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Commuting Double Sugeno Integrals

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In decision problems involving two dimensions (like several agents in uncertainty) the properties of expected utility ensure that the result of a two-stepped procedure evaluation does not depend on the order with which the aggregations of local evaluations are performed (e.g., agents first, uncertainty next, or the converse). We say that the aggregations on each dimension *commute*. In a previous conference paper, Ben Amor, Essghaier and Fargier have shown that this property holds when using pessimistic possibilistic integrals on each dimension, or optimistic ones, while it fails when using a pessimistic possibilistic integral on one dimension and an optimistic one on the other. This paper studies and completely solves this problem when more general Sugeno integrals are used in place of possibilistic integrals, leading to double Sugeno integrals. The results show that there are capacities other than possibility and necessity measures that ensure commutation of Sugeno integrals. Moreover, the relationship between two-dimensional capacities and the commutation property for their projections is investigated.

Keywords: Capacities, Sugeno integrals, possibility theory, commutation

1. Introduction and Motivation

In various applications where information fusion or multifactorial evaluation is needed, an aggregation process is carried out as a two-stepped procedure whereby several local fusion operations are performed in parallel and then the partial results are merged. It may sometimes be natural to demand that the final result does not depend on the order with which we perform the aggregation steps, because there may be no reason to perform either of the steps first.

For instance, consider a multi-person decision problem under uncertainty, where it is assumed that the same knowledge is shared among the persons and a collective decision must be found. Each alternative is evaluated by a matrix of ratings where the rows represent evaluations by persons and the columns represent evaluations for states of the world. One may, for each row, merge the ratings according to each column with some aggregation operation and thus get a rating for each person across all states, and then merge the persons opinions into a collective one, using another aggregation operation (i.e., follow the so-called *ex-ante* approach). Alternatively, one may decide first to merge the ratings in each column, thus forming the collective rating according to each state, and then merge these collective ratings across possible states, taking the uncertainty into account (i.e., adopt an *ex-post* approach).

The same considerations apply when we consider multi-criteria multiple-agent problems. Should we aggregate the ratings of agents for each fixed criterion first, and find the global evaluation, or find the global evaluation for each agent and then compute the collective overall rating? Even if we find it natural that the two procedures should deliver the same results in any sensible approach, the problem is that this natural outcome is not mathematically obvious at all. When the two procedures yield the same results, the aggregation operations are said to *commute*.[20]

In decision under risk for instance, the *ex-ante* and *ex-post* approaches are equivalent (the aggregations commute) if and only if the agent preferences are merged under a utilitarian view:[16,19] this is because the expected utility of a sum is equal to the sum of the expected utilities. With an egalitarian collective utility function this is no longer the case, which leads to a timing effect: the *ex-ante* approach (minimum of the expected utilities) is not equivalent to the *ex-post* one (the expected utility of the minimum of the utilities). Some authors[16,19] have proposed representation theorems stating that in a classical decision-theoretic setting, commutation occurs if and only if the two aggregation are weighted averages, that is the weighted average of expected utilities is the same as the expected collective utility.

More recently, Ben Amor *et al.*[2,4] have reconsidered the problem in the setting of qualitative decision theory under uncertainty. In this framework, utility values and plausibility degrees of states belong to a finite bounded chain. These authors have pointed out that commuting alternatives to weighted average operations exist, namely qualitative possibilistic integrals,[10] that is Sugeno integrals with respect to possibility or necessity measures, respectively corresponding to optimistic and pessimistic possibilistic integrals. Equivalently, they are weighted versions of minimum and maximum, first considered by Yager,[24] Whalen,[23] as well as Dubois and Prade.[7] Namely pessimistic possibilistic integrals commute, as well as optimistic ones, but a pessimistic possibilistic integral generally does not commute with an optimistic one.

The question considered in this paper is whether there exist other uncertainty measures, in the qualitative setting of Sugeno integrals, for which this commutation result holds, replacing pessimistic and optimistic utility functionals by Sugeno integrals with respect to general capacities. In other terms, we consider the commutation problem in *double* Sugeno integrals. Moreover we introduce the concept of decomposable two-dimensional (2D) capacity and we show the relationship between the decomposability of such a capacity and the commutation of Sugeno integrals based on its projections.[a]

The paper is organized as follows. Section 2 recalls commutation results obtained so-far in the setting of possibilistic integrals for qualitative decision under uncertainty. Section 3 provides the necessary background on Sugeno integrals, which are more general aggregation operations than weighted min and max, and introduces double Sugeno integrals. Section 4 provides necessary and sufficient condition for their commutation. In particular it lays bare the explicit form of capacities that commute. Section 5 focuses on decomposable 2D capacities on a finite space, which can be recovered from their projections. This section shows the link with the commutation problem, and proves that when two Sugeno integrals commute, they can be expressed in a symmetric way as a 2D Sugeno integral with respect to a decomposable 2D capacity.

2. Multiagent Decision Evaluation under Possibilistic Uncertainty

In the framework of possibilistic decision under uncertainty,[10] let $\mathcal{X} = \{x_1, \cdots, x_n\}$ be a set of states and $\mathcal{Y} = \{y_1, \cdots, y_p\}$ be a set of agents. Let L be a finite totally ordered scale with top 1, bottom 0, equipped with an order-reversing operation denoted by $1 - (\cdot)$ (it is involutive and such that $1 - 1 = 0$ and $1 - 0 = 1$). A possibility distribution π captures the common knowledge of the agents: the possibility degree π_i represents the plausibility of state x_i for the agents.[b] A weight vector $w = (w_1, \cdots, w_p) \in [0, 1]^p$ represents the importance degrees of agents. It is also formally modeled as a possibility distribution on $\mathcal{Y}$: the possibility degree w_i represents the importance of agent y_j. By definition, a normalization condition holds, of the form $\max_{i=1}^n \pi_i = \max_{i=1}^n w_i = 1$.

