Necessity-based Choquet integrals for sequential decision making under uncertainty

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Abstract. Possibilistic decision theory is a natural one to consider when information about uncertainty cannot be quantified in probabilistic way. Different qualitative criteria based on possibility theory have been proposed, the definition of which requires a finite ordinal, non compensatory, scale for evaluating both utility and plausibility. In presence of heterogeneous information, i.e. when the knowledge about the state of the world is modeled by a possibility distribution while the utility degrees are numerical and compensatory, one should rather evaluate each decision on the basis of its Necessity-based Choquet value. In the present paper, we study the use of this criterion in the context of sequential decision trees. We show that it does not satisfy the monotonicity property on which rely the dynamic programming algorithms classically associated to decision trees. Then, we propose a Branch and Bound algorithm based on an optimistic evaluation of the Choquet value of possibilistic decision trees.

1 Introduction

Decision under uncertainty is one of the main fields of research in decision theory, due to its numerous applications (e.g. medicine, robot control, strategic decision, games...). In such problems, the consequence of a decision depends on uncertain events. In decision under risk, it is assumed that a precise probability is known for each event. A decision can thus be characterized by a lottery over possible consequences. In multistage decision making, one studies problems where one has to make a sequence of decisions conditionally to observable states. The problem is to choose a strategy assigning a decision (i.e. a lottery) to each state.

A popular criterion to compare lotteries and therefore strategies is the expected utility (EU) model axiomatized by von Neumann and Morgenstern [9]. This model relies on a probabilistic representation of uncertainty, while the preferences of the decision maker are supposed to be captured by a utility function assigning a numerical value to each outcome. The evaluation of a lottery is then performed via the computation of its expected utility (the greater, the better). Since strategies can be viewed as compound lotteries, they can also be compared on the basis of their expected utility. When the decision problem is sequential, the number of possible strategies grows exponentially. Hopefully, the EU model satisfies a *monotonicity property* that guarantees completeness of a polytime algorithm of dynamic programming.

When information about uncertainty cannot be quantified in a simple, probabilistic, way the topic of possibilistic decision theory is often a natural one to consider [2] [4] [6]. Giving up the probabilistic quantification of uncertainty has led to give up the EU criterion as well. In [4], two qualitative criteria based on possibility theory, are proposed and axiomatized whose definitions only require a finite ordinal, *non compensatory* scale for evaluating both utility and plausibility. This yielded the development of sophisticated qualitative models for sequential decision making, e.g. possibilistic markov decision processes [13] [12], possibilistic ordinal decision trees [5] and possibilistic ordinal influence diagrams [5].

In presence of heterogeneous information, i.e. when the knowledge about the state of the world is possibilistic while the utility degrees are numerical and compensatory, the previous models do not apply anymore. Following [14] and [7], Choquet integrals [1] appear as a right way to extend expected utility to non Bayesian models. Like the EU model, this model is a numerical, compensatory, way of aggregating uncertain utilities. But it does not necessarily resort on a Bayesian modeling of uncertain knowledge. Indeed, this approach allows the use of any monotonic set function[3], and thus of a necessity measure (integrals based on a possibility measure are generally given up since too adventurous). Unfortunately, the use of Necessity-based Choquet integrals in sequential decision making is not straightforward: Choquet integral does not satisfy the principle of monotony in the general case. As a consequence, the optimality of the solution provided by dynamic programming is not granted. Hence a question arises: *do the Necessity-based Choquet integral satisfy the monotony principle and if not, which algorithm should we use to compute an optimal strategy*?

In the present paper, we show that the Necessity-based Choquet Integrals do not satisfy the monotonicity property and propose a Branch and Bound algorithm based on an optimistic evaluation of the Choquet value of possibilistic decision trees. This paper is organized as follows: the background notions are recalled in Section 2. Possibilistic decision trees are developed in Section 3. Section 4 is devoted to the algorithmic issues.

