2) &= \min(1, 0.8) = 0.8
\end{aligned}
$$

and for agent 2:

$$
\begin{aligned}
U^+(L_1) &= \max(\min(1, 0.1), \min(0.9, 0.8)) = 0.8 \\
U^+(L_2) &= \min(1, 0.8) = 0.8
\end{aligned}
$$

while:

$$
\begin{aligned}
U^{+\min}_{post}(L_1) &= \max(\min(1, \min(0.8, 0.1), \min(0.9, \min(0.1, 0.8)))) = 0.1 \\
U^{+\min}_{post}(L_2) &= \max(\min(1, \min(0.8, 0.8))) = 0.8
\end{aligned}
$$

Hence $L_1 \sim_1 L_2$, $L_1 \sim_2 L_2$ but $L_2 \succ L_1$, which contradicts Pareto Unanimity.

A representation theorem for $U^{-\min}$

Let us now show that the relation that satisfies both Continuity, Pareto Unanimity and the axioms of pessimistic utility can be captured by $U^{-\min}$ - thus providing a counterpart of Harsanyi's theorem that allows (weighted) equity. For the sake of brevity (and also because they look less interesting: they suffer from the Timing Effect and can violates important axioms), we let $U^{+\min}_{ante}$ and $U^{+\min}_{post}$ for further research. We first sketch the proof in the general case, allowing more or less important agents: this provides a characterization of $U^{-\min}$ in its full generality. We impose equity between agents in a second step - and rule out the weights.

We have seen in the previous Section that the set of axioms is consistent since satisfied by $U^{-\min}$. Let us now go the reverse way. Consider a relation $\succeq$ on $\mathcal{L}$ (built on X and V) that satisfies Axioms 1 to 6, a set of relations $\succeq_i$ on the same $\mathcal{L}$ that also satisfy these axioms, and suppose that the axioms of Pareto unanimity and continuity of X hold.

First of all, since Axioms 1-6 are satisfied by $\succeq$ and $\succeq_i$, these relations can be represented by pessimistic utilities - this is Dubois and Prade's theorem of representation.

Let us now consider for any agent $i \in \mathcal{A}$, the set $\top_i = \{x \in X : \forall y, x \succeq_i y\}$ of the best consequences according to i (this set cannot be empty because the $\succeq_i$'s are preorders). Thanks to Axiom **C**, there exists a consequence x^* that belongs to all the $\top_i$'s. By Pareto unanimity, $x^* \succeq y, \forall y \in X$. In the same way, there exists a x_* such that $y \succeq x_*, \forall y \in X$.

Thanks to Axiom **C**, we can define the constant act x^i for any agent i:

Definition 2. *For any $x \in X$ and any agent i, let x^i be the constant lottery s.t. $x^i \sim_i x$ and $x^i \sim_j x^*$ for each $j \neq i$.*

x^i will be identified with the *utility of x according to agent i*: the influence of the other agents is neutralized (they get their best outcome, which behaves as a neutral element in the pessimistic approach).

Let $\Delta_i = \{x^i, x \in X\}$. $(x^*)^i$ and $(x_*)^i$ belong to Δ_i by definition. The union of the Δ_i's, that is to say $\Delta = \{x^i : x \in X, i \in \mathcal{A}\}$, plays an important role in our proof - it allows the construction of a common evaluation scale. Δ is naturally ordered by $\succeq$ and each Δ_i is ordered by $\succeq_i$. By construction, we have:

Proposition 5. $\forall x \in X, (x^*)^i \succeq_i x \succeq_i (x_*)^i$.

Moreover, we can show that

Proposition 6. $\forall x^i, (x^*)^i \succeq x^i \succeq (x_*)^i$

$(x^*)^i$ is one of the best consequences for i and $(x_*)^i$ is one of her worst ones. It may happen that one of the x^i be indifferent w.r.t. $\succeq$ to $(x_*)^i$: i prefers x^i to $(x_*)^i$, but the collectivity does not; this is due to the fact that agent i is not so important, so the elements of X that are bad for her (e.g. $(x_*)^i$) are considered as not so bad for the collectivity.

Let us denote $B_i = \{x^i \in \Delta_i : x^i \sim (x_*)^i\}$ the set of the elements of Δ_i that are indifferent to $(x_*)^i$ according to the collectivity, and this even if agent i makes a difference; the elements of B_i form an equivalence class according to $\succeq$ - but, again, not necessarily according to $\succeq_i$.
Let m_i denotes the best of the elements of B_i (according to $\succeq_i$) [4]. It reflects the importance of the agent: the greatest m_i, the lower the importance of i. Formally[5]:

Definition 3. *For any $i \in \mathcal{A}$, let $m_i = argmax_{\succeq_i}\{x^i : x^i \sim (x_*)^i\}$ be the* discount degree *of i.*

Lemma 1. $\forall x \in X, i = 1, p,\ x^i \sim \max_{\succeq_i}(m_i, x^i)$.