A potential decision[c] is evaluated by a function u: $\mathcal{X} \times \mathcal{Y} \to L$ where $u(x_i, y_j)$ is the degree of utility of the decision for the agent y_j, if the state of the world is x_i. Note that for the sake of simplicity, we make the assumption that L is common finite scale for utility ratings, agent importance weights and plausibilities of states, which can be challenged in practice.

[a] Some results in this paper have been presented at the 2018 MDAI conference.[6]

[b] One might have equivalently considered a multicriteria multiagent setting, interpreting X as a set of criteria, and π_i as the degree of relevance of criterion x_i.

[c] In this paper, we do not use a special symbol for decisions. In decision under uncertainty, a decision is a mapping f from a state space $\mathcal{S}$ to a set of consequences $\mathcal{X}$. Utilities are attached to consequences, and the utility of decision f if the state is s is $u(x)$ where $x = f(s)$. In this paper, we do not distinguish between states and their consequences for decisions, assuming the decision is fixed, and only use the set $\mathcal{X} = f(\mathcal{S})$.

In order to evaluate the merit of a decision for an agent y_j under possibilistic uncertainty described by the possibility distribution π, two qualitative possibilistic integrals have been used[10] depending on whether agent y_j has a pessimistic or an optimistic attitude in the face of uncertainty:

Pessimistic possibilistic integral: $U_j^-(\pi, u) = \min_{x_i \in \mathcal{X}} \max(1 - \pi_i, u(x_i, y_j))$

Optimistic possibilistic integral: $U_j^+(\pi, u) = \max_{x_i \in \mathcal{X}} \min(\pi_i, u(x_i, y_j))$

Now fixing the state x_i, we consider its collective utility obtained by merging agent preferences. In the following we consider an egalitarian attitude expressed by an aggregation operation of the form $U_i^{min}(w, u) = \min_{y_j \in \mathcal{Y}} \max(1 - w_j, u(x_i, y_j))$. As a consequence, there are two possible approaches for egalitarian aggregations of pessimistic decision-makers, and two possible approaches for egalitarian aggregations of pessimistic decision-makers.[2]

ex-post pessimistic approach

$$\begin{aligned} U_{post}^{-min}(\pi, w, u) &= U^-(\pi, U_i^{\min}(w, u(x_i, \cdot))) \\ &= \min_{x_i \in \mathcal{X}} \max(1 - \pi_i, \min_{y_j \in \mathcal{Y}} \max(u(x_i, y_j), 1 - w_j)) \,. \end{aligned}$$

ex-ante pessimistic approach

$$\begin{aligned} U_{ante}^{-min}(\pi, w, u) &= U^{\min}(w, U_j^-(\pi, u(\cdot, y_j)) \\ &= \min_{y_j \in \mathcal{Y}} \max(1 - w_j, \min_{x_i \in \mathcal{X}} \max(u(x_i, y_j), 1 - \pi_i)) \,. \end{aligned}$$

ex-post optimistic approach

$$\begin{aligned} U_{post}^{+min}(\pi, w, u) &= U^+(\pi, U_i^{\min}(w, u(x_i, \cdot))) \\ &= \max_{x_i \in \mathcal{X}} \min(\pi_i, \min_{y_j \in \mathcal{Y}} \max(u(x_i, y_j), 1 - w_j)) \,. \end{aligned}$$

ex-ante optimistic approach

$$\begin{aligned} U_{ante}^{+min}(\pi, w, u) &= U^{\min}(w, U_j^+(\pi, u(\cdot, y_j)) \\ &= \min_{y_j \in \mathcal{Y}} \max(1 - w_j, \max_{x_i \in \mathcal{X}} \min(u(x_i, y_j), \pi_i)) \,. \end{aligned}$$

Essghaier *et al.*[2] noticed that the first two quantities are equal to $\min_{x_i \in \mathcal{X}, y_j \in \mathcal{Y}} \max(1 - \pi_i, u(x_i, y_j), 1 - w_j))$, so that the two aggregations U_j^- and U_i^{min} commute.

In the optimistic case, the two integrals do no coincide: Essghaier *et al.*[2] have shown that we only have an inequality $U_{ante}^{+min}(\pi, w, u) \geq U_{post}^{+min}(\pi, w, u)$, with no equality in general. The following counterexample shows that the latter inequality can be strict and even extreme in simple cases, even when the rating scale is reduced to $\{0, 1\}$:

Example 1. Counterexample

Let $\mathcal{X} = \{x_1, x_2\}$, $\mathcal{Y} = \{y_1, y_2\}$. Suppose $\pi_i = 1$, and $w_i = 1, \forall i = 1, 2$, and a Boolean utility function $u(x_1, y_1) = u(x_2, y_2) = 1$ and $u(x_2, y_1) = u(x_1, y_2) = 0$. $U^{+min}_{post}(\pi, w, u) = 0$ since:

$\max(\min(1, \min(\max(1-1, 1), \max(1-1, 0)), \min(1, \min(\max(1-1, 0), \max(1-1, 1))) = 0.$

But $U^{+min}_{ante}(\pi, w, u) = 1$, since computed as

$\min(\max(1-1, \max(\max(1, 1), \max(0, 1)), \max(1-1, \max(\max(0, 1), \max(1, 1))))$
$= 1.$ □

To get insights into these results, we shall study the generalized form of the weighted minimum and maximum, namely Sugeno integral.

3. Composition of Sugeno Integrals

We first recall definitions and properties of Sugeno integrals and then consider the composition of Sugeno integrals on two spaces, yielding double Sugeno integrals.

3.1. *Sugeno integral*

Consider a set $\mathcal{X} = \{x_1, \cdots, x_n\}$ and L a totally ordered scale as previously.

A decision to be evaluated is represented by a function u: $\mathcal{X} \to L$ where $u(x_i)$ is, for instance, the degree of utility of the decision in state x_i.

