International Journal of Approximate Reasoning 116 (2020) 1–18

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

Sequential decision making under ordinal uncertainty: A qualitative alternative to the Hurwicz criterion

Hélène Fargier, Romain Guillaume *

IRIT, CNRS and University of Toulouse, France

ARTICLE INFO

Article history:
Received 30 December 2018
Received in revised form 1 October 2019
Accepted 4 October 2019
Available online 9 October 2019

Keywords:
Qualitative decision
Uncertainty
Decision tree

ABSTRACT

This paper focuses on sequential qualitative decision problems, where no probability distribution on the states that may follow an action is available. New qualitative criteria that are based on ordinal uninorms and namely R_* and R^* are proposed. Like the Hurwicz criterion, the R_* and R^* uninorms arbitrate between pure pessimism and pure optimism, and generalize the Maximin and Maximax criteria. But contrarily to the Hurwicz criterion they are associative, purely ordinal and compatible with *Dynamic Consistency* and *Consequentialism*. These important properties allow the construction of an optimal strategy in polytime, following an algorithm of Dynamic Programming. Making a step further, we then generalize R_* to qualitative decision under possibilistic uncertainty, proposing an alternative to the classical optimistic and pessimistic criteria used for the computation of optimal strategies in possibilistic decision trees.

1. Introduction

In a sequential decision problem under uncertainty, a decision maker (DM in the following) faces a sequence of decisions, each decision possibly leading to several different states, where further decisions have to be made. A strategy is a conditional plan which assigns a (possibly nondeterministic) action to each state were a decision has to be made (also called "decision node"), and each strategy leads to a compound lottery, following Von Neuman and Morgenstern's terminology [15] - roughly, a tree representing the different possible scenarios, and thus the different possible final states that the plan/strategy may reach. The optimal strategies are then the ones which maximize a criterion applied to the resulting compound lottery.

Three assumptions are desired to accept the optimal strategy without discussions on the meaning of optimal strategy. Those assumptions are:

- *Dynamic Consistency*: when reaching a decision node by following an optimal strategy, the best decision at this node is the one that had been considered so when computing this strategy, i.e. prior to applying it.
- *Consequentialism*: the best decision at each step of the problem only depends on potential consequences at this point.
- *Tree Reduction*: a compound lottery is equivalent to a simple one.

Those three assumptions are linked to the possibility of computing an optimal strategy using an algorithm of dynamic programming [11].

* Corresponding author.
E-mail addresses: fargier@irit.fr (H. Fargier), guillaum@irit.fr (R. Guillaume).

https://doi.org/10.1016/j.ijar.2019.10.001
0888-613X/

When the preference about the final states is purely qualitative (ordinal), i.e., we cannot assume more than a preference order on the consequences (on the leaves of the tree), captured by satisfaction degrees on an ordinal scale (the scale $[0, 1]$ is chosen for these degrees, but any ordered set can be used). Then the pessimistic maximin approach is often presented as a way to capture the behavior of (very) cautious DMs - the utility of a decision is the minimum of the utilities it may lead to. The Hurwicz criterion [7] is then advocated since it generalizes the pessimistic maximin and the optimistic maximax approaches and makes a "compromise" between these approaches, through the use of a coefficient α of optimism - the Hurwicz value being the linear combination, according to this coefficient, of the two criteria. Nevertheless, this approach does not suit qualitative, ordinal, utilities: the Hurwicz criterion proceeds to an additive compensation of the min value by the max value. Moreover, the criterion turns out to be incompatible with the above assumptions: it can happen that none of the optimal strategies is dynamically consistent nor consequentialist - as a consequence the optimization of this criterion cannot be carried out using dynamic programming.

Some authors tend to privilege Dynamic Consistency and Tree Reduction and are ready to give up Consequentialism (e.g., the Resolute Choice approach [1]). Other insists on the fact that Resolute Choice is not acceptable since a normally behaved decision-maker is consequentialist [9] - this leads them to use other approaches, based on Veto-process [9] and Ego-dependent process [2] (see also [8], [10] who follow the same idea - quitting Resolute Choice and applying consequentialism - in nonqualitative problems). Then, the fundamental axiom of tree reduction is dropped, the structure of the decision tree affects the choices of the decision-maker, and the semantics of the criterion which is eventually optimized is defined in an operational way only.

In the present paper, rather than choosing which axiom to drop, we are looking for a new *qualitative* criterion which can take into account the level optimism/pessimism of the DM, like Hurwicz's criterion, and satisfies the three properties stated above (*Dynamic Consistency, Consequentialism* and *Tree Reduction*). We then show that, because ordinal in essence (it uses min and max functions only), this criterion can be generalized to possibilistic decision trees, where both the utility degrees and their likelihood levels are evaluated on a qualitative scale.

The paper is structured as follows. The next Section presents the Hurwicz criterion, the background on decision trees under ignorance and the principle of dynamic programming. Section 3 then advocates the use of two qualitative uninorms, R^* and R_*, as alternatives to the Hurwicz criterion. Drowning them in the context of sequential decision making, we show that R^* and R_* are compatible with *Dynamic Consistency* and *Consequentialism*, and propose to apply an algorithm of dynamic programming to compute an optimal, consequentialist and dynamically consistent strategy. Section 4 finally presents a generalization of R_* to the possibilistic case.

2. Background

2.1. The Hurwicz criterion

Let us first consider simple, non-sequential decision problems under ignorance: each decision δ_i is characterized by the multi set of final states $E_{\delta_i} = \{s^i_1, ..., s^i_{m^i}\}$ it can lead to. Given a utility function u capturing the attractiveness of each of these final states, δ_i can be identified with a simple lottery over the utility levels that may be reached: in decision under ignorance, where no probability distribution over the consequences of an act is available, a simple lottery is indeed the multiset[1] of the utility levels of the s^i_j, i.e. $L_{\delta_i} = \langle u^i_1, ..., u^i_{m^i}\rangle$ (where $u^i_j = u(s^i_j)$).

A usual way to take the optimism of the DM into account is to use the Hurwicz criterion [7]. The worth of δ_i is then:

$$H(\delta_i) = H(L_{\delta_i}) = (1-\alpha)\cdot \min(u^i_1, ..., u^i_{m^i}) + \alpha \cdot \max(u^i_1, ..., u^i_{m^i}) \quad (1)$$

where $\alpha \in [0, 1]$ is the degree of optimism. H indeed collapses with the max aggregation when $\alpha = 1$ (and with the min aggregation when $\alpha = 0$).

2.2. Decision trees

A convenient language to introduce sequential decision problems is through decision trees [11]. This framework proposes an explicit modeling in a graphical way, representing each possible scenario by a path from the root of the tree to one of its leaves. Formally, a decision tree $\mathcal{T} = (\mathcal{N}, \mathcal{E})$ is such that $\mathcal{N}$ contains three kinds of nodes (see Fig. 1 for an example):

- $\mathcal{D} = \{d_0, \dots, d_m\}$ is the set of decision nodes (depicted by rectangles).
- $\mathcal{LN} = \{ln_1, \dots, ln_k\}$ is the set of leaves, that represent final states in $\mathcal{S} = \{s_1, \dots, s_k\}$; such states can be evaluated thanks to a utility function: $\forall s_i \in \mathcal{S}$, $u(s_i)$ is the degree of satisfaction of being eventually in state s_i (of reaching node ln_i). For the sake of simplicity we assume, without loss of generality, that only leaf nodes lead to utilities.
- $\mathcal{X} = \{x_1, \dots, x_n\}$ is the set of chance nodes (depicted by circles).

[1] A given utility level may be present several times, since labeling on several leaves.

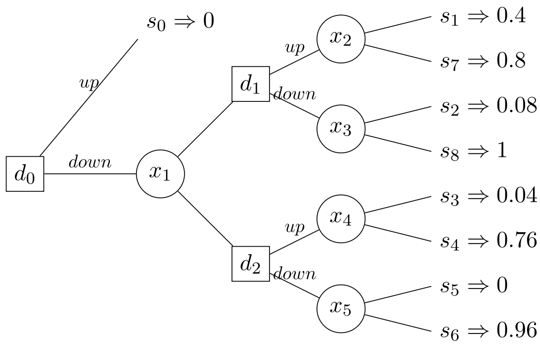

Fig. 1. A decision tree.

For any node $n_i \in \mathcal{N}$, $Succ(n_i) \subseteq \mathcal{N}$ denotes the set of its children. In a decision tree, for any decision node d_i, $Succ(d_i) \subseteq \mathcal{X}$: $Succ(d_i)$ is the set of actions that can be chosen when d_i is reached. For any chance node x_i, $Succ(x_i) \subseteq \mathcal{LN} \cup \mathcal{D}$: $Succ(x_i)$ is the set of possible outcomes of action x_i - either a leaf node is observed, or a decision node is reached (and then a new action should be chosen).

When the problem is a problem of *qualitative* decision making under ignorance:

- the information at chance nodes is a list of potential outcomes - this suits situations of total ignorance, where no probabilistic distribution is available.
- the preference about the final states is purely qualitative (ordinal), i.e., we cannot assume more than a preference order on the consequences (on the leaves of the tree), captured by the satisfaction degrees. The scale [0, 1] is chosen for these degrees, but any ordered set can be used.

Solving a decision tree consists in building a *strategy*, i.e. a function δ that associates to each decision node d_i an action (i.e. a chance node) in $Succ(d_i)$: $\delta(d_i)$ is the action to be executed when decision node d_i is reached. Let Δ be the set of strategies that can be built for $\mathcal{T}$. We shall also consider the subtree $\mathcal{T}_n$ of $\mathcal{T}$ rooted at node n, and denote by Δ_n its strategies: they are substrategies of the strategies of Δ.

Any strategy in Δ can be viewed as a connected subtree of $\mathcal{T}$ where there is exactly one edge (and thus one chance node) left at each decision node - skipping the decision nodes, we get a chance tree or, using von Neuwman and Morgern-stern's terminology, a compound lottery.[2]

Simple lotteries indeed suit the representation of decisions made at the last step of the tree: $\langle u_1, ..., u_k\rangle$ is the multiset of the utilities of the leaf nodes $(ln_1, ..., ln_k)$ that may be reached when some decision x is executed. Consider now a decision x made at the penultimate level: it may lead to any of the decision nodes d_i in $Succ(x)$, and thus to any of the simple lotteries $L_i = \langle u^i_1, ..., u^i_{m^i}\rangle$, $d_i \in Succ(x)$ - the substrategy rooted in x defines the compound lottery $\langle L_i,\ \text{s.t.}\ d_i \in Succ(x)\rangle$. The reasoning generalizes for decisions x at any level of the tree, hence the definition of the (possibly multi level) compound lottery L_δ associated to δ.