2 Background on possibility theory and possibilistic decision making under uncertainty

The basic building block in possibility theory is the notion of *possibility distribution* [3]. Let x be a variable whose value is ill-known and denote Ω the domain of x. The agent's knowledge about the value of x can be encoded by a possibility distribution $\pi : \Omega \to [0,1]$; $\pi(\omega) = 1$ means that value ω is totally possible for variable x and $\pi(\omega) = 0$ means that $x = \omega$ is impossible. From π, one can compute the possibility $\Pi(A)$ and necessity $N(A)$ of an event "$x \in A$":

$$\Pi(A) = sup_{v \in A} \pi(v) \quad (1)$$

[3] This kind of set function is often called capacity or fuzzy measure.

$$N(A) = 1 - \Pi(\bar{A}) = 1 - sup_{v \notin A}\pi(v) \tag{2}$$

Measure $\Pi(A)$ evaluates at which level A is *consistent* with the knowledge represented by π while $N(A)$ corresponds to the extent to which $\neg A$ is impossible and thus evaluates at which level A is certainly implied by the knowledge.

Given n non interactive (independent) possibilistic variables x_1,...,x_n respectively restricted by π_1,...,π_n, the joint possibility distribution π on Ω_1,...,Ω_n is a combination of π_1,...,π_n:

$$\pi(\omega_1, ..., \omega_n) = \pi_1(\omega_1) \otimes ... \otimes \pi_n(\omega_n). \tag{3}$$

The particularity of the possibilistic scale is that it can be interpreted in twofold: when the possibilistic scale is interpreted in an *ordinal* manner, i.e. when the possibility degree reflect only an ordering between the possible values, the *minimum* operator is used to combine different distributions ($\otimes = \min$). In a *numerical* interpretation, possibility distributions are related to upper bounds of imprecise probability distributions - $\otimes$ then corresponds to *product* operator ($\otimes = *$).

Following [4][2]'s possibilistic approach of decision making under uncertainty a decision can be seen as a possibility distribution over a finite set of states. In a single stage decision making problem, a utility function maps each state to a utility value in a set $U = \{u_1, ..., u_n\} \subseteq \mathbb{R}$ (we assume without loss of generality that $u_1 \leq \cdots \leq u_n$). This function models the attractiveness of each state for the decision maker. An act can then be represented by a possibility distribution on U, also called a (simple) *possibilistic lottery*, and denoted by $(\lambda_1/u_1, ..., \lambda_n/u_n)$: λ_i is the possibility that the decision leads to a state of utility u_i.

In the following, $\mathcal{L}$ denotes the set of simple lotteries (i.e. the set of possibility distributions over U). We shall also distinguish the set $\mathcal{L}_c \subseteq \mathcal{L}$ of constant lotteries over $\mathcal{L}$. Namely, $\mathcal{L}_c = \{\pi \text{ s.t. } \exists u_i, \pi(u_i) = 1 \text{ and } \forall u_j \neq u_i, \pi(u_j) = 0\}$. A possibilistic lottery $L \in \mathcal{L}$ is said to *overcome* a lottery $L' \in \mathcal{L}$ iff:

$$\forall u_i, N(L \geq u_i) \geq N(L' \geq u_i) \tag{4}$$

A *possibilistic compound lottery* $(\lambda_1/L^1, ..., \lambda_m/L^m)$ is a possibility distribution over $\mathcal{L}$. The possibility $\pi_{i,j}$ of getting a utility degree $u_j \in U$ from one of its sub-lotteries L^i depends on the possibility λ_i of getting L^i and on the possibility λ^i_j of getting u_j from L^i i.e. $\pi_{i,j} = \lambda_j \otimes \lambda^i_j$. More generally, the possibility of getting u_j from a compound lottery $(\lambda_1/L^1, ..., \lambda_m/L^m)$ is simply the *max*, over all L^i, of $\pi_{i,j}$. Thus, [4][2] have proposed to reduce $(\lambda_1/L^1, ..., \lambda_m/L^m)$ into a simple lottery defined by:

$$\lambda_1 \otimes L^1 \oplus ... \oplus \lambda_m \otimes L^m = (max_{j=1,m}\lambda^j_1 \otimes \lambda_j/u_1, \ldots, max_{j=1,m}\lambda^j_n \otimes \lambda_j/u_n) \tag{5}$$

where $\otimes = \min$ (resp. $\otimes = *$) if the possibilistic scale is interpreted in an ordinal (resp. numerical) way. $\lambda_1 \otimes L^1 \oplus ... \oplus \lambda_m \otimes L^m$ is considered as equivalent to $(\lambda_1/L^1, ..., \lambda_m/L^m)$ and is called the *reduction* of the compound lottery.