In the definition of Sugeno integral the relative likelihood of subsets of states is represented by a capacity (or fuzzy measure), which is a set function μ: $2^{\mathcal{X}} \to L$ that satisfies $\mu(\emptyset) = 0$, $\mu(\mathcal{X}) = 1$ and $A \subseteq B$ implies $\mu(A) \leq \mu(B)$. The conjugate capacity of μ is defined by $\mu^c(A) = 1 - \mu(A^c)$ where A^c is the complement of A. Sugeno integral is originally defined as follows.[21,22]

Definition 1. The Sugeno integral (S-integral for short) of a function u: $\mathcal{X} \to L$ with respect to a capacity μ is defined by:

$$S_\mu(u) = \max_{\alpha \in L} \min(\alpha, \mu(u \geq \alpha))$$

where $\mu(u \geq \alpha) = \mu(\{x_i \in \mathcal{X} | u(x_i) \geq \alpha\})$.

A Sugeno integral can be equivalently written under various forms,[17,21] especially:

$$S_\mu(u) = \max_{A \subseteq \mathcal{X}} \min(\mu(A), \min_{x_i \in A} u(x_i)) = \min_{A \subseteq \mathcal{X}} \max(\mu(A^c), \max_{x_i \in A} u(x_i)) \,. \tag{1}$$

Recall that a (*lattice*) *polynomial function* on L is any map q: $L^n \to L$ obtained as a composition of the lattice operations $\wedge$ and $\vee$, the projections (here, $u(x_i)$) and the constant functions $\lambda \in L$. Here, min and max on a finite chain. As observed by Marichal,[17] Sugeno integrals exactly coincide with those lattice polynomials which are *idempotent*, that is, which satisfy $q(\lambda, \dots, \lambda) = \lambda$, for every $\lambda \in L$. In fact, it suffices to verify this identity for $\lambda \in \{0, 1\}$, that is, $q(\mathbf{1}_X) = 1$ and $q(\mathbf{1}_\emptyset) = 0$, where $\mathbf{1}_A$ denotes the characteristic function of set A ($\mathbf{1}_A(x) = 1$ if $x \in A$ and 0 otherwise). As shown by Goodstein,[12] polynomial functions over bounded distributive lattices (in particular, over bounded chains) have very neat normal form representations, namely disjunctive and conjunctive normal forms given, for idempotent ones, by the above expressions in (1).

In particular, every idempotent polynomial function $q\colon L^n \to L$ is uniquely determined by its restriction to $\{0, 1\}^n$. Also, since every lattice polynomial function is order-preserving, the coefficients in the disjunctive normal form are monotone increasing as well, i.e., $q(\mathbf{1}_I) \le q(\mathbf{1}_J)$ whenever $I \subseteq J$, i.e., they yield the capacity at work in the S-integral. Moreover, a function $f\colon \{0, 1\}^n \to L$ can be extended to a polynomial function over L if and only if it is order-preserving. These results only point out that

$$S_\mu(u) = \mu(A) \text{ if } u = \mathbf{1}_A\,.$$

When μ is a possibility measure Π,[25] such that $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$, i.e., a maxitive capacity, it is is entirely defined by a function $\pi\colon \mathcal{X} \to L$, called the possibility distribution associated to Π, by: $\Pi(A) = \max_{i \in A} \pi_i$. The conjugate $N(A) = 1 - \Pi(A^c)$ of a possibility measure is a necessity measure N,[8] i.e., a capacity such that $N(A \cap B) = \max(N(A), N(B))$ (minitivity) and it can be expressed in terms of the possibility distribution as $N(A) = \min_{i \notin A} 1 - \pi_i$. We thus get the following special cases of the Sugeno integral:

$$S_\Pi(u) = \max_{x_i \in \mathcal{X}} \min(\pi_i, u(x_i))\,, \tag{2}$$

$$S_N(u) = \min_{x_i \in \mathcal{X}} \max(1 - \pi_i, u(x_i))\,. \tag{3}$$

These are the weighted maximum and minimum operations that are used in the previous section to model optimistic and pessimistic qualitative utility functionals respectively.

The S-integral can be expressed in a non-redundant format by means of the qualitative Möbius transform of μ:[13]

$$\mu_\#(T) = \begin{cases} \mu(T) & \text{if } \mu(T) > \max_{x \in T}\ \mu(T \backslash \{x\}) \\ 0 & \text{otherwise} \end{cases}$$

as

$$S_\mu(u) = \max_{T \subseteq \mathcal{X} : \mu_\#(T) > 0} \min(\mu_\#(T), \min_{x_i \in T} u(x_i))\,.$$

Indeed, $\mu_\#$ enables the reconstruction of the capacity μ as $\mu(A) = \max_{T \subseteq A} \mu_\#(T)$. Subsets T of $\mathcal{X}$ for which $\mu_\#(T) > 0$ are called *focal sets* of μ (the set of focal sets of μ is denoted $\mathcal{F}(\mu)$). $\mathcal{F}(\mu)$ is the minimal family of sets which allow to recover the capacity (via $\mu_\#$). By construction, if $E, F \in \mathcal{F}(\mu)$ and $E \subset F$ then $\mu_\#(F) > \mu_\#(E) > 0$. As a matter of fact, it is clear that the qualitative Möbius transform of a possibility measure coincides with its possibility distribution: $\Pi_{\#}(A) = \pi(s)$ if $A = \{s\}$ and 0 otherwise.[9]

Lastly, the S-integral can be expressed in terms of Boolean capacities (i.e., of capacities that take their values in $\{0, 1\}$), namely to the λ-*cuts* of μ. Given a capacity μ on $\mathcal{X}$, for all $\lambda > 0, \lambda \in L$, let μ_λ: $2^{\mathcal{X}} \to \{0, 1\}$ (the λ-cut of μ) be the Boolean capacity defined by

$$\mu_\lambda(A) = \begin{cases} 1 & \text{if } \mu(A) \geq \lambda\,, \\ 0 & \text{otherwise.} \end{cases}$$

for all $A \subseteq \mathcal{X}$. It is clear that the capacity μ can be reconstructed from the μ_λ's as follows:

$$\mu(A) = \max_{\lambda>0} \min(\lambda, \mu_\lambda(A))\,.$$

Observe that the focal sets of a Boolean capacity μ_λ form an antichain of subsets (there cannot be any inclusion between them).