In order to apply a criterion, e.g. Hurwicz's, a simple lottery is needed. To this extent the *Reduction* of the compound lottery relative to the strategy is computed, which is the simple lottery which gathers all the utilities reached by the inner lotteries. Formally, the reduction of a compound lottery $L = \langle L_1, ..., L_k\rangle$ composed of lotteries L_i is defined by:

$$Reduction(L) = \langle Reduction(L_1), \ldots, Reduction(L_k)\rangle \tag{2}$$

where the reduction of a simple lottery is the simple lottery itself. For instance, if L is composed of simple lotteries $(L_1, ..., L_k)$, with $L_i = \langle u^i_1, ..., u^i_{n^i}\rangle$:

$$Reduction(L) = \langle u^1_1, ..., u^1_{n^1}, ..., u^k_1, ..., u^k_{n^k}\rangle \tag{3}$$

The principle of reduction makes the comparison of compound lotteries (and thus of strategies) possible: to compare compound lotteries by some criteria O, simply apply it to their reductions:

$$O(L) = O(Reduction(L)) \tag{4}$$

For instance, considering the Hurwicz criterion, the preference relation over strategies is defined by:

$$\delta \preceq_H \delta' \text{ iff } H(Reduction(L_\delta)) \preceq H(Reduction(L_{\delta'})) \tag{5}$$

Optimality can now be soundly defined, at the global and the local levels:

[2] Recall that a simple lottery $L = \langle u_1, ..., u_k\rangle$ is a multiset of utilities; a compound Lottery $L = \langle L_1, \ldots, L_k\rangle$ is a multiset of (simple or compound) lotteries.

Algorithm 1: Dynamic programming.

Input: decision tree $\mathcal{T}$ of depth $p > 1$, criterion O
Output: A strategy δ which is optimal for O, its value $O(\delta)$
for $ln \in \mathcal{LN}$ **do**
  $L(ln) = u(ln)$
for $t = p - 1$ *to* 0 **do**
  for $d \in \mathcal{D}_t$ **do**
    // $\mathcal{D}_t$ denotes the decision nodes at depth t
    for $n \in Succ(d)$ **do**
      $L(n) = Reduction(\langle L(n'), n' \in Succ(n)\rangle)$
      $V(n) = O(L(n))$
    $\delta(d) = arg\max_{n \in Succ(d)} V(n)$
    $L(d) = L(\delta(d))$
Return $(\delta, (d_0))$

- $\delta \in \Delta$ is optimal for $\mathcal{T}$ iff
  $\forall \delta' \in \Delta,\ O(Reduction(L_\delta)) \succeq O(Reduction(L_{\delta'}))$
- $\delta \in \Delta_n$ is optimal for $\mathcal{T}_n$ iff
  $\forall \delta' \in \Delta_n,\ O(Reduction(L_\delta)) \succeq O(Reduction(L_{\delta'}))$

In all the approaches that follow Equation (4), and in particular in the approach considered in this paper, *Tree Reduction* is thus obeyed by construction.

Let us now consider *Dynamic Consistency*. An optimal strategy δ is said to be dynamically consistent iff for any decision node n, δ_n, the restriction of δ to node n and its descendent, is optimal for the subtree rooted in n. A criterion is said to be compatible with *Dynamic Consistency* if there is always an optimal strategy that is dynamically consistent.

The purely optimist (resp. pessimist) criterion, max (resp. min) is compatible with *Dynamic Consistency* - there always exist an optimal strategy whose substrategies are optimal. Unfortunately, the Hurwicz criterion is not compatible with *Dynamic Consistency*. Let us give a counter example:

Example 1. Consider the decision tree of Fig. 1 and $\alpha = 0.9$; Strategy ($d_0 \leftarrow down$, $d_1 \leftarrow down$, $d_2 \leftarrow up$) is optimal, with a Hurwicz value of $0.1 \cdot 0.04 + 0.9 \cdot 1 = 0.904$; as a matter of fact ($d_0 \leftarrow down$, $d_1 \leftarrow down$, $d_2 \leftarrow down$) has a Hurwicz value of 0.9 and all the strategies with $d_0 \leftarrow up$ or $d_1 \leftarrow up$ have a lower value. Hence the (only) optimal strategy prescribes "*up*" for d_2. On the other hand, considering the tree rooted in d_2, "*up*" has a H value equal to 0.684, while "*down*" has a H value equal to 0.864 - *up* is not the optimal strategy in this subtree. This counter example shows that Hurwicz is not compatible with *Dynamic Consistency*.

2.3. Dynamic programming

Consequentialism prescribes that the DM selects a plan looking only at the possible futures (regardless of the past or counterfactual history). This is the case when choosing, at each node n, the decision that maximizes O. Hence a consequentialist strategy can be built starting from the anticipated future decisions and rolling back to the present. This is the idea implemented in the algorithm of dynamic programming (see Algorithm 1 where the depth of a node in the number of its predecessors, which simulates the behavior of such a consequentialist DM: the algorithm builds the best strategy by a process of backward induction, optimizing the decisions from the leaves of the tree to its root. Since each edge/node is passed through only one this algorithm is linear in the size of the tree, provided that both the reduction of lotteries and the computation of the value associated to a simple lottery (i.e. functions *Reduction* and O) can be run in linear time.

As to correctness, one can roughly say that a transitive criterion is coherent with *Consequentialism* iff the strategy returned by the algorithm of dynamic programming is optimal according to this criterion.

Unfortunately this is not always the case when optimality is based on the principle of *Tree Reduction*: rolling back the Hurwicz optimization at each node of the tree of Fig. 1 leads to strategy ($d_0 \leftarrow down$, $d_1 \leftarrow down$, $d_2 \leftarrow down$) which is *not* optimal according to Equation (4).

The correctness of dynamic programming actually relies on an important property, called weak monotonicity:

Definition 1. A preference criterion O over lotteries is said to be weakly monotonic iff whatever L, L' and L'':

$$L \succeq_O L' \Rightarrow \langle L, L''\rangle \succeq_O \langle L', L''\rangle \tag{6}$$

Weak Monotonicity is an important property; indeed, when $\succeq_O$ is complete and transitive, then the strategy returned by dynamic programming is optimal according to O. By construction, this strategy is dynamically consistent (any of its substrategies is optimal in its subtree), consequentialist and equivalent, according to O, to its reduction. In short, if a transitive criterion O satisfies *weak monotonicity* then strategy returned by dynamic programming is consequentialist and dynamically consistent.

3. R_* and R^* as criteria for decision making under ignorance

As we have seen in the previous Section, the Hurwicz criterion which is often advocated for decision making under ignorance suffers from several drawbacks for ordinal decision making. First of all it is neither so qualitative, since performing an compensation between the min value and the max value. Moreover, it fails to obey *Dynamic Consistency* and *Consequentialism*. This is regrettable from a prescriptive point of view: when optimizing this criterion, the decision planned for a node is not necessarily the one that would be the best one if the tree rooted at this node were considered - when reaching this node, a Hurwicz maximizer would be tempted not to follow the plan. That is why we look for alternative generalizations of the maximax and maximin rules, which are qualitative and which, like Hurwicz, allow a balance between pure pessimism and pure optimism.

A first idea could be to adapt the formulation of the Hurwicz criterion to the qualitative setting, replacing the product by the min operator and the sum by the max operator.

$$H^Q(\delta_i) = \max(\min((1-\alpha), \min(u_1^i, \dots, u_{m^i}^i)), \min(\alpha, \max(u_1^i, \dots, u_{m^i}^i))) \quad (7)$$

Unfortunately this simple adaptation of the Hurwicz criterion is not very satisfactory. Let us consider two decisions $\square = (0.1, 1)$ and $\bigcirc = (0, 0.1)$ with $\alpha = 0.1$ i.e. a quite pessimistic DM. Remark that $\square$ is worse than $\bigcirc$ for both the maximin criterion and the maximax criterion. On the other hand $H^Q(\bigcirc) = \max(\min(0.9, 0), \min(0.1, 0.1)) = 0.1$ and $H^Q(\square) = \max(\min(0.9, 0.05), \min(0.1, 1)) = 0.1$. Hence $\square \sim_{H^Q} \bigcirc$ while $\square \succ_{\max\min} \bigcirc$ and $\square \succ_{\max\max} \bigcirc$. Moreover, H^Q does not satisfy the monotony principle, as shown by the following counter example:

Example 2. Let $L = (0, 1)$, $L' = (0.5, 0.6)$ and $L'' = (0.5, 1)$ hence for $\alpha = 0.2$ we have $H(L) = 0.8 > H(L') = 0.58$ but $H(L \cup L'') = 0.8 < H(L' \cup L'') = 0.9$.

In other terms, H^Q does not satisfy the dynamic consistency principle- when reaching this node, a Hurwicz or a H^Q maximizer would be tempted not to follow the plan. We develop in the following another alternative to the Hurwicz criterion, based on the notion of uninorm.

3.1. An refresher on the R_* and R^* uninorms

The uninorm aggregators [16] are generalization of t-norms and t-conorms. These operators allow the identity element (e) to lay anywhere in the unit interval - it is not necessarily equal to zero nor to one, as required by t-norms or t-conorms, respectively.

Definition 2. [16] A uninorm R is a mapping $R : [0, 1] \times [0, 1] \to [0, 1]$ having the following properties:

1. $R(a, b) = R(b, a)$ (Commutativity)
2. $R(a, b) \geq R(c, d)$ if $a \geq c$ and $b \geq d$ (Monotonicity)
3. $R(a, R(b, c)) = R(R(a, b), c)$ (Associativity)
4. There exists some element $e \in [0, 1]$, called the identity element, such that for all $x \in [0, 1]$ $R(x, e) = x$.

It can be checked that the definition of t-norms (resp. t-conorms) is recovered when $e = 1$ (resp. $e = 0$). In this paper we focus on two ordinal uninorms proposed by Yager [16]:

1. $R_* : [0, 1]^n \to [0, 1]$:
   - $R_*(a_1, \dots, a_n) = \min(a_1, \dots, a_n)$ if $\min(a_1, \dots, a_n) < e$
   - $R_*(a_1, \dots, a_n) = \max(a_1, \dots, a_n)$ if $\min(a_1, \dots, a_n) \geq e$
2. $R^* : [0, 1]^n \to [0, 1]$:
   - $R^*(a_1, \dots, a_n) = \min(a_1, \dots, a_n)$ if $\max(a_1, \dots, a_n) < e$
   - $R^*(a_1, \dots, a_n) = \max(a_1, \dots, a_n)$ if $\max(a_1, \dots, a_n) \geq e$

R_* specifies that if one of the a_i's is lower than e then the min operator is applied, otherwise max is applied. R^* specifies that if one of the a_i's is greater than e then the max operator is applied, otherwise min is applied. One can see that both R_* and R^* generalize the min and max uninorms, as Hurwicz does (min is recovered when $e = 1$, max when $e = 0$).

R_* and R^* constitute two different ways of generalizing the maximin and maximax criterion, and capture different types of behaviors of the DM. In the context of decision making under ignorance, we propose to interpret $[0, e[$ as an interval of hazards and $[e, 1]$ as interval of desirable opportunities:

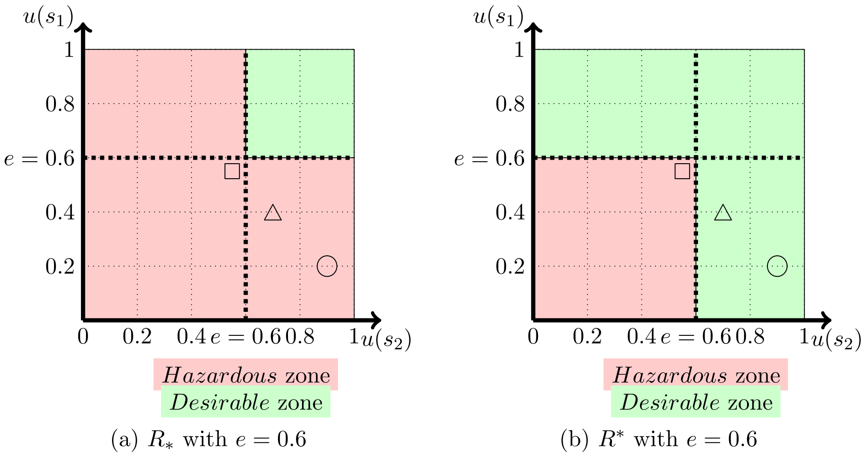

Fig. 2. Illustration of R_* and R^*. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

1. When all the possible utilities lay in the hazardous interval, both R_* and R^* behave in a pessimistic way and evaluate the lottery by its worst outcome.
2. When all the possible utilities lay in the interval of opportunity, both R_* and R^* behave in an optimistic way and evaluate the lottery by its best outcome.
3. When some possible utility belongs to the hazardous interval and others in the interval of opportunities, R_* returns a pessimistic value (the worst one) while R^* returns the best, optimistic, one.