Lemma 2. $\forall x \in X,\ x \sim\ argmin_{\succeq}\ \{x^i : i \in \mathcal{A}\}$.

From Lemmas 1 and 2 we get:

Corollary 1. $x \sim\ argmin_{\succeq}\{\max_{\succeq_i}(m_i, x^i) : i \in \mathcal{A}\}$.

In order to show that a relation satisfying Axioms 1-6 is a pessimistic utility, (Dubois and Prade 1995) built the scale $U = \{[x] : x \in X\}$ where $[x]$ is the equivalence class of x according to $\succeq$. U is totally ordered by $\succeq$ and these authors set $u(x) = [x]$. Here, we use the set $\Delta = \{x^i : x \in X, i \in \mathcal{A}\}$, partially ordered by the relation $\trianglerighteq$ defined by:

Definition 4. $\quad x^i \trianglerighteq y^i$ *iff* $x^i \succeq_i y^i$

[4]If $|B_i| > 1$, m_i can be any one of its elements.

[5]In the following, there are many relations (preorders). For the sake of clarity, we indicate for each minimum or maximum operation the preorder it relies on.

$x^i \trianglerighteq y^j$ *iff* $x_i \succeq_i m_i$, $y_j \succeq_j m_j$ *and* $x^i \succeq y^j$, *for all* $i \neq j$.

This relation is a partial preoder (it is reflexive and transitive) but if $x_i \prec m_i$ and $i \neq j$, x_i and y_j are not comparable: neither $x^i \trianglerighteq y^j$ nor $y^j \trianglerighteq x^i$ hold, but this is harmeless. What is important is that (i) the restriction of $\trianglerighteq$ to each Δ_i is a preorder (on Δ_i, $\trianglerighteq = \succeq_i$) and (ii) that any x_i that is as least as good as m_i (according to i) is comparable to any x_j that is as least as good as m_j (according to j) . Properties (i) and (ii) ensure that $v(x) = \min_{\trianglerighteq}\{\max_{\trianglerighteq}(m_i, x^i) : i \in \mathcal{A}\}$ exists. Then from Corollary 1 and Definition 4 it follows that: $x \sim v(x)$. Let k be the agent for which the min is reached in the expression of $v(x)$: $v(x) = \max_{\trianglerighteq}(m_k\ x^k)$ belongs to Δ_k and is such that $v(x) \succeq_k m_k$. Hence $v(x)$ and $v(y)$ are comparable w.r.t. $\trianglerighteq$, whatever x, y. This allows us to write:

Lemma 3. $x \succeq y$ *iff* $argmin_{\trianglerighteq}\{\max_{\trianglerighteq}(m_i, x^i) : i \in \mathcal{A}\} \trianglerighteq argmin_{\trianglerighteq}\{\max_{\trianglerighteq}(m_i, y^i) : i \in \mathcal{A}\}$.

Because working with a partial preorder is not so convenient, we shall use any complete preorder $\trianglerighteq'$ on Δ such that $x \trianglerighteq y \implies x \trianglerighteq' y$ (there always exists one). Then we get:

Lemma 4. $x \succeq y$ *iff* $argmin_{\trianglerighteq'}\{\max_{\trianglerighteq'}(m_i, x^i) : i \in \mathcal{A}\} \trianglerighteq' argmin_{\trianglerighteq'}\{\max_{\trianglerighteq'}(m_i, y^i) : i \in \mathcal{A}\}$.

Since $\succeq$ satisfies Axioms 1-6, Dubois and Prade's result applies: there exists an order reversing function n s.t.:

$$L \succeq L' \text{ iff } \min_{\succeq, x\in X} \max_{\succeq}(n(L[x]), u(x)) \succeq \min_{\succeq, x\in X} \max_{\succeq}(n(L'[x]), u(x)).$$

Let us denote $u(x) = argmin_{\trianglerighteq'}\{\max_{\trianglerighteq'}(m_i, x^i) : i \in \mathcal{A}\}$ and $n^{ext}(v) = u(n(v))$ ($n(v)$ is an element of Δ). By applying Lemma 4, we can write:

$$L \succeq L' \text{ iff } \min_{\trianglerighteq', x\in X} \max_{\trianglerighteq'}(n^{ext}(L[x]), u(x)) \trianglerighteq' \min_{\trianglerighteq', x\in X} \max_{\trianglerighteq'}(n^{ext}(L'[x]), u(x)).$$

$n^{ext}(v)$, m_i and x^i, $u(x)$ belong to Δ. In order to get a total order, we consider the equivalence classes of Δ, i.e. the set $U^{ext} = \{[x] : x \in X\}$ where $[x]$ is the equivalence class of x w.r.t. $\trianglerighteq'$. Because $x = argmin_{\succeq}\{x^i : i \in \mathcal{A}\}$ (Lemma 2) U^{ext} contains the equivalence classe of each $x \in X$ to $\succeq$, in particular, the equivalence classe $[x^i]$ of each x^i; U^{ext} is ordered by $\trianglerighteq'$ and is equipped with a maximal and a minimal elements ($[x^*]$ and $[x_*]$, respectively).