We can express S-integrals with respect to μ by means of the cuts of μ:

Proposition 1. $S_\mu(u) = \max_{\lambda>0} \min(\lambda, S_{\mu_\lambda}(u))$

Proof:

$$\begin{aligned} S_\mu(u) &= \max_{A\subseteq\mathcal{X}} \min(\max_{\lambda>0} \min(\lambda, \mu_\lambda(A)), \min_{i\in A} u(x_i)) \\ &= \max_{A\subseteq\mathcal{X}} \max_{\lambda>0} \min(\lambda, \mu_\lambda(A), \min_{i\in A} u(x_i)) \\ &= \max_{\lambda>0} \max_{A\subseteq\mathcal{X}} \min(\lambda, \mu_\lambda(A), \min_{i\in A} u(x_i)) \\ &= \max_{\lambda>0} \min(\lambda, \max_{A\subseteq\mathcal{X}} \min(\mu_\lambda(A), \min_{i\in A} u(x_i))\,, \end{aligned}$$

which is $\max_{\lambda>0} \min(\lambda, S_{\mu_\lambda}(u))$. □

Note that the expression $S_\mu(u) = \max_{\alpha\in L} \min(\alpha, \mu(u \geq \alpha))$ uses cuts of the utility function. It can be combined with Proposition 1 to yield:

$$S_\mu(u) = \max_{\alpha,\lambda\in L} \min(\alpha, \lambda, \mu_\lambda(u \geq \alpha))\,. \tag{4}$$

This expression can be simplified as follows:

Proposition 2. $S_\mu(u) = \max_{\lambda\in L} \min(\lambda, \mu_\lambda(u \geq \lambda))$.

Proof: Note that $\mu_\lambda(u \geq \alpha)$ does not increase with α nor λ. Suppose then that $S_\mu(u) = \min(\alpha^*, \lambda^*, \mu_{\lambda^*}(u \geq \alpha^*))$. If $\mu_{\lambda^*}(u \geq \alpha^*) = 1$, and $\alpha^* > \lambda^*$, then notice that $\mu_{\lambda^*}(u \geq \lambda^*) = 1$ as well. Likewise, if $\alpha^* < \lambda^*, \mu_{\alpha^*}(u \geq \alpha^*)) = 1$. If $\mu_{\lambda^*}(u \geq \alpha^*) = 0$, this is also true for $\mu_\lambda(u \geq \alpha)$ with $\alpha > \alpha^*$ and $\lambda > \lambda^*$. So we can assume $\alpha = \lambda$ in Eq. (4). □

These results, to our knowledge, have never been highlighted, and will be instrumental in the sequel.

3.2. *Double Sugeno integrals*

Let us now generalize the qualitative setting for multi-agent decision under uncertainty using aggregation based on capacities. Consider two sets: $\mathcal{X} = \{x_1, \cdots, x_n\}$, $\mathcal{Y} = \{y_1, \cdots, y_p\}$ and two fuzzy measures $\mu_{\mathcal{X}}$ on $\mathcal{X}$ and $\mu_{\mathcal{Y}}$ on $\mathcal{Y}$.

Definition 2. The *double Sugeno* integral of a function u: $\mathcal{X} \times \mathcal{Y} \to L$, denoted by $S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}(u))$ is the Sugeno integral $S_{\mu_{\mathcal{X}}}(f)$, according to $\mu_{\mathcal{X}}$ of the function f: $L^n \to L$ defined by $f(x_i) = S_{\mu_{\mathcal{Y}}}(u(x_i, \cdot))$.

There are two double S-integrals, according to whether we apply S-integral on $\mathcal{X}$ first, or on $\mathcal{Y}$ first (as presented in Figure 1):

- $S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}(u))$ denotes the Sugeno integral, according to $\mu_{\mathcal{X}}$, of the function defined as $x_i \mapsto S_{\mu_{\mathcal{Y}}}(u(x_i, \cdot)), x_i \in \mathcal{X}$.
- $S_{\mu_{\mathcal{Y}}}(S_{\mu_{\mathcal{X}}}(u))$ denotes the Sugeno integral, according to $\mu_{\mathcal{Y}}$, of the function defined as $y_j \mapsto S_{\mu_{\mathcal{X}}}(u(\cdot, y_j)), y_j \in \mathcal{Y}$.

$$
\begin{array}{cccc}
 & \begin{array}{ccc} y_1 & \dots & y_p \end{array} & & \\
\begin{array}{c} x_1 \\ \\ x_n \end{array} & \begin{pmatrix} u(x_1, y_1) & \dots & u(x_1, y_p) \\ \vdots & \ddots & \vdots \\ u(x_n, y_1) & \dots & u(x_n, y_p) \end{pmatrix} & \begin{array}{c} \to \\ \vdots \\ \to \end{array} & \begin{pmatrix} S_{\mu_{\mathcal{Y}}}(u(x_1, \cdot)) \\ \vdots \\ S_{\mu_{\mathcal{Y}}}(u(x_n, \cdot)) \end{pmatrix} \\
 & & & \downarrow \\
 & \downarrow \dots \downarrow & & S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}(u(x_1, \cdot)), \cdots, S_{\mu_{\mathcal{Y}}}(u(x_n, \cdot))) \\
 & \left(S_{\mu_{\mathcal{X}}}(u(\cdot, y_1)), \cdots, S_{\mu_{\mathcal{X}}}(u(\cdot, y_p))\right) & \to & S_{\mu_{\mathcal{Y}}}(S_{\mu_{\mathcal{X}}}(u(\cdot, y_1)), \cdots, S_{\mu_{\mathcal{X}}}(u(\cdot, y_p)))
\end{array}
$$

Fig. 1. Two double Sugeno integrals.