Hence, in the simultaneous presence of hazards and opportunities, R_* focuses on the hazards while R^* focuses on the opportunities. If we summarize, the comparison of strategies by R_* and R^* is made as follows:

- R^*: if one of the two strategies may lead to (at least) one opportunity, the DM prefers the strategy with the greatest opportunity. If both lead surely into the interval of hazards, the DM prefers the more robust strategy.
- R_*: if one of the two strategies may lead to (at least) one hazardous utility, the DM prefers the more robust of the strategies. If both are exempt of hazards, the DM prefers the one with the greatest opportunity.

In robust decision making, where performance guarantees are looked for, one will obviously apply the R_* uninorm because of its cautiousness. R^* indeed appears as too adventurous: one single possible opportunity carries the final decision, and this even if all the other utilities lay in the hazard interval. On the contrary, R_* looks for opportunity only when the required level of satisfaction, e, is guaranteed for all the possible outcomes.

Example 3. Let us consider three decisions $\square = (0.55, 0.55)$, $\triangle = (0.7, 0.39)$ and $\bigcirc = (0.9, 0.2)$ and $e = 0.6$ (see Fig. 2). The red zone contains the decisions that the DM would like to avoid because it is too hazardous; the decision in the green zone are desirable, since they lead to opportunities for sure. One can see (Fig. 2(a)) that if the DM uses R_*, all the solutions are in the red zone hence she/he will select $\square$. Conversely, if the DM uses R^* (Fig. 2(b)), decision $\square$ is the only decision in the red zone and $\bigcirc$ will be selected.

Depending on the value of $e \in [0, 1]$, the optimal solutions are:

- $\forall e \in [0, 0.2]$ the optimal solution is $\bigcirc$ for both R_* and R^*.
- $\forall e \in]0.2, 0.39]$ for R_*: $\triangle$ and for R^*: $\bigcirc$
- $\forall e \in]0.39, 0.9]$ for R_*: $\square$ and for R^*: $\bigcirc$
- $\forall e \in]0.9, 1]$ the optimal solution is $\square$ for both uninorms.

Notice that $\triangle$ is favored by R_* when the degree of guaranteed performance, e, is moderate ($e \leq 0.39$). If a higher degree of performance must be ensured, R_* chooses $\square = (0.55, 0.55)$.

3.2. R_* and R^* in the sequential context

Let us now study the two uninorms in the context of sequential decision. Applying the principle of lottery reduction, we have:

$$\delta \succeq_{R_*} \delta' \text{ iff } R_*(Reduction(\delta)) \succeq R_*(Reduction(\delta')) \tag{8}$$

$$\delta \succeq_{R^*} \delta' \text{ iff } R^*(Reduction(\delta)) \succeq R^*(Reduction(\delta')) \tag{9}$$

Example 4. Let us go back to the example of Fig. 1 and focus first on criterion R_*. The strategies that decide *down* for d_2 are hazardous (they may reach s_5, which has a utility of 0) and have a R_* equal to 0 whatever the value of e. This is also the case for all the strategies that decide *up* for d_0. Now,

- if $e \in]0, 0.04]$ $(d_0 \leftarrow down,\ d_1 \leftarrow down,\ d_2 \leftarrow up)$ is optimal, with $R_* = 1$.
- if $e \in]0.04, 1]$ there are two optimal strategies, $(d_0 \leftarrow down,\ d_1 \leftarrow up,\ d_2 \leftarrow up)$ and $(d_0 \leftarrow down,\ d_1 \leftarrow down,\ d_2 \leftarrow up)$, both with $R_* = 0.04$.

It can be checked that each optimal strategy is dynamically consistent. For instance, $R_*(d_2 \leftarrow up)$, which is at least equal to 0.04 (whatever e), is always greater than $R_*(d_2 \leftarrow down)$, which is always equal to 0.

If we consider R^*, both $(d_0 \leftarrow down,\ d_1 \leftarrow down,\ d_2 \leftarrow down)$ and $(d_0 \leftarrow down,\ d_1 \leftarrow down,\ d_2 \leftarrow up)$ are optimal: their R^* values are equal to 1, whatever the value e (and both are dynamically consistent).

Beyond this example, R_* and R^* behave well for sequential problems in the general case; indeed, both are compatible with *Dynamic Consistency* and *Consequentialism*. The reason is that, contrarily to the Hurwicz criterion, they satisfy weak monotonicity:

Proposition 1. *R_* and R^* satisfies weak monotonicity.*

A direct consequence of Proposition 1 is that both uninorms can be optimized by dynamic programming. The optimization leads to strategies which are consequentialist and dynamically consistent; it follows that the uninorms R^* and R_* are compatible with *Dynamic Consistency*, *Consequentialism* and *Tree Reduction*.

As already outlined, compatibility with *Dynamic Consistency* guarantees that the DM cannot be tempted to deviate from the plan during its execution. Because R_* is consequentialist, the evaluation of a decision can be conservative at some node in the tree (because hazard cannot be excluded) and become optimistic when some safer point is reached (e.g. at node d_1 when $e \leq 0.08$). On the example of Fig. 1, with $e = 0.05$, R_* compares the min values of the two candidate decisions at node d_2, but is optimistic at node d_1: all the outputs that can be reached from d_1 are greater than 0.05, i.e. all the decisions are safe when d_1 is reached. Similar examples can be built for R^* (which is nevertheless less in accordance with the intuition, since pessimism is taken into account only when no opportunity is available).

A last algorithmic advantage of R^* and R_* is that they are associative (like any uninorm). This allows dynamic programming to memorize, for each node, the *value* of the corresponding reduced lottery rather than the lottery itself.

Definition 3. A criterion O satisfies the decomposition principle iff whatever L, L', $O(\langle L, L' \rangle) = O(\langle O(L), O(L') \rangle)$.

Proposition 2. *R^* and R_* satisfy the decomposition principle.*

Algorithm 1 thus directly applies, replacing O by R_* (resp. R^*).

3.3. Discussion: R_* and R^* vs. Hurwicz

Let us now focus on the comparison between the uninorms (and especially on R_*, which has a well founded interpretation in terms of robustness) and the Hurwicz criterion. All are generalization of the maximax and maximin criteria, allow a tuning between optimism and pessimism, and extend to sequential problems through the application of the lottery reduction principle.

The first remark is that R_* can capture the desiderata of a DM who is looking for guarantees of performance, the level of performance being represented by e. This kind of requirement cannot be captured by the Hurwicz criterion, unless $\alpha = 0$, i.e. unless Hurwicz collapses with the min (and also collapses with R_* and with R^*, setting $e = 0$).

Our running example also shows that Hurwicz can be very adventurous even for small positive α's: $(d_0 \leftarrow down,\ d_1 \leftarrow down,\ d_2 \leftarrow up)$ might reach a very low utility (0.08) is indeed optimal for Hurwicz as soon as $\alpha > 0$. This strategy will on the contrary be considered as too hazardous for R_*, unless a low level ($e < 0.08$) of guaranteed performance is looked for.

Moreover, the behavior of Hurwicz's approach may appear chaotic in its way to move from pessimism to optimism. Consider again Example 3: $\square = (0.55, 0.55)$ and $\bigcirc = (0.2, 0.9)$ are the min optimal and max optimal solutions, respectively. The max (resp. the min) value of $\triangle$ lays between the ones of $\square$ and $\bigcirc$, so $\triangle = (0.39, 0.7)$ appears as an intermediate solution between $\square$ and $\bigcirc$ (see Fig. 2). Nevertheless, $\triangle$ is never optimal for Hurwicz. It can indeed be checked than $H(\square) = 0.55$ whatever α. $H(\triangle) = 0.545$ at $\alpha = 0.5$. When $\alpha \leq 0.5$, $H(\triangle) < 0.55 = H(\square)$; when $\alpha \geq 0.5$ $H(\bigcirc) \geq H(\triangle)$, because $H(\bigcirc)$ increases faster than $H(\triangle)$. Hence a slight variation of α makes Hurwicz jump directly from the pessimistic solution $\square$ to the very optimistic solution $\bigcirc$, without considering $\triangle$, which is Pareto optimal and intermediate between $\square$ and $\bigcirc$.

If we look at the formal properties that may be sought for, the first difference is that the uninorms are purely ordinal. They do not need to assume that the utilities are additive to some extent, while Hurwicz is basically an additive criterion. The second one is their associativity - a basic property that is not satisfied by the Hurwicz's aggregation (for instance $H((1,0),(0)) = \alpha^2$ while $H((1,0,0)) = \alpha$). Because of the property of associativity, the application of R_* and R^* to compound lotteries satisfies decomposition, while this is not the case when Huwicz's criterion is used. Last but not least, R_* and R^* are compatible with *Dynamic Consistency* and *Consequentialism*, while Hurwicz is not.

A first, practical consequence is that a polynomial algorithm of dynamic programming can be designed to find consequentialist and dynamically consistent optimal solutions. Moreover, decomposition allows dynamic programming to memorize, for each node, the R_* and R^* value of the corresponding reduced lottery rather than the lottery itself.

Dynamic Consistency and *Consequentialism* are also important from a prescriptive point of view. Because the R_* and R^* optimal strategies are dynamically consistent, the DM will never be tempted to deviate from it - we have seen on Example 1 that Hurwicz does not prevent for such deviations.

Consequentialism says that the value of a (sub)strategies only depends on the future consequences - R_* and R_* never care of "parallel", counter factual worlds. As we have seen, Hurwicz is not compatible with this principle: what happens in a world (e.g., in Example 1 in d_2 when *up* is chosen for d_2) may influence the decision in an independent, parallel world (here, in d_1). Indeed, Hurwicz will always prefer $d_1 \leftarrow down$ to $d_1 \leftarrow up$ even in case of a very low - but positive - degree of optimism. This is due to the fact the low value (0.04) for s_3, which is not a descendent of d_1 but of d_2, masks the 0.08 utility of s_2.

4. Possibilistic generalizations of uninorm R_*

In the previous sections, we have made an assumption of total ignorance: all the consequences of a given lottery are equally possible. Possibility theory [17,3] allows to capture more information while staying in the ordinal context - the idea is to rank the consequences from the totally possible ones to the impossible ones. Following Dubois and Prade's possibilistic approach of decision making under ordinal uncertainty [4], a decision can be represented by a normalized possibility distribution on a set of utility degrees and evaluated either in a pure pessimistic way, using a Sugeno integral based on a necessity measure or on a pure optimistic way, using a Sugeno integral based on a possibility measure. We show in the following how to generalize the R_* uninorm to possibilistic lotteries, thus providing a new possibilistic criterion which takes the level optimism/pessimism of the DM into account.