Under the assumption that the utility scale and the possibility scale are commensurate and *purely ordinal*, [4] have proposed the following qualitative pessimistic and optimistic utility degrees for evaluating any simple lottery $L = (\lambda_1/u_1, ..., \lambda_n/u_n)$ (possibly issued from the reduction of a compound lottery):

$$U_{pes}(L) = max_{i=1,n} \min(u_i, N(L \geq u_i)) \text{ and } U_{opt}(L) = max_{i=1,n} \min(u_i, \Pi(L \geq u_i)) \quad (6)$$

where $\Pi(L \geq u_i) = max_{j=i,n}\lambda_j$ and $N(L \geq u_i) = 1 - \Pi(L < u_i) = 1 - max_{j=1,i-1}\lambda_j$ are the possibility and necessity degree that L reaches at least the utility value u_i. The U_{pes} degree estimates to what extend it is certain (i.e. necessary according to measure N) that L reaches a good utility. Its optimistic counterpart, U_{opt}, estimates to what extend it is possible that L reaches a good utility. Both are instances of the Sugeno integral [15] expressed as follows:

$$S_\mu(L) = max_{i=1,n} \min(u_i, \mu(L \geq u_i)) \quad (7)$$

where μ is any capacity function, i.e. any set function s.t. $\mu(\emptyset) = 0, \mu(\Omega) = 1, A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$. U_{pes} is recovered when μ is a necessity measure.

Under the same assumption of commensurability, but assuming that the utility degrees have a richer, cardinal interpretation, one shall synthesize the utility of L by a Choquet integral:

$$Ch_\mu(L) = \Sigma_{i=n,1}(u_i - u_{i-1}) \ . \ \mu(L \geq u_i) \quad (8)$$

If μ is a probability measure then $Ch_\mu(L)$ is simply the expected utility of L. In the present paper, we are interested by studying Choquet decision criterion in the possibilistic framework - this lead to let the capacity μ be a necessity measure N (integrals based on a possibility measure are generally given up since too adventurous). In this case, Equation (8) is expressed by $Ch_N(L) = \Sigma_{i=n,1}(u_i - u_{i-1}) \ . \ N(L \geq u_i)$.

3 Possibilisitic decision trees

Decision trees [11] are graphical representations of sequential decision problems under the assumption of full observability (i.e. once a decision has been executed, its outcome is known and observed). A decision tree (see e.g. Figure 1) is a tree $GT = (\mathcal{N}, \mathcal{E})$. The set of nodes $\mathcal{N}$ contains three kinds of nodes:

- $\mathcal{D} = \{D_0, \ldots, D_m\}$ is the set of decision nodes (represented by rectangles). The labeling of the nodes is supposed to be in accordance with the temporal order i.e. if D_i is a descendant of D_j, then $i > j$. The root node of the tree is necessarily a decision node, denoted by D_0.
- $\mathcal{LN} = \{LN_1, \ldots, LN_k\}$ is the set of leaves, also called utility leaves: $\forall LN_i \in \mathcal{LN}$, $u(LN_i)$ is the utility of being eventually in node LN_i.
- $\mathcal{C} = \{C_1, \ldots, C_n\}$ is the set of chance nodes represented by circles. Chance nodes represent the possible actions.

For any $X_i \in \mathcal{N}$, $Succ(X_i) \subseteq \mathcal{N}$ denotes the set of its children. Moreover, for any $D_i \in \mathcal{D}, Succ(D_i) \subseteq \mathcal{C}$: $Succ(D_i)$ is the set of actions that can be decided when D_i is observed. For any $C_i \in \mathcal{C}, Succ(C_i) \subseteq \mathcal{LN} \cup \mathcal{D}$: $Succ(C_i)$ is indeed the set of outcomes of action C_i - either a leave node is observed, or a decision node is observed (and then a new action should be executed).