That is to say:

$$
\begin{aligned}
S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}(u)) &= \max_{A \subseteq \mathcal{X}} \min(\mu_{\mathcal{X}}(A), \min_{x \in A} S_{\mu_{\mathcal{Y}}}(u(x, \cdot))) \\
&= \max_{A \subseteq \mathcal{X}} \min(\mu_{\mathcal{X}}(A), \min_{x \in A} \max_{B \subseteq \mathcal{Y}} \min(\mu_{\mathcal{Y}}(B), \min_{y \in B}(u(x, y)))) ,
\end{aligned}
$$

$$
\begin{aligned}
S_{\mu_{\mathcal{Y}}}(S_{\mu_{\mathcal{X}}}(u)) &= \max_{B \subseteq \mathcal{Y}} \min(\mu_{\mathcal{Y}}(B), \min_{y \in B} S_{\mu_{\mathcal{X}}}(u(\cdot, y))) \\
&= \max_{B \subseteq \mathcal{Y}} \min(\mu_{\mathcal{Y}}(B), \min_{y \in B} \max_{A \subseteq \mathcal{X}} \min(\mu_{\mathcal{X}}(A), \min_{x \in A}(u(x, y)))) .
\end{aligned}
$$

When $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(u)) = S_{\mu_\mathcal{Y}}(S_{\mu_\mathcal{X}}(u))$, the two double S-integrals are said to commute for function u. When this commutation property holds for all functions u, we shall simply say that the two Sugeno integrals commute.

In the terminology of multi-agent decision under uncertainty, the capacity $\mu_\mathcal{X}$ describes knowledge about the actual state of nature and $\mu_\mathcal{Y}$ on $\mathcal{Y}$ evaluates the importance of groups of agents. The double Sugeno integral $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(u))$ evaluates the collective worth of a decision under uncertainty, characterized by utility function u according to all agents whose importance is described by $\mu_\mathcal{Y}$, and given some common knowledge $\mu_\mathcal{X}$ about the states of the world.

It is clear that the egalitarian aggregations for pessimistic agents are of the form

$$U^{-min}_{post}(\pi, w, u) = S_{N_\mathcal{X}}(S_{N_\mathcal{Y}}(u)); \quad U^{-min}_{ante}(\pi, w, u) = S_{N_\mathcal{Y}}(S_{N_\mathcal{X}}(u))\,,$$

and the commutation of the two expressions, proved by Ben Amor *et al.*[2] and recalled in Section 2, can be expressed as $S_{N_\mathcal{X}}(S_{N_\mathcal{Y}}(u)) = S_{N_\mathcal{Y}}(S_{N_\mathcal{X}}(u))$. Likewise it can be checked that $S_{\Pi_\mathcal{X}}(S_{\Pi_\mathcal{Y}}(u)) = S_{\Pi_\mathcal{Y}}(S_{\Pi_\mathcal{X}}(u)) = \max_{x_i\in\mathcal{X}, y_j\in\mathcal{Y}} \min(\pi_i, w_j, u(x_i, y_j))$.

In the optimistic case, qualitative decision theory prescribes the use of a Sugeno integral based on a possibility measure on $\mathcal{X}$, and we get

$$U^{+min}_{post}(\pi, w, u) = S_{\Pi_\mathcal{X}}(S_{N_\mathcal{Y}}(u)); \quad U^{+min}_{ante}(\pi, w, u) = S_{N_\mathcal{Y}}(S_{\Pi_\mathcal{X}}(u))\,.$$

It has been pointed out that $S_{\Pi_\mathcal{X}}(S_{N_\mathcal{Y}}(u)) < S_{N_\mathcal{Y}}(S_{\Pi_\mathcal{X}}(u))$ in general, so that no commutation result can be obtained when one of the capacity is a necessity measure and the other one is a possibility measure. In summary, double S-integrals may commute, e.g., when the two capacities are both necessity measures or both possibility measures, but generally they do not — the previous example shows that the difference between the two double S-integrals can be maximal.

The interesting question is then whether commutation of double S-integrals takes place only when the capacities $\mu_\mathcal{X}$ and $\mu_\mathcal{Y}$ are both necessity measures or both possibility measures.

4. The Commutation of Sugeno Integrals

In this section, given two capacities on finite sets $\mu_\mathcal{X}$ on $\mathcal{X}$ and $\mu_\mathcal{Y}$ on $\mathcal{Y}$, we check for necessary and sufficient conditions under which commutation takes place, namely the following identity holds:

$$S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(u(x_1, \cdot)), \cdots, S_{\mu_\mathcal{Y}}(u(x_n, \cdot))) = S_{\mu_\mathcal{Y}}(S_{\mu_\mathcal{X}}((u(\cdot, y_1)), \ldots, S_{\mu_\mathcal{X}}(u(\cdot, y_p)))$$

or for short $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(u)) = S_{\mu_\mathcal{Y}}(S_{\mu_\mathcal{X}}(u))$. This question was considered from two points of view: for which functions u do S-integrals commute for all capacities on $\mathcal{X}$ and $\mathcal{Y}$? For which capacities do the S-integrals commute *for all functions u* (we then write $S_{\mu_\mathcal{Y}} \perp S_{\mu_\mathcal{X}}$)?

The first question is considered by Narukawa and Torra[18] for more general fuzzy integrals, and the second one by Behrisch *et al.*,[1] albeit in the larger setting of distributive lattices, for general lattice polynomials. However, their results are not easy to exploit. It is of interest to give an independent proof of these results for S-integrals valued on chains, as it is easier to grasp and it enables an explicit description of capacities ensuring commutation. Very recently, Halas *et al.*[15] have solved the problem on Cartesian products of 2-element spaces. Our proof solves the question for S-integrals on any finite sets. Like the one by Behrisch *et al.*,[1] it requires several lemmas, but is easier to read and simpler. In particular we can then explicitly lay bare pairs of commuting S-integrals, i.e., such that $S_{\mu_\mathcal{Y}} \perp S_{\mu_\mathcal{X}}$, based on capacities other than possibility measures and necessity measures.