4.1. A refresher on possibilistic decision making

The basic component of possibility theory is the notion of possibility distribution. It is a representation of a state of knowledge of an agent about the more or less possible values of a variable x taking its values on a referential S. Let $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ be a bounded ordered scale; by convention and without any loss of generality, we set $\lambda_i > \lambda_{i+1}$; typically $\lambda_1 = 1$ and $\lambda_n = 0$. A possibility distribution π is simply a mapping from S to Λ: for a value $s \in S$, $\pi(s) = 1$ means that s is totally possible and $\pi(s) = 0$ means that s is an impossible value. It is generally assumed that there exists at least one value s which is totally possible: π is said then to be *normalized*.

In the possibilistic framework, extreme forms of knowledge can be captured, namely:

- Complete knowledge i.e. $\exists\ s \in S$ s.t. $\pi(s) = 1$ and $\forall\ s' \neq s, \pi(s') = 0$.
- Total ignorance i.e. $\forall s \in S, \pi(s) = 1$ (all values in S are possible).

From π one can compute the possibility and the necessity of any event $A \subseteq S$:

$$\Pi(A) = \max_{s \in A} \pi(s) \tag{10}$$

$$N(A) = 1 - \Pi(\bar{A}) = 1 - \max_{s \notin A} \pi(s) \tag{11}$$

Π estimates to what extent A is compatible with the knowledge captured by π, and its conjugate, the necessity measure, estimates to what extent A is implied (to what extent a value outside A is unlikely).

Following Dubois and Prade's possibilistic approach of decision making under uncertainty [4], a simple possibilistic lottery L is a normalized possibility distribution on a set of utility degrees, both being expressed in the same ordered scale (Λ). **L** will denote the set of lotteries that can be built on Λ. We often write the lotteries as vectors $L = \langle \pi_1/\lambda_1, \dots, \pi_n/\lambda_n \rangle$ with $\pi_i \in \Lambda$. π_i is the possibility degree of getting utility λ_i according to the decision captured by L. For the sake of brevity, the λ_i such that $\pi_i = 0$ are often omitted in the notation of a lottery (e.g. $\langle 1/0.8 \rangle$ denotes the lottery that provides utility 0.8 for sure, all the other utility degrees being impossible). The normalization conditions imposes that one of the π_i is equal to λ_1 (to 1). The set of nonimpossible utility degrees is called the support of the lottery (we denote it $\mathcal{S}_L$).

Dubois and Prade [4,5] propose to use the possibilistic Sugeno integrals to evaluate the global utility of a possibilistic lottery. Recall that for any capacity function γ:

$$Sug_\gamma(L) = \max_{\lambda_i \in \Lambda} \min(\lambda_i, \gamma(L \geq \lambda_i)) \tag{12}$$

$\gamma(L \geq \lambda_i)$ estimates to what extent it is likely that L leads to a utility greater than λ_i. In the possibilistic case, two measures, N and Π shall be used. Hence the definition of two possibilistic global utilities:

$$Sug_{PES}(L) = \max_{\lambda_i \in \Lambda} \min(\lambda_i, N(L \geq \lambda_i)) \tag{13}$$

$$Sug_{OPT}(L) = \max_{\lambda_i \in \Lambda} \min(\lambda_i, \Pi(L \geq \lambda_i)) \tag{14}$$

where, according to Equations (10) and (11):

$$N(L \geq \lambda_i) = 1 - \max_{j<i} \pi_j \tag{15}$$

$$\Pi(L \geq \lambda_i) = \max_{j \leq i} \pi_j \tag{16}$$

When the lottery is normalized, the two possibilistic Sugeno integrals can be rewritten directly with respect to the possibility distribution [5,6]:

$$Sug_{PES}(L) = \min_{\lambda_i} \max(1 - \pi_i, \lambda_i) \tag{17}$$

$$Sug_{OPT}(L) = \max_{\lambda_i} \min(\pi_i, \lambda_i) \tag{18}$$

In other terms, a lottery is attractive according to Sug_{PES} when the possibility of getting a low utility degree is low - this measure suits cautious DMs. The second measure rather suits adventurous, optimistic DMs; indeed, a lottery is attractive according to Sug_{OPT} as soon as a good utility degree is possible. Of course, $Sug_{PES}(L) \geq \min_{\lambda \in S_L} \lambda$ and $Sug_{OPT}(L) \geq \min_{\lambda \in S_L} \lambda$.

After having defined the two criteria on simple lotteries, [4] generalizes the notion of composition of lotteries to the possibilistic case: a compound lottery $\langle \pi_1/L_1, \ldots, \pi_m/L_m \rangle$ is a possibility distribution over a set of lotteries.

Sug_{PES} and Sug_{OPT} can be extended to compound lotteries thanks to the possibilistic principle of lottery reduction: for any compound lottery $L = \langle \pi_1/L_1, \ldots, \pi_m/L_m \rangle$, $Reduction(L)$ is the simple lottery that associates to each of the λ_i the possibility degree

$$\pi_i = \max_{L_j \in L} \min(\pi_j, \pi_i^j) \tag{19}$$

π_i^j denoting the possibility of getting λ_i though lottery L_j and π_j denoting the possibility of getting L_j.

The principle of lottery reduction allows the comparison of compound lotteries to any other lottery: L is preferred to L' iff its reduction is preferred to the one of L'.

$$L \succeq_{PES} L' \text{ iff } Sug_{PES}(Reduction(L)) \succeq Sug_{PES}(Reduction(L'))$$

$$L \succeq_{OPT} L' \text{ iff } Sug_{OPT}(Reduction(L)) \succeq Sug_{OPT}(Reduction(L'))$$

The principle of monotonicity and the principle of decomposition extend easily to the possibilistic case, and are satisfied by Sug_{PES} and Sug_{OPT}:

Definition 4. A preference criterion O over possibilistic lotteries is said to be weakly monotonic iff whatever L, L' and L'', whatever α, β such that $\max(\alpha, \beta) = 1$:

$$L \preceq_O L' \Rightarrow \langle \alpha/L, \beta/L'' rangle \preceq_O \langle \alpha/L', \beta/L'' \rangle \tag{20}$$

Definition 5. A preference criterion O over possibilistic lotteries is said to satisfy the principle of decomposition iff whatever L and L', whatever α, β such that $\max(\alpha, \beta) = 1$:

$$O(\langle \alpha/L, \beta/L'' \rangle) = O(\langle \alpha/O(L), \beta/O(L') \rangle) \tag{21}$$

Proposition 3. *Whatever α, β, L, L', it holds that*

- $Sug_{PES}(Reduction(\langle \alpha/L, \beta/L' \rangle) = Sug_{PES}(\langle \alpha/Sug_{PES}(L), \beta/Sug_{PES}(L') \rangle)$
- $Sug_{OPT}(Reduction(\langle \alpha/L, \beta/L' \rangle) = Sug_{OPT}(\langle \alpha/Sug_{OPT}(L), \beta/Sug_{OPT}(L') \rangle)$
- *if* $L \succeq_{PES} L'$ *then* $\langle \alpha/L, \beta/L'' \rangle \succeq_{PES} \langle \alpha/L', \beta/L'' \rangle$
- *if* $L \succeq_{OPT} L'$ *then* $\langle \alpha/L, \beta/L'' \rangle \succeq_{OPT} \langle \alpha/L', \beta/L'' \rangle$

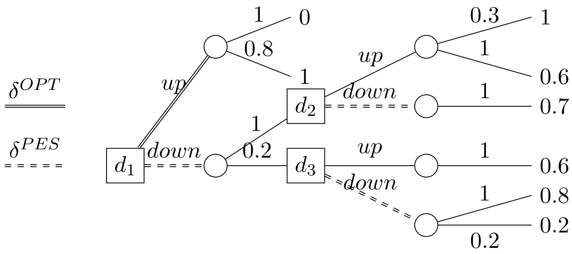

Fig. 3. A possibilistic decision tree.

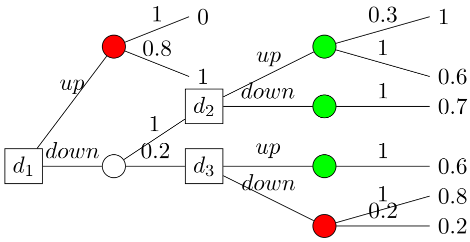

Fig. 4. Hazardous (in red) and desirable (in green) lotteries in a possibilistic decision tree.

This proposition is due to [4]; in this seminal paper, Dubois and Prade propose a representation theorem for Sug_{PES} and Sug_{OPT} which is precisely based on the properties and monotonicity, lottery reduction and certainty equivalence.

[14] have extended decision trees to the possibilistic case. In a possibilistic decision tree (Π-tree) the edges outgoing from the chance nodes are labeled by the possibility of being in the subsequent node when the decision represented by the chance node is executed (see Fig. 3). Together with lottery reduction, monotonicity and decomposition allow the computation of a Sug_{PES} (resp. Sug_{OPT}) optimal policy by dynamical programming, as shown by [12]. Algorithm 1 applies, replacing O by Sug_{PES} (resp. Sug_{OPT}) and reducing the lotteries according to Equation (19).

Example 5. Fig. 3 describes a possibilistic decision tree with three decision nodes. At each decision node, two decisions are available, *up* and *down*. The optimistic optimal decision at d_1 is *up* since this decision can lead with great possibility to utility 1. On the contrary, the pessimistic optimal decision at d_1 is *down* since $d_1 \leftarrow up$ can lead with possibility 1 to utility 0. The pessimistic and optimistic optimal strategies (δ^{PES} and δ^{OPT}) are also represented on Fig. 3.

4.2. A possibilistic generalization of R_*: U_{R_*}

The pessimistic possibilistic Sugeno integral generalizes the min uninorm - indeed, using Equation (17) it is easy to show that when all the degrees in $\mathcal{S}_L$ have a possibility degree equal to 1, $Sug_{PES}(L)$ simply computes the minimun of these degrees. Likewise, Equation (18) shows that the optimistic Sugeno integral generalizes the max uninorm. In the previous sections, we have advocated the use of R_* as way to arbitrate between the max and min uninorms in decision making under total ignorance. In the following, we are looking for generalization of R_* to possibilistic lotteries, extending the principles defined by this uninorm to decision under ordinal uncertainty.

Recall that the principle at work in R_* is to look for opportunity only when the required level of satisfaction, e, is guaranteed for all the possible outcomes: R_* distinguishes two families of decisions, the desirable ones and the hazardous ones, with respect to a neutral level e. The hazardous ones are evaluated in a pessimistic way (according to the min uninorm) and the desirable ones in an optimistic way (according to the max uninorm). Following this principle, we define in the present section a possibilistic generalization of R_* which compares the utilities to which a possibilistic lottery may lead to a neutral level, and propose to use a pessimistic possibilistic Sugeno integral for hazardous levels of utility, and to use an optimistic one for desirable ones. Formally, for any $L = \langle \pi_1/\lambda_1, \ldots, \pi_n/\lambda_n \rangle$:

Definition 6.