In classical, probabilistic, decision trees the uncertainty pertaining to the more or less possible outcomes of each C_i is represented by a probability distribution on $Succ(C_i)$. Here, we obviously use a possibilistic labeling, i.e. for any $C_i \in \mathcal{C}$, the uncertainty pertaining to the more or less possible outcomes of each C_i is represented by a *possibility degree* $\pi_i(X), \forall X \in Succ(C_i)$.

Solving the decision tree amounts at building a *strategy* that selects an action (i.e. a chance node) for each reachable decision node. Formally, we define a strategy as a function δ from $\mathcal{D}$ to $\mathcal{C} \cup \{\perp\}$. $\delta(D_i)$ is the action to be executed when a decision node D_i is observed. $\delta(D_i) = \perp$ means that no action has been selected for D_i (because either D_i cannot be reached or the strategy is partially defined). Admissible strategies must be:

- *sound*: $\forall D_i, \delta(D_i) \in Succ(D_i) \cup \{\perp\}$
- *complete*: (i) $\delta(D_0) \neq \perp$ and (ii) $\forall D_i$ s.t. $\delta(D_i) \neq \perp, \forall X_j \in Succ(\delta(D_i))$, either $\delta(X_j) \neq \perp$ or $X_i \in \mathcal{LN}$

Let Δ be the set sound and complete strategies that can be built from the decision tree. Any strategy in Δ can be viewed as a connected subtree of the decision tree whose arcs are of the form $(D_i, \delta(D_i))$.

In the present paper, we interpret utility degrees in a numerical, compensatory, way and we are interested in strategies in Δ that maximize the Necessity-based Choquet criterion. The Choquet value of a (sound and complete) strategy can be determined thanks to the notion of lottery reduction. Recall indeed that leave nodes ln in $\mathcal{LN}$ are labeled with utility degrees, or equivalently constant lotteries in $\mathcal{L}_c$. Then a chance node can be seen as either a lottery in $\mathcal{L}$, or as a compound lottery. The principle of the evaluation of a strategy is to reduce it in order to get an equivalent simple lottery, the Choquet value of which can then be computed. Formally, the composition of lotteries will be applied from the leaves of the strategy to its root, according to the following recursive definition:

$$\forall X_i \in \mathcal{N} L(X_i, \delta) = \begin{cases} < 1/u(X_i) > \quad if \quad X_i \in \mathcal{LN} \\ L(\delta(X_i), \delta) \quad if \quad X_i \in \mathcal{D} \\ Max_{X_j \in Succ(X_i)} \pi_i(X_j) \otimes L(X_j, \delta) \quad if \quad X_i \in \mathcal{C} \end{cases} \tag{9}$$

Depending on the interpretation of the possibility degrees labeling the arcs of the tree, we can distinguish between ordinal, min-based possibilistic decision trees (for which $\otimes = \min$) and numerical, product-based possibilistic decision trees (for which $\otimes = *$). Equation 9 is simply the adaptation to strategies of lottery reduction (Equation 5). We can then compute $L(\delta) = L(D_0, \delta) : L(\delta)(u_i)$ is simply the possibility of getting utility u_i when δ is applied from D_0. The Choquet value of δ can then be computed:

$$Ch_N(\delta) = Ch_N(L(D_0, \delta)) \tag{10}$$

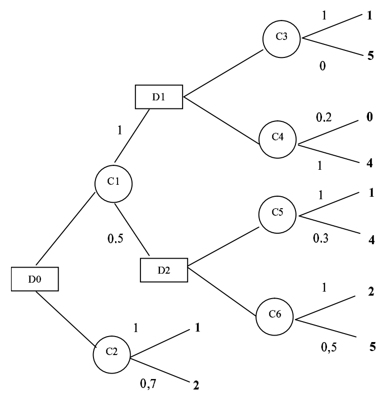

Fig. 1. Example of possibilistic decision tree with $\mathcal{C} = \{C_1, C_2, C_3, C_4, C_5, C_6\}$, $\mathcal{D} = \{D_0, D_1, D_2\}$ and $\mathcal{LN} = U = \{0, 1, 2, 3, 4, 5\}$.