4.1. *Necessary and sufficient conditions for commutation*

It is easy to see that a double S-integral $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(u))$ is actually an idempotent lattice polynomial p: $L^{|\mathcal{X}|+|\mathcal{Y}|} \to L$, since it is built using $\min, \max$, constants and variable terms of the form $u(x_i, y_j)$. Moreover, $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_{\mathcal{X}\times\mathcal{Y}}))=1$, $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_\emptyset)) = 0$. Using results on lattice polynomials, we conclude that *a double S-integral is always an S-integral on* $\mathcal{X} \times \mathcal{Y}$, based on a capacity that maps each $R \subseteq \mathcal{X} \times \mathcal{Y}$ to L the value $\kappa_{\mathcal{X}\mathcal{Y}}(R) = S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R))$. Namely noticing that $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R)) = S_{\mu_\mathcal{X}}(f_R)$, where $f_R(x_i) = \mu_\mathcal{Y}(\{y \in \mathcal{Y}\colon x_i R y\})$, is a monotonic function of R only, i.e., a capacity $\kappa_{\mathcal{X}\mathcal{Y}}$ on $\mathcal{X}\times\mathcal{Y}$, we can write any double S-integral as

$$S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(u)) = \max_{R\subseteq\mathcal{X}\times\mathcal{Y}} \min(\kappa_{\mathcal{X}\mathcal{Y}}(R), \min_{(x_i,y_j)\in R} u(x_i, y_j))\,. \tag{5}$$

The double S-integral is thus an S-integral based on the so-defined capacity $\kappa_{\mathcal{X}\mathcal{Y}}$ on the two dimensional space. This result was independently proved very recently by Halas *et al.*,[15] but it also straightforwardly follows from considerations by Goodstein[12] and Behrisch *et al.*[1] This capacity, by definition, is unique. Then it is clear that the commutation of the double S-integrals takes place if and only if the two capacities $\kappa_{\mathcal{X}\mathcal{Y}}$ and $\kappa_{\mathcal{Y}\mathcal{X}}$ coincide.

From this result, we conclude that commutation holds for all functions u: $\mathcal{X} \times \mathcal{Y} \to L$ whatever the capacities if and only if commutation holds for all Boolean-valued functions u: $\mathcal{X} \times \mathcal{Y} \to \{0, 1\}$, that is relations $R \subseteq \mathcal{X} \times \mathcal{Y}$, because double S-integrals are polynomial lattices.

Proposition 3. $S_{\mu_\mathcal{Y}} \perp S_{\mu_\mathcal{X}}$ *if and only if* $\forall R \subseteq \mathcal{X} \times \mathcal{Y}$, $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R)) = S_{\mu_\mathcal{Y}}(S_{\mu_\mathcal{X}}(\mathbf{1}_R))$.

Proof: We have shown above (Eq. (5)) that any double S-integral $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(u))$ is actually an S-integral associated to a capacity $\kappa_{\mathcal{X}\mathcal{Y}}$ on $\mathcal{X} \times \mathcal{Y}$, namely $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(u)) = S_{\kappa_{\mathcal{X}\mathcal{Y}}}(u)$. Likewise $S_{\mu_\mathcal{Y}}(S_{\mu_\mathcal{X}}(u)) = S_{\kappa_{\mathcal{Y}\mathcal{X}}}(u)$ for another capacity $\kappa_{\mathcal{Y}\mathcal{X}}$. So $S_{\mu_\mathcal{Y}} \perp S_{\mu_\mathcal{X}}$ if and only if $S_{\kappa_{\mathcal{X}\mathcal{Y}}}(u) = S_{\kappa_{\mathcal{Y}\mathcal{X}}}(u)$, if and only if $\kappa_{\mathcal{X}\mathcal{Y}} = \kappa_{\mathcal{Y}\mathcal{X}}$. And note that $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R)) = \kappa_{\mathcal{X}\mathcal{Y}}(R) = S_{\kappa_{\mathcal{X}\mathcal{Y}}}(\mathbf{1}_R)$, for all $R \subseteq \mathcal{X} \times \mathcal{Y}$. □

So if commutation holds for all relations, it holds for all functions and conversely. Now consider relations. In the following we first consider relations in the form of a Cartesian product $A \times B, A \subset \mathcal{X}, B \subset \mathcal{Y}$, then in the form of the union of two Cartesian products $R = (A_1 \times B_1) \cup (A_2 \times B_2)$. We shall show that if commutation occurs for the latter kind of relations, it holds for all functions.

The case of a Cartesian product: Consider the case when $R = A \times B$:

Proposition 4. *If $R = A \times B$ then $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R)) = \min(\mu_\mathcal{X}(A), \mu_\mathcal{Y}(B))$ for any capacities $\mu_\mathcal{X}$ and $\mu_\mathcal{Y}$.*

Proof:
$S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R))$ stands for:

$$\max_{C \subseteq \mathcal{X}} \min(\mu_\mathcal{X}(C), \min_{x \in C} \max_{C' \subseteq \mathcal{Y}} \min(\mu_\mathcal{Y}(C'), \min_{y \in C'} \mathbf{1}_R(x, y)))\,.$$

Because $\mathbf{1}_R(x, y) = 1$ if $(x, y) \in A \times B$, and $\mathbf{1}_R(x, y) = 0$ otherwise, we get:
$S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(u)) = \max_{C \subseteq A} \min(\mu_\mathcal{X}(C), \min_{x \in C} \max_{C' \subseteq B} \min(\mu_\mathcal{Y}(C'), \min_{y \in C'} 1))$
Then
$S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(u)) = \max_{C \subseteq A} \min(\mu_\mathcal{X}(C), \min_{x \in C} \max_{C' \subseteq B} \min(\mu_\mathcal{Y}(C'), 1))$
$= \max_{C \subseteq A} \min(\mu_\mathcal{X}(C), \min_{x \in C} \mu_\mathcal{Y}(B)))$
$= \max_{C \subseteq A} \min(\mu_\mathcal{X}(C), \mu_\mathcal{Y}(B))) = \min(\mu_\mathcal{X}(A), \mu_\mathcal{Y}(B)))$. □

This is a Fubini theorem for S-integrals, which is a special case of a result proved by Narukawa and Torra.[18] As a consequence we can identify a family of functions $u \colon \mathcal{X} \times \mathcal{Y} \to L$ for which commutation holds whatever the capacities involved.