- L is hazardous (with respect to $e \in \Lambda$) iff $\exists \lambda_i < e$ such that $\pi_i > 0$;
- L is desirable (with respect to $e \in \Lambda$) iff $\forall \lambda_i$ such that $\pi_i > 0$, $\lambda_i \geq e$.

In other terms, a lottery is said to be hazardous if and only if getting a utility lower than e is not impossible. A lottery is desirable if it always leads to a utility at least equal to e. Fig. 4 distinguishes the hazardous and desirable lotteries of the decision tree of Fig. 3.

We can now propose the following possibilistic generalization of R_*:

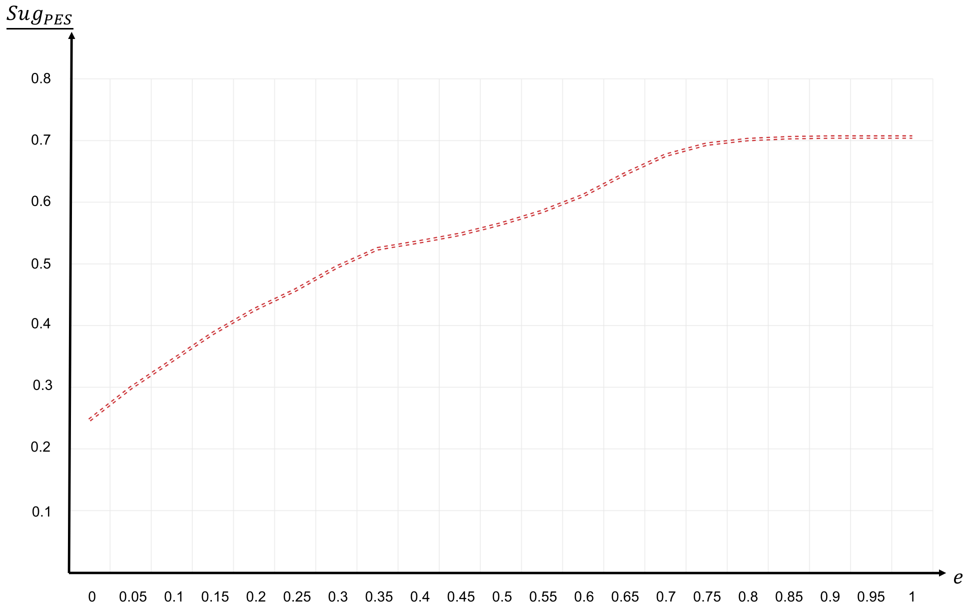

Fig. 5. Evolution of the worst $\underline{Sug_{PES}}$ with respect to e (U_{R_*} criterion).

Definition 7. $U_{R_*} : \mathbf{L} \to \Lambda$:

$$\begin{cases} U_{R_*}(L) = \max_{\lambda_i|\lambda_i<e}(\min(\lambda_i, N(L \geq \lambda_i)) \text{ if } L \text{ is hazardous} \\ U_{R_*}(L) = \max_{\lambda_i|\lambda_i \geq e}(\min(\lambda_i, \Pi(L \geq \lambda_i)) \text{ otherwise} \end{cases} \tag{22}$$

Example 6. Let $\Lambda = \{1, 0.75, 0.5, 0.25, 0\}$ and consider the three following lotteries $L_1 = \langle 0.75/0.25, 1/0.5, 1/1\rangle$, $L_2 = \langle 1/0, 0.5/0.5, 1/0.75\rangle$ and $L_3 = \langle 0.25/0.25, 1/0.7, 0.75/1\rangle$. Suppose that the DM would like avoiding the utility degrees below 0.5 - so $e = 0.5$. So, the three lotteries are hazardous and we have $U_{R_*}(L_1) = 0.25$, $U_{R_*}(L_2) = 0$ $U_{R_*}(L_3) = 0.25$ $L_1 \sim L_3 \succ L_2$. L_1 and L_3 are equivalent while L_1 is more hazardous. Lottery $L_4 = \langle 1/0.5, 0.75/0.75, 0.75/1\rangle$ is desirable, with $U_{R_*}(L_4) = 0.75$.

According to this criterion, a hazardous lottery is evaluated on the basis of the hazardous utilities *only* and according to the pessimistic possibilistic Sugeno integral (i.e. following the necessity measure). When the lottery is desirable, then it is evaluated on the basis of the desirable utilities and according to the optimistic posibilistic Sugeno integral.

We now check that U_{R_*} is a generalization of Sug_{PES}, Sug_{OPT} and R_*:

Proposition 4.

- *If $e = 0$, then $U_{R_*} = Sug_{OPT}$*
- *If $e = 1$, then $U_{R_*} = Sug_{PES}$*
- *If all the utilities in the support of L have a possibility degree equal to* 1, $U_{R_*}(L) = R_*(S_L)$.

To better understand the relationships between U_{R_*} and Sug_{PES}, we have randomly generated 500 samples of simple decision problems, each containing 50 simple lotteries over Λ. For each problem, and each value of e in Λ, all U_{R_*} optimal solutions have been computed, and the worst of their Sug_{PES} values retained - the different U_{R_*} optimal solutions to a given problem may indeed lead to different values of Sug_{PES}, unless $e = 1$ (in this case indeed, U_{R_*} is equivalent to Sug_{PES}). We denote this worst value $\underline{Sug_{PES}}$. The average (over the 500 problems) of $\underline{Sug_{PES}}$ is a function of e, which is reported at Fig. 5 for Λ containing 20 positive level (the size of scale do not modify the shape of the curve). We can see that when e increases the minimal value of Sug_{PES} over the set of optimal solutions increases too. At $e = 1$ (where U_{R_*} is equivalent to Sug_{PES}) U_{R_*} reach the maximal value of Sug_{PES} (0.719 in average) and decreasing slowly when e decrease - this shows that U_{R_*} really capture a notion of robustness.

We now give the main results of this Section: like R_*, Sug_{PES} and Sug_{OPT}, U_{R_*} does satisfy the principles of weak monotonicity and decomposition:

Proposition 5. $U_{R_*}(Reduction(\langle \alpha/L, \beta/L'\rangle)) = U_{R_*}(\langle \alpha/U_{R_*}(L), \beta/U_{R_*}(L')\rangle)$

Proposition 6. *U_{R_*} is weakly monotonic*

These two propositions allow the computation of R_* optimal strategies by Dynamical Programming: Algorithm 1 can be adapted, replacing O by U_{R_*} and reducing the lotteries according to Equation (19).

Algorithm 2: Applying Dynamic Programming to optimize U_{R_*} in a possibilistic decision tree.

```
Input: decision tree T of depth p > 1, criterion O
Output: A strategy δ which is optimal for O, its value O(δ)
for ln ∈ LN do
    V(ln) = u(ln)
for t = p − 1 to 0 do
    for d ∈ D_t do
        // D_t denotes the decision nodes at depth t
        for n ∈ Succ(d) do
            // Reduction
            for λ_i ∈ Λ do
                π^n[λ_i] = 0
            for n' ∈ Succ(n) do
                π^n[V(n')] = max(π^n[V(n')], π^n_{n'})
            V(n) = U_{R_*}(π^n)
        δ(d) = arg max_{n∈Succ(d)} V(n)
        V(d) = V(δ(d))
    Return (δ, (d_0))
```

4.3. Alternative generalizations of R_* and discussion

4.3.1. U_{R_*} vs Hurwicz

If we would like to apply the principles of the Hurwicz criterion to possibilistic lotteries, a first way to balance between optimism and pessimism could be to combine Sug_{PES} and Sug_{OPT} following Hurwicz's aggregation:

$$U_H(L) = (1-\alpha).Sug_{PES}(L) + \alpha.Sug_{OPT}(L))) \tag{23}$$

Hurwicz's criterion is recovered when all the degrees in $\mathcal{S}_L$ have a possibility degree equal to 1. In a sequential and qualitative context, this proposition is as unsatisfactory as Hurwicz's criterion: it proceeds to an additive compensation of two ordinal evaluations, and does not satisfy weak monotonicity. More generally, since U_{R_*} does satisfy these two properties, and is a generalization of R_*, all the arguments developed in the comparison of R_* and Hurwicz's criterion (Section 3.3) apply here.

An extension of the qualitative variant of the Hurwicz criterion, H^Q (Equation (7)) can also be proposed:

$$U_{H^Q}(L) = \max(\min(1-\alpha), Sug_{PES}(L), \min(\alpha, Sug_{OPT}(L))) \tag{24}$$

But even if purely ordinal, this proposition is not satisfactory either in the sequential context: as a generalization of Hurwicz criterion, H^Q does not obey weak monotonicity (Counter-Example 2 applies).

In summary, U_{R_*} overcomes the drawbacks of Hurwicz's approaches of possibilistic decision, just like R_* does in decision under ignorance.

4.3.2. Alternative generalizations of R_*

We present in this subsection alternative generalizations of R_* that could seem more direct, and compare them to U_{R_*}

The spirit of the first one, is the same than the one of U_{R_*}: be pessimistic if the lottery is hazardous, and optimistic otherwise.

Definition 8. $OR_* : \mathbf{L} \to \Lambda$:

$$\begin{cases} OR_*(L) = Sug_{PES}(L) \text{ if } L \text{ is hazardous} \\ OR_*(L) = Sug_{OPT}(L) \text{ otherwise} \end{cases}$$

R_* is recovered when L is a zero/one lottery ($\pi_i = 1$ if $\lambda_i \in S_L$, $\pi_i = 0$ otherwise). The slight difference between OR_* and U_{R_*} is that the pessimistic evaluation uses *all* the degrees of utility in the first case, while only the degrees lower than e are used in the second case - when L is hazardous $U_{R_*}(L) = \max_{\lambda_i|\lambda_i<e}(\min(\lambda_i, N(L \geq \lambda_i))$ while $OR_*(L) = \max_{\lambda_i}(\min(\lambda_i, N(L \geq \lambda_i))$.

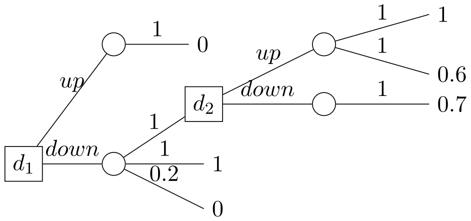

Fig. 6. A counter example to the monotonicity of OR_*.

It should be noticed that when L and L' are hazardous OR_* refines U_{R_*}, i.e.: $U_{R_*}(L) > U_{R_*}(L') \implies OR_*(L) > OR_*(L')$. But OR_* can exhibit a counter-intuitive behavior when hazardous lotteries are compared to desirable ones: a hazardous lottery can be preferred to a desirable one ! For instance, let $e = 0.2$ $L = \langle 0.1/0, 1/1\rangle$ and $L' = \langle 1/0.2\rangle$ hence L is hazardous and L' is desirable but $OR_*(L) = 0.9 > OR_*(L') = 0.2$. Last but not least drawback, OR_* does not satisfy dynamic consistency and monotonicity, as shown by the following counter example:

Example 7. Let us consider a sequential problem with 2 decisions represented in Fig. 6, with $\Lambda = \{1, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3, 0.2, 0.1, 0\}$ and $e = 0.4$. The strategy optimal for OR_* is $d_1 \leftarrow down$ and $d_2 \leftarrow down$ ($OR_* = 0.7$); But if the DM reaches decision node d_2, decision $d_2 \leftarrow up$ receives a higher OR_* value than decision $d_2 \leftarrow down$. In other terms, dynamic consistency and monotonicity are not satisfied.