4 Finding the Choquet optimal strategy in possibilistic decision trees

Given a possibilistic decision tree encoding a set of admissible strategies $\Delta = \{\delta_1 \ldots \delta_n\}$, we are looking for a strategy δ^* such that $\forall \delta \in \Delta$, $Ch_N(\delta^*) \geq Ch_N(\delta)$. Unfortunately, finding optimal strategies via an exhaustive enumeration of Δ is a highly computational task. For instance, in a decision tree with n binary decision nodes, the number of potential strategies is in $O(2^{\sqrt{n}})$.

For standard probabilistic decision trees, where the goal is to maximize expected utility [11], an optimal strategy can be computed in polytime (with respect to the size of the tree) thanks to an algorithm of dynamic programming which builds the best strategy backwards, optimizing the decisions from leaves of the tree its root. Regarding possibilistic decision trees, Garcia and Sabbadin [5] have shown that such a method can also be used to get a strategy maximizing U_{pes}. The reason is that like EU, U_{pes} satisfies the following key property of montonicity:

Definition 1. *Let V be a decision criterion. V is said to be monotonic iff whatever L, L' and L, whatever the normalized distribution (α,β):*

$$V(L) \geq V(L') \quad \Rightarrow \quad V((\alpha \otimes L) \oplus (\beta \otimes L")) \geq V((\alpha \otimes L') \oplus (\beta \otimes L")). \qquad (11)$$

This property states that the combination of L (resp. L') with a third one, $L"$, does not change the order induced by V between L and L' - this allows dynamic programming to decide in favor of L before considering the compound decision.

Unfortunately monotonicity is not satisfied any criterion. Some Choquet integrals, e.g. the one encoding the Rank Dependent Utility model, may fail to fulfill this condition (see e.g. [8]). We show in the following counter examples that this can also be the case when using Necessity-based Choquet integrals:

Counter example 1 (Numerical setting) *Let* $L =< 0.1/1, 1/2, 0/3 >$, $L' =< 0.9/1, 0/2, 1/3 >$ *and* $L" =< 1/1, 0.1/2, 0/3 >$*; let* $L_1 = (\alpha \otimes L) \oplus (\beta \otimes L")$ *and* $L_2 = (\alpha \otimes L') \oplus (\beta \otimes L")$*, with* $\alpha = 1$ *and* $\beta = 0.9$.
Using equation (5) with $\otimes = *$ *we have:* $L_1 =< 0.9/1, 1/2, 0/3 >$
and $L_2 =< 0.9/1, 0.09/2, 1/3 >$
It is easy to show that $Ch_N(L) = 1.9$ *and* $Ch_N(L') = 1.2$*, then* $L \succ L'$*. But* $Ch_N(L_1) = 1.1 < Ch_N(L_2) = 1.2$*: this contradicts the monotonicity property.*

Counter example 2 (Ordinal setting) *Let* $L =< 0.2/0, 0.5/0.51, 1/1 >$,
$L' =< 0.1/0, 0.6/0.5, 1/1 >$ *and* $L" =< 0.01/0, 1/1 >$*; ; let* $L_1 = (\alpha \otimes L) \oplus (\beta \otimes L")$ *and* $L_2 = (\alpha \otimes L') \oplus (\beta \otimes L")$*, with* $\alpha = 0.55$ *and* $\beta = 1$.
Using equation (5) with $\otimes = min$ *we have:* $L_1 =< 0.2/0, 0.5/0.51, 1/1 >$ *and* $L_2 =< 0.1/0, 0.55/0.5, 1/1 >$.
Computing $Ch_N(L) = 0.653$ *and* $Ch_N(L') = 0.650$ *we get* $L \succ L'$*. But* $Ch_N(L_1) = 0.653 < Ch_N(L_2) = 0.675$*: this contradicts the monotonicity property.*

As a consequence, the application of dynamic programming to the case of the Necessity-based Choquet integral may lead to a suboptimal strategy. As an alternative, we have chosen to proceed by implicit enumeration via a Branch and Bound algorithm, following [8] for the case of another (non possibilistic) Choquet integral, namely the one encoding the Rank Dependent Utility criterion. The fact that implicit enumeration performs better for RDU than the resolute choice approach proposed in [10] encourages us to adapt it to our case.