Corollary 1. *If $u(x, y) = \min(u_\mathcal{X}(x), u_\mathcal{Y}(y))$, commutation holds for any pair of capacities, i.e.,*

$$S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(u)) = S_{\mu_\mathcal{Y}}(S_{\mu_\mathcal{X}}(u)) = \min(S_{\mu_\mathcal{X}}(u_\mathcal{X}), S_{\mu_\mathcal{Y}}(u_\mathcal{Y}))\,.$$

Proof: Note that by assumption $R = \{(x, y) : u(x, y) \geq \lambda\}$ is of the form, $S_\lambda \times T_\lambda$, where $S_\lambda = \{x : u_\mathcal{X} \geq \lambda\}$ and $T_\lambda = \{y : u_\mathcal{Y} \geq \lambda\}$. Then we can use the fact that a double S-integral is a simple one on $\mathcal{X} \times \mathcal{Y}$ (Eq.(5) in the form of Definition 1):

$$\begin{aligned} S_{\mu_\mathcal{Y}}(S_{\mu_\mathcal{X}}(u)) &= \max_{\lambda \in L} \min(\kappa_{\mathcal{Y}\mathcal{X}}(u \geq \lambda), \lambda) \\ &= \max_{\lambda \in L} \min(\mu_\mathcal{X}(S_\lambda), \mu_\mathcal{Y}(T_\lambda), \lambda) \text{ (by Prop. 4)} \\ &= \min(\max_{\lambda \in L} \min(\mu_\mathcal{X}(S_\lambda), \lambda), \max_{\lambda \in L} \min(\mu_\mathcal{Y}(T_\lambda), \lambda)) \\ &= \min(S_{\mu_\mathcal{X}}(u_\mathcal{X}), S_{\mu_\mathcal{Y}}(u_\mathcal{Y}))\,. \end{aligned}$$

□

The case of the union of two Cartesian products: Now let us consider relations $R = (A_1 \times B_1) \cup (A_2 \times B_2)$. Let us compute $\kappa_{\mathcal{X}\mathcal{Y}}(R) = S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R))$ in that case:

Lemma 1.

$$S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R)) = \max \begin{cases} \min(\mu_\mathcal{X}(A_1 \cap A_2), \mu_\mathcal{Y}(B_1 \cup B_2)) \\ \min(\mu_\mathcal{X}(A_1), \mu_\mathcal{Y}(B_1)) \\ \min(\mu_\mathcal{X}(A_2), \mu_\mathcal{Y}(B_2)) \\ \min(\mu_\mathcal{X}(A_1 \cup A_2), \mu_\mathcal{Y}(B_1), \mu_\mathcal{Y}(B_2)) \,. \end{cases}$$

Proof: First note that if $R = (A_1 \times B_1) \cup (A_2 \times B_2)$, then $xR = \{y \in \mathcal{Y} : xRy\}$ is of the form

- $xR = B_1$ if $x \in A_1 \setminus A_2$ and $xR = B_2$ if $x \in A_2 \setminus A_1$
- $xR = B_1 \cup B_2$ if $x \in A_1 \cap A_2$
- $xR = \emptyset$ otherwise.

Now $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R)) = \max_{S \subseteq \mathcal{X}} \min(\mu_\mathcal{X}(S), \min_{x \in S} \mu_\mathcal{Y}(xR)) = \max_{S \subseteq \mathcal{X}} \phi(S)$ for short. We compute the term $\min_{x \in S} \mu_\mathcal{Y}(xR)$ according to the position of S with respect to A_1 and A_2:

- If $S \subseteq A_1 \cap A_2$ then $\min_{x \in S} \mu_\mathcal{Y}(xR) = \mu_\mathcal{Y}(B_1 \cup B_2)$ so, $\max_{S \subseteq A_1 \cap A_2} \phi(S) = \min(\mu_\mathcal{X}(A_1 \cap A_2), \mu_\mathcal{Y}(B_1 \cup B_2))$.
- If $S \subseteq A_1$ and $S \not\subseteq A_2$, then $\min_{x \in S} \mu_\mathcal{Y}(xR) = \mu_\mathcal{Y}(B_1)$, so, $\max_{S \subseteq A_1, S \not\subseteq A_2} \phi(S) = \min(\mu_\mathcal{X}(A_1), \mu_\mathcal{Y}(B_1))$.
- If $S \subseteq A_2$ and $S \not\subseteq A_1$, then $\min_{x \in S} \mu_\mathcal{Y}(xR) = \mu_\mathcal{Y}(B_2)$, so, $\max_{S \subseteq A_2, S \not\subseteq A_1} \phi(S) = \min(\mu_\mathcal{X}(A_2), \mu_\mathcal{Y}(B_2))$.
- If $S \subseteq A_1 \cup A_2$ and $S \not\subseteq A_1, S \not\subseteq A_2$, then $\min_{x \in S} \mu_\mathcal{Y}(xR) = \min(\mu_\mathcal{Y}(B_1), \mu_\mathcal{Y}(B_2))$, so, $\max_{S \subseteq A_1 \cup A_2, S \not\subseteq A_1, S \not\subseteq A_2} \phi(S) = \min(\mu_\mathcal{X}(A_1 \cup A_2), \mu_\mathcal{Y}(B_1), \mu_\mathcal{Y}(B_2))$.
- If $S \not\subseteq A_1 \cup A_2$, then $\min_{x \in S} \mu_\mathcal{Y}(xR) = 0$ and $\max_{S \not\subseteq A_1 \cup A_2} \phi(S) = 0$. □