Notice that the OR_* degree of the strategy optimal for U_{R_*}, namely $d_1 \leftarrow down$ and $d_2 \leftarrow up$, is equal to 0.6, i.e. is lower than the one of $d_1 \leftarrow down$ and $d_2 \leftarrow down$.

Another idea could be to take the degrees of possibility of the elements of S_L into account while deciding whether L is hazardous or not (contrarily to U_{R_*} and OR_* which consider the elements of the support of the lottery, but not their possibility degrees). This leads to the following straightforward generalization of R_*, which evaluates a lottery L by its pessimistic Sugeno value, $Sug_{PES}(L)$, when this value is lower than the degree e of optimism, and by its optimistic Sugeno value otherwise:

Definition 9. $PR_* : \mathbf{L} \to \Lambda$:

$$\begin{cases} PR_*(L) = Sug_{PES}(L) \text{ if } Sug_{PES}(L) < e \\ PR_*(L) = Sug_{OPT}(L) \text{ otherwise} \end{cases}$$

If $\pi_i \in \{1, 0\}, \forall i = 1, \ldots, n$ then we get back to the uninorm R_*: in this case indeed $Sug_{PES}(L) = \min_{i|\pi_i=1}(\lambda_i)$ and $Sug_{OPT}(L) = \max_{i|\pi_i=1}(\lambda_i)$.

Example 8. Let $\Lambda = \{1, 0.75, 0.5, 0.25, 0\}$ and consider the three following lotteries $L_1 = \langle 0.75/0.25, 1/0.5, 1/1\rangle$, $L_2 = \langle 0.25/0, 0.5/0.5, 1/0.75\rangle$ and $L_3 = \langle 0.25/0.5, 1/1\rangle$.

Suppose that the DM would like avoiding values of Sug_{PES} below 0.5 - so $e = 0.5$. It is easy to check that $Sug_{PES}(L_1) = 0.25$, $Sug_{PES}(L_2) = 0.5$, and $Sug_{PES}(L_3) = 0.75$. So, L_1 is evaluated in a cautious way (by Sug_{PES}) while the optimistic aggregation is used for L_2 and L_3: $Sug_{OPT}(L_2) = 0.75$ and $Sug_{OPT}(L_3) = 1$. We get $PR_*(L1) = 0.25$, $PR_*(L2) = 0.75$ and $PR_*(L3) = 1$. The preference is thus $L_3 \succ_{PR_*} L_2 \succ_{PR_*} L_1$.

Notice that both L_1 and L_3 can lead to a high utility (both consider that utility 1 is totally possible) but L_1 is very hazardous (the possibility of getting 0.25 by L_1 is equal to 0.75) while the lowest nonimpossible utility with L_3 is 0.5. L_2 is preferable to L_1 since the possibility of getting a bad utility with L_2 does not exceeds 0.25.

If the DM were more optimistic, setting for instance $e = 0.25$ (from level 0.25 all the utility degrees are considered as desirable opportunities) the preference relation would rather be $L_3 \sim_{PR_*} L_1 \succ_{PR_*} L_2$.

This criterion is radically different from the two other generalizations of R_*, namely OR_* and U_{R_*}, since with PR_* the hazardous area is defined from the aggregation of possibility and utility degrees. The following definition allows us to depict the area of lotteries which are hazardous for PR_* in a dimensional space (see Fig. 7(a)).

Definition 10. A lottery is hazardous for criterion PR_* iff $\exists i = 1, \ldots, n$ s.t.$\lambda_i < e$ and $\pi_i > 1 - e$.

Proposition 7. *$Sug_{PES}(L) < e$ iff L is hazardous.*

Example 9. Let us illustrate the two dimensional visualization of hazardous and desirable lotteries (Fig. 7). Suppose that $e = 0.75$. Consider lotteries $L_1 = (\langle 0.75/0.25, 1/0.5, 1/0.75\rangle$ and $L_2 = \langle 0.25/0, 0.5/0.25, 0.25/0.5, 1/0.75\rangle$. L_1 is in the PR_*

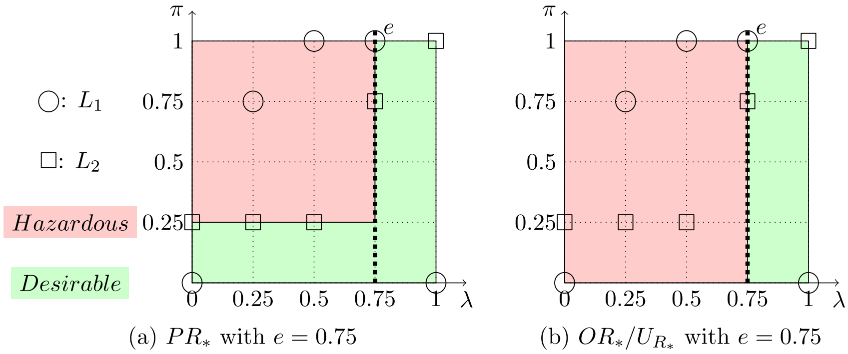

Fig. 7. A two dimensional view of the hazardous areas (with $e = 0.75$); a lottery (for instance ○ ans □) is represented by points (λ_i, π_i) in $\Lambda \times \Lambda$.

hazardous area while L_2 is not (see Fig. 7(a)) This area is different from the hazardous area of OR_*/U_{R_*} (depicted in Fig. 7(b)).

Nevertheless, like OR_*, PR_* fails to satisfy dynamic consistency and monotonicity, as shown by the following counter example. That is why it cannot be used in the sequential context.

Example 10. Let $\Lambda = \{1, 0.8, 0.6, 0.4, 0.2, 0\}$, $e = 0.4$ $L = \langle 1/0.2 \rangle$, $L' = \langle 1/1, 1/0 \rangle$. $Sug_{PES}(L') = 0 < e$ and $Sug_{PES}(L) = 0.2 < e$: L and L' are hazardous. So Sug_{PES} is used to rank the lotteries and we get $L' \prec_{PR_*} L$.

Consider now the two compound lotteries $\langle 0.6/L, 1/L'' \rangle$ and $\langle 0.6/L', 1/L'' \rangle$ where $L'' = \langle 1/0.4 \rangle$ so $Reduction(\langle 0.6/L, 1/L'' \rangle) = \langle 0.6/0.2, 1/0.4 \rangle$ and $Reduction(\langle 0.6/L', 1/L'' \rangle) = \langle 0.6/1, 0.6/0, 1/0.4 \rangle$. We have:

$Sug_{PES}(Reduction(\langle 0.6/L, 1/L'' \rangle)) = Sug_{PES}(Reduction(\langle 0.6/L', 1/L'' \rangle)) = 0.4$: the two compound lotteries are desirable.

So PR_* uses Sug_{OPT} to rank them.

Since $Sug_{OPT}(Reduction(\langle 0.6/L, 1/L'' \rangle)) = 0.4$ and $Sug_{OPT}(Reduction(\langle 0.6/L', 1/L'' \rangle)) = 0.6$ we get $\langle 0.6/L', 1/L'' \rangle \succ_{PR_*} \langle 0.6/L, 1/L'' \rangle$ while $L' \prec_{PR_*} L$.

5. Conclusion

In this paper, we have shown how the R_* and R^* uninorms can be used for sequential decision under qualitative uncertainty. They constitute an appealing alternative to Hurwicz's criterion to model the behavior of a DM who is neither purely optimistic nor purely pessimistic: an optimal strategy can be computed in polytime, which satisfies *Tree reduction*, *Consequentialism* and *Dynamic Consistency*. Moreover, these utilities are purely qualitative. It is then natural to extend them to possibilistic (qualitative) decision trees [14]. We have thus proposed a possibilistic generalization of R_*, and shown that it preserves the main properties of R_*, namely decomposability and weak monotonicity. An optimal strategy can be computed in polytime (by dynamic programming) which satisfies the three natural assumptions of sequential decision making. In robust decision making, where performance guarantees are looked for, is a natural domain of application for the R_* uninorm (R^* is on the contrary too adventurous: one single possible opportunity carries the final decision, and this even if all the other utilities lay in the hazard interval). The possibilistic generalization of R^* has not been presented in the paper for the sake of brevity, and above all because it is less interesting in decision domains, but it is easy to develop and obeys the same properties.

The present work relies on a simple sequential framework, namely (possibilistic) decision trees. It can directly be applied to finite horizon possibilistic Markov decision processes [14] which are acyclic oriented decision graphs - the sole difference with decision trees is that a given decision node can be reached from several chance nodes - the algorithm is unchanged. The case of infinite horizon possibilistic Markov decision process [13] is one of the short term perspectives of the present work.

Beyond the application to sequential decision making, the present work extends a uninorm (R_*), which aggregates a vector, to a bi-dimensional problem (a possibilistic lottery is matrix: the first line contains the utilities and the second line their degrees of possibility). The generalization of uninorms to bi-dimensional problems, and more generally to multi-dimensional problems, is an exciting question for further research.

Declaration of competing interest

The authors confirm that the manuscript has been read and approved by all named authors and that there are no other persons who satisfied the criteria for authorship but are not listed. We further confirm that the order of authors listed in the manuscript has been approved by all of us.

Appendix A

Proofs of Section 3

Proof of Proposition 1. For the sake of brevity, $R^*(Reduction(\langle L, L'\rangle))$ will be written $R^*(\langle L, L'\rangle)$ in the following.

Because R^* and R_* are associative, $R^*(\langle L, L'\rangle) = \max(R^*(L), R^*(L')$ and $R_*(\langle L, L'\rangle) = \max(R_*(L)), R_*(L'))$.

Suppose that $L_1 \preceq_{R_*} L_2$. This happens in three cases:

- $\max(u^1_1, ..., u^1_{n^1}) \leq \max(u^2_1, ..., u^2_{n^2})$: (a) L_3 has its smallest element (u^*) smaller than e hence: $R_*(\langle L_1, L_3\rangle) = u^* = R_*(\langle L_2, L_3\rangle)$ or (b) all elements are greater than e hence $R_*(\langle L_1, L_3\rangle) = \max(R_*(L_1), R_*(L_3)) \leq \max(R_*(L_2), R_*(L_3)) = R_*(\langle L_2, L_3\rangle)$.
- $\min(u^1_1, ..., u^1_{n^1}) \leq \min(u^2_1, ..., u^2_{n^2})$ whatever L_3 we have: $R_*(\langle L_1, L_3\rangle) = \min\ (R_*(L_1), R_*(L_3)) \leq \min(R_*(L_2), R_*(L_3)) = R_*(\langle L_2, L_3\rangle)$.
- $\min(u^1_1, ..., u^1_{n^1}) \leq \max(u^2_1, ..., u^2_{n^2})$ that implies that $\min(u^2_1, ..., u^2_{n^2}) \geq e$ so whatever L_3 we have $R_*(\langle L_1, L_3\rangle) = \min(R_*(L_1), R_*(L_3))$ and $R_*(\langle L_2, L_3\rangle) = R_*(L_3)$ or $R_*(L_2)$ hence $R_*(\langle L_1, L_3\rangle) \leq R_*(\langle L_2, L_3\rangle)$.