The Branch and Bound algorithm (outlined by Algorithm 1.1) takes as argument a partial strategy δ and an upper bound of the best Choquet value it can reach. It returns the value Ch_N^{opt} of the best strategy found so far, δ^{opt}). As initial value for δ we will choose the empty strategy ($\delta(D_i) = \perp, \forall D_i$). For δ^{opt}, we can choose the one provided by the dynamic programming algorithm. Indeed, even not necessarily providing an optimal strategy, this algorithm may provide a good one, at least from a consequentialist point of view.

At each step, the current partial strategy, δ, is developed by the choice of an action for some unassigned decision node. When several decision nodes need to be developed, the one with the minimal rank (i.e. the former one according to the temporal order) is developed first. The recursive procedure stops when either the current strategy is complete (then δ^{opt} and Ch_N^{opt} may be updated) or proves to be worst than δ^{opt} in any case. To this extend, we call a function that computes a lottery $Lottery(\delta)$ that overcomes all those associated with the complete strategies compatible with δ and use $Ch_N(Lottery(\delta))$ as an upper bound of the Choquet value of the best strategy compatible with δ - the evaluation is sound, because whatever L, L', if L overcomes L', then $Ch_N(L) \geq Ch_N(L')$. Whenever $Ch_N(Lottery(\delta)) \leq Ch_N^{opt}$, the algorithms backtracks, yielding the choice of another action for the last decision nodes considered. Moreover when δ is complete, $Lottery(\delta)$ returns $L(D_0, \delta)$; the upper bound is equal to the Choquet value when computed for a complete strategy.

Function *Lottery* (see algorithm 1.2) inputs a partial strategy. It proceeds from backwards, assigning a simple lottery $< 1/u(NL_i) >$ to each terminal

Algorithm 1.1: BB

Data: A (possibly partial) strategy δ, the evaluation its Choquet value, Ch_N^{δ}

Result: Ch_N^{opt} *% also memorizes the best strategy found so far,* δ^{opt}

begin

  if $\delta = \emptyset$ **then** $\mathcal{D}_{pend} = \{D_1\}$ **else**

    $D_{pend} = \{D_i \in \mathcal{D}$ s.t. $\begin{array}{l}\delta(D_i) = \bot \text{ and} \\ \exists D_j, \delta(D_j) \neq \bot \text{ and } D_i \in Succ(\delta(D_j))\end{array} \}$

  if $\mathcal{D}_{pend} = \emptyset$ *(% δ is a complete strategy)* **then**

    if $Ch_N^{\delta} > Ch_N^{opt}$ **then**

    $\delta^{opt} \leftarrow \delta$

    return Ch_N^{δ}

  else

    $D_{next} \leftarrow arg\ min_{D_i \in D_{pend}}\ i$

    foreach $C_i \in Succ(D_{next})$ **do**

      $\delta((D_{next}) \leftarrow C_i)$

      $Eval \leftarrow Ch_N(Lottery(D_0, \delta))$

      if $Eval > Ch_N^{opt}$ **then**

        $Ch_N^{opt} \leftarrow \max(Ch_N^{opt}, \delta, Eval)$

  return Ch_N^{opt}

end

Algorithm 1.2: Lottery

Data: a node X, a (possibly partial) strategy δ

Result: L^X % $L^X[u_i]$ *is the possibility degree to have the utility* u_i

begin

  for $i \in \{1, .., n\}$ **do** $L^X[u_i] \leftarrow 0$

  if $X \in \mathcal{LN}$ **then** $L^X[u(X)] \leftarrow 1$

  if $X \in \mathcal{C}$ **then**

    foreach $Y \in Succ(X)$ **do**

      $L^Y \leftarrow Lottery(Y, \delta)$

      for $i \in \{1, .., n\}$ **do** $L^X[u_i] \leftarrow \max(L^X[u_i], \pi_X(Y) \otimes L^Y[u_i])$