Setting $u_i(x) = [x^i]$, $nw_i = [m_i]$ and $n(v) = [n^{ext}(v)]$, we get:

$$L \succeq L \text{ iff } \min_{\trianglerighteq', x\in X} \max_{\trianglerighteq'}(n(L[x]), \min_{\trianglerighteq', i\in\mathcal{A}} max_{\trianglerighteq'}(u_i(x), nw_i)) \trianglerighteq' \min_{\trianglerighteq', x\in X} \max_{\trianglerighteq'}(n(L[x]), \min_{\trianglerighteq', i\in\mathcal{A}} max_{\trianglerighteq'}(u_i(x), nw_i)).$$

Hence the main result of this paper:

Theorem 1. *If the collective preference and individual preference relations satisfy Axioms 1-6, Pareto unanimity (**P**) and the axiom of continuity of X (**C**) then there exists a scale* U^{ext} *totally ordered by* $\trianglerighteq'$*, a distribution of weights* $nw : \mathcal{A} \mapsto U^{ext}$*, a series of functions* $u_i : X \mapsto U^{ext}, i = 1, n$ *and an order reversing function* $n : V \mapsto U^{ext}$ *s.t. for each couple of lotteries L and* L'*:*

$$L \succeq L' \text{ iff } \min_{x\in X} \max(n(L[x]), \min_{i\in\mathcal{A}} \max(nw_i, u_i(x))) \trianglerighteq' \min_{x\in X} \max(n(L'[x]), \min_{i\in\mathcal{A}} \max(nw_i, u_i(x))).$$

We can add two axioms that leads to pure egalitarianism.

Axiom E: $\forall i, j, (x_*)^i \sim (x_*)^j$

Axiom PW: $\forall i$, if $x \succ_i y$ then $x^i \succ y^i$

By **E**, the dissatisfaction of one agent has no more power than the one of another agent. A direct consequence is that the agents have the same discount degree. By **PW**, each agent has some power (she makes the decision at least when every other one is totally happy with both x and y). It implies $m_i \sim_i x_*$, for each i. Because all i share the same discount m_i, Pareto unanimity implies that $m_i \sim x_*$.

This provides a characterization of the full egalitarian CUF:

Theorem 2. *If the collective preference and individual preference relations satisfy Axioms **1-6**, **P**, **C**, **E** and **PW** then there exists a scale* U^{ext} *totally ordered by* $\trianglerighteq'$*, a series of functions* $u_i : X \mapsto U^{ext}$ *and an order reversing function* $n : V \mapsto U^{ext}$ *such that:*

$$L \succeq L' \text{ iff } \min_{x\in X} \max(n(L[x]), \min_{i\in\mathcal{A}} u_i(x)) \trianglerighteq' \min_{x\in X} \max(n(L'[x]), \min_{i\in\mathcal{A}} u_i(x)).$$

Conclusion

In conclusion, not only egalitarianism and decision under uncertainty are compatible and can escape the Timing Effect, but egalitarianism is compulsory when the decision is to be made on a possibilistic and cautious basis. This is interpreted as a justification of egalitarianism, just like Harsanyi's theorem can be interpreted as a justification of utilitarianism.

The present work, like the seminal work of Harsanyi, assumes that all the agents share the same knowledge, which is seldom the case. This consideration has been the topic of several works, always in a probabilistic context (Harsanyi 1967; Hammond 1992): the *ex-ante* approach can be compatible with the existence of different quantifications of the uncertainty (while the ex-post approach clearly requires a unique knowledge); the Timing Effect shall not always be understood as a paradox. The next step of our work is to characterize egalitarianism in the context of a non homogeneous qualitative knowledge of the agents.

Another way of seeing Harsanyi' Theorem - and ours - is to say that the axioms simply transfer the additive nature of probabilities on the collective utility function and the cautiousness of pessimistic utility into a maximization of the least satisfied agent. The problem has to be also studied without any explicit a priori assumption on the type of knowledge - following Savage's (1954) or Arrow's (1950) (see also (Hammond 1987)) ways rather than von Neumann and Morgenstern's. We probably then get that when egalitarianism is required, the collective decision is made as if each one were deciding in a qualitative, ordinal way. The probabilities may exist in the mind of (some) decision makers, but are not fully exploited. This is an exciting topic of further research.

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