So, R_* is satisfies weak monotonicity.

Suppose that $L_1 \preceq_{R^*} L_2$. This happens in three cases:

- $\max(u^1_1, ..., u^1_{n^1}) \leq \max(u^2_1, ..., u^2_{n^2})$ whatever L_3 we have: $R^*(\langle L_1, L_3\rangle) = \max\ (R^*(L_1), R^*(L_3)) \leq \max\ (R^*(L_2), R^*(L_3)) = R^*(\langle L_2, L_3\rangle)$.
- $\min(u^1_1, ..., u^1_{n^1}) \leq \min(u^2_1, ..., u^2_{n^2})$ so we have: $\max(u^1_1, ..., u^1_{n^1}, u^2_1, ..., u^2_{n^2}) < e$ hence if L_3 has its largest element greater than or equal to e we have $R^*(\langle L_1, L_3\rangle) = R^*(L_2, L_3)$ else $R^*(\langle L_1, L_3\rangle) = \min(R^*(L_1), R^*(L_3)) \leq \min(R^*(L_2), R^*(L_3)) = R^*(\langle L_2, L_3\rangle)$.
- $\min(u^1_1, ..., u^1_{n^1}) \leq \max(u^2_1, ..., u^2_{n^2})$ that implies that: $\max(u^1_1, ..., u^1_{n^2}) < e$ so we have: $R^*(\langle L_1, L_3\rangle) = R^*(L_3)$ if $\max(u^3_1, ..., u^3_{n^2}) \geq e$ else $\min(R^*(L_1), R^*(L_3))$ and $R_*(\langle L_2, L_3\rangle) = \max(R^*(L_2), R^*(L_3))$ hence $R_*(\langle L_1, L_3\rangle) \leq R_*(\langle L_2, L_3\rangle)$.

So, R^* is satisfies weak monotonicity. □

Proof of Proposition 2. A criterion O satisfies the decomposition principle iff whatever L, L', $O(\langle L, L'\rangle) = O(\langle O(L), O(L')\rangle)$.

We must prove that $R^*(\langle L1, L2\rangle) = R^*(\langle R^*(L1), R^*(L2)\rangle)$ (resp. $R_*(\langle L1, L2\rangle) = R_*(\langle R_*(L1), R_*(L2)\rangle)$).

For R^* we can distinguish three cases: (1) only one of the lotteries has an element greater or equal than e (suppose that is $L1$); (2) both have an element greater or equal than e ; (3) none of them has an element greater than e.

1 $R^*(L1) = \max(u^1_1, ..., u^1_{n^1}) \geq e$ and $R^*(L2) = \min(u^2_1, ..., u^2_{n^1}) < e$ so $R^*(\langle R^*(L1), R^*(L2)\rangle) = \max(R^*(L1), R^*(L2)) = R^*(\langle L1, L2\rangle)$
2 $R^*(L1) = \max(u^1_1, ..., u^1_{n^1})$ and $R^*(L2) = \max(u^2_1, ..., u^2_{n^1})$ so $R^*(\langle R^*(L1), R^*(L2)\rangle) = R^*(\langle L1, L2\rangle)$
3 $R^*(L1) = \min(u^1_1, ..., u^1_{n^1})$ and $R^*(L2) = \min(u^2_1, ..., u^2_{n^1})$ so $R^*(\langle R^*(L1), R^*(L2)\rangle) = R^*(\langle L1, L2\rangle)$.

For R_* we can distinguish three cases: (1) only one of lotteries has all these elements greater or equal than e (suppose that is $L1$); (2) both have all these elements greater or equal than e; (3) none of them has all these elements greater than e.

1 $R_*(L1) = \max(u^1_1, ..., u^1_{n^1}) \geq e$ and $R_*(L2) = \min(u^2_1, ..., u^2_{n^1}) < e$ so $R_*(\langle R_*(L1), R_*(L2)\rangle) = \min(R_*(L1), R_*(L2)) = R_*(L2)$ and $R_*(\langle L1, L2\rangle) = \min(u^1_1, ..., u^1_{n^1}, u^2_1, ..., u^2_{n^1}) = R_*(L2)$
2 $R_*(L1) = \max(u^1_1, ..., u^1_{n^1})$ and $R_*(L2) = \max(u^2_1, ..., u^2_{n^1})$ so $R_*(\langle R_*(L1), R_*(L2)\rangle) = R_*(\langle L1, L2\rangle)$
3 $R_*(L1) = \min(u^1_1, ..., u^1_{n^1})$ and $R_*(L2) = \min(u^2_1, ..., u^2_{n^1})$ so $R_*(\langle R_*(L1, R_*(L2)\rangle) = R_*(\langle L1, L2\rangle)$. □

Proof of Proposition 3. See [4,5]. This is due to the fact that, when the lotteries are normalized, $Sug_{PES}(L) = U_{PES}(L)$ and $Sug_{OPT}(L) = U_{OPT}(L)$ where $U_{PES}(L) = \min_{\lambda_i} \max(\lambda_i, 1 - \pi_i)$ and $U_{OPT}(L) = \max_{\lambda_i} \min(\lambda_i, \pi_i)$. U_{PES} and U_{OPT} have been presented and axiomatized in [4]. Among other properties, it is shown that U_{PES} and U_{OPT} satisfy weak monotonicity, lottery reduction and certainty equivalence. □

Proofs of Section 4

The proofs of this Section are based on a series of Lemma that we detail now, before entering the proofs themselves.

Lemma 1. *If L is desirable, then $U_{R_*}(L) = Sug_{OPT}(L)$*

Indeed L is desirable, then all the utilities in its support are greater or equal to e, then $L = L^{\geq e}$. Moreover, all the $\lambda_i < e$ have a π_i equal to 0. Then $Sug_{\Pi}(L) = \max_{\lambda_i \in \Lambda} \min(\lambda_i, \Pi(L \geq \lambda_i)) = \max_{\lambda_i \geq e} \min(\lambda_i, \Pi(L \geq \lambda_i))$ which is precisely the value of U_{R_*} for desirable lotteries. So, $Sug_{\Pi}(L) = U_{R_*}(L)$.

Lemma 2. *If L is hazardous, $U_{R_*}(L) = \min(e^-, Sug_{PES}(L))$ e^- being the level just below e in Λ*

If $e = 0$, no lottery can lead to a utility strictly lower than e, so no lottery can be hazardous; So, if L is hazardous, $e > 0$ and e^- exists. Then:

$$\begin{aligned}
U_{R_*}(L) &= \max_{\lambda | \lambda < e} \min(\lambda, N(L \geq \lambda)) \\
&= \max_{\lambda | \lambda \leq e^-} \min(\lambda, N(L \geq \lambda)) \\
&= \max_{\lambda} \min(\lambda, N(L \geq \lambda), e^-) \\
&= \max_{\lambda} \min(\lambda, N(L \geq \lambda), e^-) \\
&= \min(\max_{\lambda} \min(\lambda, N(L \geq \lambda)), e^-) \\
&= \min(Sug_{PES}(L), e^-)
\end{aligned}$$

Lemma 3. *If L is hazardous and $\alpha > 0$, $\langle \alpha/L, \beta/L'' \rangle$ is hazardous.*

Proof. $L = \langle \pi_1/\lambda_1, \ldots, \pi_n/\lambda_n \rangle$ is hazardous, i. e. that at least a $\lambda_i < e$ receives a positive possibility degree π_i. So, at least one of the utilites in $\langle \alpha/L, \beta/L'' \rangle$ is lower than e - namely λ_i, coming from L. It receive possibility degree $\max(\min(\alpha, \pi_i), \min(\beta, \pi_i''))$ which is positive since $\alpha > 0$ and π_i positive. Then $\langle \alpha/L, \beta/L'' \rangle$ is hazardous. □

Lemma 4. *If L is desirable and L'' is desirable, $\langle \alpha/L, \beta/L'' \rangle$ is desirable.*

Proof. since all the utility degrees having a positive possibility by L are greater than e (L is desirable) and all the utility degrees having a positive possibility by L'' are greater than e (L'' is desirable), all the utility degrees having a positive possibility in $\langle \alpha/L, \beta/L'' \rangle$ are greater than e: $\langle \alpha/L, \beta/L'' \rangle$ is desirable. □

Lemma 5. *If L is desirable then $U_{R_*}(L) \geq e$.*

Proof. L is desirable, $\min(\lambda \in S_L) \geq e$. Because L is normalized, $\Pi(L \geq e) = 1$. That is to say, there exists a $\lambda* \geq e$ such that $\Pi(L \geq \lambda*) = 1$, thus such that $\min(\lambda*, \Pi(L \geq \lambda*)) \geq e$.

So, $U_{R_*}(L) = \max_{\lambda in S_L} \min(\lambda, \Pi(L \geq \lambda)) \geq e$. □

Lemma 6. *If L is hazardous then $U_{R_*}(L) < e$.*

Proof. When L is hazardous, $U_{R_*}(L) = \max_{\lambda_i < e} \min(\lambda_i, N(L \geq \lambda_i))$. So for each λ_i in the max, $\min(\lambda_i, N(L \geq \lambda_i)) < e$. So $U_{R_*}(L) < e$. □

Lemma 7. *If L is desirable then $Sug_{PES}(L) \geq e$*

Proof. When L is desirable $N(a \geq e) = 1$ and $\max_{\lambda_i} \min(\lambda_i, N(L \geq \lambda_i)) \geq \min(e, N(a \geq e)) = e$ □

Lemma 8.

$$Sug_{PES}(Reduction(\langle \alpha/L, \beta/L' \rangle)) \geq \min(Sug_{PES}(L), Sug_{PES}(L')).$$

Proof. We know that

$$N(a \geq \lambda) = \min(\max(1 - \alpha, N^L(a \geq \lambda)), \max(1 - \beta, N^{L'}(a \geq \lambda)))$$

so $N(a \geq \lambda) \geq \min(N^L(a \geq \lambda), N^{L'}(a \geq \lambda))$ with $N^L(a \geq \lambda)$ the necessity measure of lottery L. □

Proof of Proposition 4.