      % $\otimes = \min$ *in the ordinal setting ;* $\otimes = *$ *in the numerical setting*

  if $X \in \mathcal{D}$ **then**

    if $\delta(X) \neq \bot$ **then** $L^X = lottery(\delta(X), \delta)$ **else**

      if $|Succ(X)| = 1$ **then**

        $L^X = Lottery(\delta(Succ(X)), \delta)$

      else

        foreach $Y \in Succ(X) \cap N_{\delta}$ **do**

          $L^Y \leftarrow Lottery(Y, \delta)$

          for $i \in \{1, .., n\}$ **do** $G_Y^c[u_i] \leftarrow 1 - max_{u_j < u_i} L^Y[u_j]$

        % *Compute the upper envelop of the cumulative functions)*

        for $i \in \{1, .., n\}$ **do** $G^c[u_i] \leftarrow \max_{Y \in Succ(X) \cap N_{\delta}} G_Y^c[u_i]$

        % Compute $Rev(G^c)$

        $L^X[u_n] \leftarrow 1$

        for $i \in \{n-1, .., 1\}$ **do** $L^X[u_i] \leftarrow 1 - G^c[u_{i+1}]$

  return L^X

end

node LN_i . At each chance node C_i, we perform a composition of the lotteries in $Succ(C_i)$ according to Equation (9). At each decision node D_i, node LN_i . At each chance node C_i, we perform a composition of the lotteries in $Succ(C_i)$ according to Equation (9). At each decision node D_i we ascend a lottery that overcomes all those in $Succ(D_i)$. To this end, let us use the following notations and definitions:

- Given a simple lottery $L \in \mathcal{L}$, G^c_L is the *possibilistic decumulative* function of L: $\forall u \in U, G^c_L(u) = N(L \geq u)$
- Given a set $\mathcal{G} = \{G^c_{L_1}, \dots, G^c_{L_k}\}$ of decumulative functions, the *upper envelop* of $\mathcal{G}$ is the decumulative function $G^c_{\mathcal{G}}$ defined by:
  $\forall u \in U, G^c_{\mathcal{G}}(u) = max_{G^c_{L_i} \in \mathcal{G}} G^c_{L_i}(u)$
- Given a decumulative function G^c on U, $Rev(G^c)$ is the lottery defined by:
  $Rev(G^c)(u_i) = \begin{cases} 1 & if \quad i = n \\ 1 - G^c(u_{i+1}) & if \quad i \in \{1, \dots, n-1\} \end{cases}$

Now it is easy to showthat the possibilistic decumulative function associated to a lottery $Rev(G^c)$ is equal to G^c. As a consequence:

Proposition 1. [4] *Given a set $\{L_1, \dots, L_k\} \subseteq \mathcal{L}$ of simple lotteries over U, $\mathcal{G} = \{G^c_{L_1}, \dots, G^c_{L_k}\}$ the set of their decumulative function, we have: $Rev(G^c_{\mathcal{G}})$ overcomes any lottery $L_i \in \{L_1, \dots, L_k\}$.*

Hence, the Choquet value of $Lottery(D_0, \delta)$ is an upper bound of the Choquet value of the best complete strategy compatible with δ, which proofs the correctness of our algorithm.

Example 1. The major steps executed by the BB algorithm on the min-based possibilistic decision tree of Figure 1 can be summarized as follows (we suppose that δ^{opt} has been initialized with $((D_0, C_2))$, the Choquet value of which is 1)

- $\delta = \emptyset$ and $Ch^{opt}_N = 1$. BB calls $Ch_N(Lottery(D_0, (D_0, C_1)))$
  $L^{D_2} = (0/0, 0.2/1, 0.2/2, 1/4, 1/5)$, $L^{D_3} = (0/0, 0/1, 1/2, 1/4, 1/5)$.
  So, $Lottery(D_0, (D_0, C_1)) = (0/0, 0.2/1, 0.5/2, 0.5/3, 1/4, 1/5)$
  and $Eval = Ch_N(Lottery(D_0, (D_0, C_1))) = 2.8 > 1$.
- $\delta = (D_0, C_1)$ and $Ch^{opt}_N = 1$. BB calls $Ch_N(Lottery(D_0, ((D_0, C_1), (D_2, C_3))))$.
  $Lottery(D_0, ((D_0, C_1), (D_2, C_3))) = (0/0, 1/1, 0.5/2, 0.5/3, 0.5/4, 0.5/5)$
  and $Eval = Ch_N(Lottery(D_0, ((D_0, C_1), (D_2, C_3)))) = 1 = 1$.
  $\delta = (D_0, C_1)$ and $Ch^{opt}_N = 1$. BB calls $Ch_N(Lottery(D_0, ((D_0, C_1), (D_2, C_4))))$
  $Lottery(D_0, ((D_0, C_1), (D_2, C_4))) = (0.2/0, 0/1, 0.5/2, 0.5/3, 1/4, 0.5/5)$
  and $Eval = Ch_N(Lottery(D_0, ((D_0, C_1), (D_2, C_4)))) = 2.6 > 1$.
- $\delta = ((D_0, C_1), (D_2, C_4))$ and $Ch^{opt}_N = 1$. BB calls $Ch_N(Lottery(D_0, ((D_0, C_1), (D_2, C_4), (D_3, C_5))))$,
  $Lottery(D_0, ((D_0, C_1), (D_2, C_4), (D_3, C_5))) = (0.2/0, 0.5/1, 0/2, 0/3, 1/4, 0/5)$
  and $Eval = Ch_N(Lottery(D_0, ((D_0, C_1), (D_2, C_4), (D_3, C_5)))) = 2.3 > 1$.
- $\delta = ((D_0, C_1), (D_2, C_4), (D_3, C_5))$ and $Ch^{opt}_N = 1$.
  There is no more pending decision node. $\delta^{opt} \leftarrow ((D_0, C_1), (D_2, C_4), (D_3, C_5))$, $Ch^{opt}_N \leftarrow 2.3$
- $\delta = ((D_0, C_1), (D_2, C_4))$ and $Ch^{opt}_N = 2.3$. BB calls $Ch_N(Lottery(D_0, ((D_0, C_1), (D_2, C_4), (D_3, C_6))))$,
  $Lottery(D_0, ((D_0, C_1), (D_2, C_4), (D_3, C_6))) = (0.2/0, 0/1, 0.5/2, 0/3, 1/4, 0.5/5)$
  and $Eval = Ch_N(Lottery(D_0, ((D_0, C_1), (D_2, C_4), (D_3, C_6)))) = 2.6 > 2.3$.
- $\delta = ((D_0, C_1), (D_2, C_4), (D_3, C_5))$ and $Ch^{opt}_N = 2.3$.
  There is no more pending decision node. $\delta^{opt} \leftarrow ((D_0, C_1), (D_2, C_4), (D_3, C_6))$, $Ch^{opt}_N \leftarrow 2.6$
- etc.

The algorithm eventually terminates with $\delta^{opt} = ((D_0, C_1), (D_2, C_4), (D_3, C_6))$ and $Ch^{opt}_N = 2.6$.

[4] Obviously, $G^c_{Rev(G^c)}(u_1) = 1 = G^c(u_1)$. Note that $\forall i = 2, n, Rev(G^c)(u_i) \geq Rev(G^c)(u_{i-1})$. Hence $G^c_{Rev(G^c)}(u_i) = 1 - max_{j=1,i-1} Rev(G^c)(u_j) = 1 - Rev(G^c)(u_{i-1})$. Since $Rev(G^c)u_{i-1} = 1 - G^c(u_i)$, we get $G^c_{Rev(G^c)}(u_i) = G^c(u_i)$. Thus $G^c_{Rev(G^c)} = G^c$.

5 Conclusion

In this paper, we have proposed to use the Necessity-based Choquet Integral to optimize decision in *heterogeneous* possibilistic decision trees, where the utility levels of consequences are numerical in essence. We have shown that dynamic programming cannot be applied to find optimal strategies since the monotonicity property on which this algorithm relies is not satisfied by this criterion. As an alternative solution, we have developed a Branch and Bound algorithm based on an optimistic evaluation of the Choquet value (namely by taking the upper envelop of the decumulative functions of the concurrent possible actions). The implementation of this approach is under progress.

The further development of this work deals with the optimization of Necessity-based Choquet integrals and Sugeno Integrals in heterogeneous possibilistic influence diagrams, considering, again, both the numerical and the purely ordinal interpretation of possibility degrees.

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