- If $e=0$, then no lottery can be hazardous (there is no utility level below 0). So, whatever L, L is desirable. Applying Lemma 1 then $U_{R_*}(L)=Sug_{OPT}(L)$
- If $e=1$, the only desirable lottery is $\langle 1/1\rangle$; it is easy to check that $Sug_{PES}(\langle 1/1\rangle)=1=Sug_{OPT}(\langle 1/1\rangle)=U_{R_*}(\langle 1/1\rangle)$. Consider now a hazardous lottery. Then $U_{R_*}(L)=\min(e^-, Sug_{PES}(L))$. Because L is hazardous, there exits $\lambda_i\le e^-$ such that $\pi_i>0$. So, $Sug_{PES}(L)\le\max(1-\pi_i,\lambda_i)$ with $\lambda_i\le e^-$ and $1-\pi_i<1$ i. e. $1-\pi_i\le e^-$. So $Sug_{PES}(L)\le e^-$. Thus $U_{R_*}(L)=\min(e^-, Sug_{PES}(L))=Sug_{PES}(L)$
- Suppose that all the utilities in the support of L are totally possible. Then $\Pi(L\ge\lambda_i)=1$ for any λ_i in the support S_L of L. If L is desirable, $U_{R_*}(L)=\max_{\lambda_i\in S_L}\min(\lambda_i,\Pi(L\ge\lambda_i))=\max_{\lambda_i\in S_L}\min(\lambda_i,1)=\max_{\lambda_i\in S_L}\lambda_i=R_*(S_L)$.
  Suppose now that L is hazardous and let $\lambda_j=arg\min S_L$ - so $R_*(S_L)=\lambda_j$ and $N(L\ge\lambda_j)=1$; because all the utilities in S_L are totally possible, $\pi_j=1$ and thus $N(L\ge\lambda_i)=0$ for any $\lambda_i\rangle\lambda_j$. Thus $U_{R_*}(L)=\max_{\lambda_i<e}\min(\lambda_i, N(L\ge\lambda_i))=\min(\lambda_j,1)=\lambda_j=R_*(S_L)$. □

Proof of Proposition 5. From Lemmas 3, 5 and Proposition 3 we can reduce the set of cases to study only two cases:
$U_{R_*}(Reduction(\langle\alpha/U_{R_*}(L),\beta/U_{R_*}(L')\rangle))$ is equal either to
$\min(Sug_{PES}(\langle\alpha/\min(Sug_{PES}(L),e^-),\beta/\min(Sug_{PES}(L'),e^-)\rangle)),e^-)$,
or to
$\min(Sug_{PES}(\langle\alpha/\min(Sug_{PES}(L),e^-),\beta/Sug_{OPT}(L')\rangle)),e^-)$.

Case 1: Suppose that: $U_{R_*}(Reduction(\langle\alpha/U_{R_*}(L),\beta/U_{R_*}(L')\rangle))=\min(Sug_{PES}(\langle\alpha/\min(Sug_{PES}(L),e^-),\beta/\min(Sug_{PES}(L'),e^-)\rangle)),e^-)$.

From Lemma 8 we have:

$$\begin{aligned}&\min(Sug_{PES}(\langle\alpha/\min(Sug_{PES}(L),e^-),\beta/\min(Sug_{PES}(L'),e^-)\rangle)),e^-)\\&=\min(Sug_{PES}(\langle\alpha/Sug_{PES}(L),\beta/Sug_{PES}(L')\rangle)),e^-)\\&=\min(Sug_{PES}(Reduction(\langle\alpha/L,\beta/L'\rangle),e^-)\\&=U_{R_*}(Reduction(\langle\alpha/L,\beta/L'\rangle)\end{aligned}$$

Case 2: Suppose that $U_{R_*}(Reduction(\langle\alpha/U_{R_*}(L),\beta/U_{R_*}(L')\rangle))=\min(Sug_{PES}(\langle\alpha/\min(Sug_{PES}(L),e^-),\beta/Sug_{OPT}(L')\rangle)),e^-)$.

From Lemmas 6 and 7 we have $\min(Sug_{OPT}(L),e^-)=\min(Sug_{PES}(L),e^-)=e^-\ge\min(Sug_{PES}(L),e^-)$ and using Lemma 8 we get:

$$\min(Sug_{PES}(\langle\alpha/\min(Sug_{PES}(L),e^-),\beta/\min(Sug_{PES}(L),e^-)\rangle)),e^-)$$

Hence, we recover Case 1 back. □

Proof of Proposition 6. We have to prove that if $U_{R_*}(L)\le U_{R_*}(L')$, then whatever α,β such that $\max(\alpha,\beta)=1$, whatever L'', $U_{R_*}(Reduction(\langle\alpha/L,\beta/L''\rangle))\le U_{R_*}(Reduction(\langle\alpha/L',\beta/L''\rangle))$.

The proof is based on the fact that $U_{R_*}(L)=Sug_{OPT}(L)$ for desirable lotteries, $U_{R_*}(L)=\min(e^-, Sug_{PES}(L))$ for hazardous ones, while Sug_{OPT} and Sug_{PES} are monotonic. Let us go step by step. We distinguish six cases, depending on whether the lotteries are hazardous or desirable.

Case 1: L, L' and L'' are hazardous. Then $\langle\alpha/L,\beta/L''\rangle$ and $\langle\alpha/L',\beta/L''\rangle$ are hazardous (Lemma 3).

From Proposition 5, Lemma 2 and the fact that Sug_{PES} is monotonic, we have:

$$\begin{aligned}U_{R_*}(Reduction(\langle\alpha/L,\beta/L''\rangle))&=\min(Sug_{PES}(Reduction(\langle\alpha/L,\beta/L''\rangle),e^-)\\&\le\min(Sug_{PES}(Reduction(\langle\alpha/L',\beta/L''\rangle),e^-)\\&=U_{R_*}(Reduction(\langle\alpha/L',\beta/L''\rangle)).\end{aligned}$$

Case 2: L and L' are hazardous, L'' is desirable. Then $\langle\alpha/L,\beta/L''\rangle$ and $\langle\alpha/L',\beta/L''\rangle$ are hazardous (Lemma 3). Using the same arguments than Case 1 we have:

$$\begin{aligned}U_{R_*}(Reduction(\langle\alpha/L,\beta/L''\rangle))&=\min(Sug_{PES}(Reduction(\langle\alpha/L,\beta/L''\rangle),e^-)\\&\le\min(Sug_{PES}(Reduction(\langle\alpha/L',\beta/L''\rangle),e^-)\\&=U_{R_*}(Reduction(\langle\alpha/L',\beta/L''\rangle)).\end{aligned}$$

Case 3: L and L'' are hazardous, L' is desirable. Then $\langle\alpha/L,\beta/L''\rangle$ and $\langle\alpha/L',\beta/L''\rangle$ are hazardous (Lemma 3).

Using the same arguments than in Case 1 we have:

$$U_{R_*}(Reduction(\langle\alpha/L,\beta/L''\rangle)) = \min(Sug_{PES}(Reduction(\langle\alpha/L,\beta/L''\rangle), e^-)$$
$$\leq \min(Sug_{PES}(Reduction(\langle\alpha/L',\beta/L''\rangle), e^-)$$
$$= U_{R_*}(Reduction(\langle\alpha/L',\beta/L''\rangle)).$$

Case 4: L is hazardous, L' and L'' are desirable. Then $\langle\alpha/L,\beta/L''\rangle$ is hazardous (Lemma 3) and $\langle\alpha/L',\beta/L''\rangle$ is desirable (Lemma 4). So $U_{R_*}(\langle\alpha/L',\beta/L''\rangle) \geq e$ (Lemma 5), and $U_{R_*}(\langle\alpha/L,\beta/L''\rangle) \leq e^-$ (Lemma 6) so we have

$$U_{R_*}(Reduction(\langle\alpha/L,\beta/L''\rangle)) \leq e^- < e \leq U_{R_*}(Reduction(\langle\alpha/L',\beta/L''\rangle)).$$

Case 5: L and L' are desirable, L'' is hazardous. Then $\langle\alpha/L,\beta/L''\rangle$ and $\langle\alpha/L',\beta/L''\rangle$ are hazardous (Lemma 3).

Adding Lemmas 7 and 8 to the arguments used in Case 1 we have:

$$U_{R_*}(Reduction(\langle\alpha/L,\beta/L''\rangle)) = \min(Sug_{PES}(Reduction(\langle\alpha/L,\beta/L''\rangle), e^-)$$
$$\leq \min(Sug_{PES}(Reduction(\langle\alpha/L',\beta/L''\rangle), e^-)$$
$$= U_{R_*}(Reduction(\langle\alpha/L',\beta/L''\rangle)).$$

Case 6: L, L' and L'' are desirable. So $\langle\alpha/L,\beta/L''\rangle$ and $\langle\alpha/L',\beta/L''\rangle$ are desirable (Lemma 4). So, $U_{R_*} = Sug_{OPT}$ for the four lotteries. The monotonicity of U_{R_*} straightforwardly follow from the monotonicity of Sug_{OPT}

Hence U_{R_*} is weakly monotonic. □

Proof of Proposition 7. If $Sug_{PES}(L) = \min_i \max(\lambda_i, 1-\pi_i) < e$ iff there exists at least one i $\max(\lambda_i, 1-\pi_i) < e$ then the lottery is hazardous. $\max(\lambda_i, 1-\pi_i) < e$ iff $\lambda_i < e$ and $\pi_i > 1-e$. If $Sug_{PES}(L) = \min_i \max(\lambda_i, 1-\pi_i) \geq e$ iff $\forall i$ $\max(\lambda_i, 1-\pi_i) \geq e$ then the lottery is not hazardous. $\max(\lambda_i, 1-\pi_i) \geq e$ iff $\lambda_i \geq e$ or $\pi_i \leq 1-e$. □

References

[1] E.F. McClennen, Rationality and Dynamic Choice: Foundational Explorations, Cambridge University Press, 1990.
[2] D. Dubois, H. Fargier, R. Guillaume, C. Thierry, Deciding under ignorance: in search of meaningful extensions of the Hurwicz criterion to decision trees, in: Strengthening Links Between Data Analysis and Soft Computing, Springer, 2014, pp. 3–11.
[3] D. Dubois, H. Prade, Possibility Theory, Plenum Press, New York, 1988.
[4] D. Dubois, H. Prade, Possibility theory as a basis for qualitative decision theory, in: Proceedings of IJCAI 1995, 1995, pp. 1924–1932.
[5] D. Dubois, H. Prade, R. Sabbadin, Qualitative decision theory with Sugeno integrals, in: Proceedings of UAI 1998, 1998, pp. 121–128.
[6] D. Dubois, J.L. Marichal, H. Prade, M. Roubens, R. Sabbadin, The use of the discrete Sugeno integral in decision-making: a survey, Int. J. Uncertain. Fuzziness Knowl.-Based Syst. 9 (5) (2001) 539–561.
[7] L. Hurwicz, Optimality Criteria for Decision Making under Ignorance, Cowles commission papers: Statistics no 370, 1951.
[8] N. Huntley, M. Troffaes, An efficient normal form solution to decision trees with lower previsions, in: Soft Methods for Handling Variability and Imprecision, 2008, pp. 419–426.
[9] J.-Y. Jaffray, Rational decision making with imprecise probabilities, in: Proceedings of ISIPTA 1999, 1999, pp. 183–188.
[10] D. Kikuti, F. Cozman, R. Filho, Sequential decision making with partially ordered preferences, Artif. Intell. 175 (2011) 1346–1365.
[11] H. Raiffa, Decision Analysis: Introductory Lectures on Choices Under Uncertainty, Addison-Wesley, Reading, MA, 1968.
[12] R. Sabbadin, A possibilistic model for qualitative sequential decision problems under uncertainty in partially observable environments, in: UAI, 1999, pp. 567–574.
[13] R. Sabbadin, Possibilistic Markov decision processes, Eng. Appl. Artif. Intell. 14 (3) (2001) 287–300.
[14] R. Sabbadin, H. Fargier, J. Lang, Towards qualitative approaches to multi-stage decision making, Int. J. Approx. Reason. 19 (1998) 441–471.
[15] J. Von Neumann, O. Morgenstern, Theory of Games and Economic Behavior, Princeton University Press, Princeton, 1947.
[16] R. Yager, A. Rybalov, Uninorm aggregation operators, Fuzzy Sets Syst. 80 (1) (1996) 111–120.
[17] L. Zadeh, Fuzzy sets as the basis for a theory of possibility, Fuzzy Sets Syst. 1 (1978) 3–28.