We denote by $g(A_1, A_2, B_1, B_2)$ the expression obtained for $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R))$ when $R = (A_1 \times B_1) \cup (A_2 \times B_2)$. We get a counterpart of Lemma 3.4 in the paper by Behrisch *et al.*:[1]

Lemma 2. *$S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R)) = S_{\mu_\mathcal{Y}}(S_{\mu_\mathcal{X}}(\mathbf{1}_R))$ for $R = (A_1 \times B_1) \cup (A_2 \times B_2)$ if and only if the equality*

$$g(A_1, A_2, B_1, B_2) = g(B_1, B_2, A_1, A_2)$$

holds. This is called the 2-rectangle condition.

Proof: It is obvious that computing $S_{\mu_\mathcal{Y}}(S_{\mu_\mathcal{X}}(\mathbf{1}_R))$ by the method of Lemma 1 comes down to swapping terms A_1 and B_1, and A_2 and B_2, in the expression of $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R))$. □

The following is a counterpart of Lemma 3.6 in the paper by Behrisch *et al.*:[1]

Lemma 3. *The 2-rectangle condition of Lemma 2 implies the two following properties*

$$\min(\mu_\mathcal{X}(A_1),\mu_\mathcal{X}(A_2),\mu_\mathcal{Y}(B_1\cup B_2)) = \max\begin{cases}\min(\mu_\mathcal{X}(A_1\cap A_2),\mu_\mathcal{Y}(B_1\cup B_2)),\\ \min(\mu_\mathcal{X}(A_1),\mu_\mathcal{X}(A_2),\max(\mu_\mathcal{Y}(B_1),\mu_\mathcal{Y}(B_2)))\end{cases}$$

$$\min(\mu_\mathcal{X}(B_1),\mu_\mathcal{X}(B_2),\mu_\mathcal{Y}(A_1\cup A_2)) = \max\begin{cases}\min(\mu_\mathcal{X}(B_1\cap B_2),\mu_\mathcal{Y}(A_1\cup A_2)),\\ \min(\mu_\mathcal{X}(B_1),\mu_\mathcal{X}(B_2),\max(\mu_\mathcal{Y}(A_1),\mu_\mathcal{Y}(A_2)))\end{cases}$$

Proof: To get the first equality the idea (from Behrisch *et al.*[1]) is to compute the conjunction of each side of the 2-rectangle condition with $\min(\mu_\mathcal{X}(A_1),\mu_\mathcal{X}(A_2))$ (applying distributivity). Indeed consider each factor of the 2-rectangle condition conjuncted with this term:

- On the left-hand side (as in Lemma 1), the conjunction of the first term with $\min(\mu_\mathcal{X}(A_1),\mu_\mathcal{X}(A_2))$ is still $\min(\mu_\mathcal{X}(A_1\cap A_2),\mu_\mathcal{Y}(B_1\cup B_2))$ since $\mu_\mathcal{X}(A_1\cap A_2)) \leq \min(\mu_\mathcal{X}(A_1),\mu_\mathcal{X}(A_2))$ by monotonicity of $\mu_\mathcal{X}$.
- The 2d term becomes $\min(\mu_\mathcal{X}(A_1),\mu_\mathcal{X}(A_2),\mu_\mathcal{Y}(B_1))$.
- The 3d term becomes $\min(\mu_\mathcal{X}(A_1),\mu_\mathcal{X}(A_2),\mu_\mathcal{Y}(B_2))$.
- The 4th term becomes $\min(\mu_\mathcal{X}(A_1),\mu_\mathcal{X}(A_2),\mu_\mathcal{X}(A_1\cup A_2),\mu_\mathcal{Y}(B_1),\mu_\mathcal{Y}(B_2))$, but this term is less than the above second and third terms and subsumed via maximum.

The right hand side of the 2-rectangle condition is handled similarly, exchanging A_1 and B_1, A_2 and B_2 in the expression in Lemma 1.

- The first term becomes $\min(\mu_\mathcal{Y}(B_1\cap B_2),\mu_\mathcal{X}(A_1),\mu_\mathcal{X}(A_2))$ due to monotonicity again.
- The second and third terms on the right-hand side are the same as in the left-hand side, but here they are less than the first term.
- The last term remains the same, i.e., $\min(\mu_\mathcal{Y}(B_1\cup B_2),\mu_\mathcal{X}(A_1),\mu_\mathcal{X}(A_2))$ but is greater the first term, so it is the maximum.

We thus get the first equality. The second equality is obtained likewise, by conjunction of each side of the equality with the term $\min(\mu_\mathcal{Y}(B_1),\mu_\mathcal{Y}(B_2))$. □

The following lemma simplifies the two obtained equalities into simpler inequalities.

Lemma 4. *The two equalities in Lemma 3 are equivalent to the two inequalities*

$$\max(\mu_{\mathcal{X}}(A_1 \cap A_2), \mu_{\mathcal{Y}}(B_1), \mu_{\mathcal{Y}}(B_2)) \geq \min(\mu_{\mathcal{X}}(A_1), \mu_{\mathcal{X}}(A_2), \mu_{\mathcal{Y}}(B_1 \cup B_2))\,, \quad (6)$$

$$\max(\mu_{\mathcal{Y}}(B_1 \cap B_2), \mu_{\mathcal{X}}(A_1), \mu_{\mathcal{X}}(A_2)) \geq \min(\mu_{\mathcal{Y}}(B_1), \mu_{\mathcal{Y}}(B_2), \mu_{\mathcal{X}}(A_1 \cup A_2))\,. \quad (7)$$

Proof: Let us apply distributivity to the right-hand side of the first equality in Lemma 3: We get a conjunction of four disjunctive terms of the form:

- $\max(\mu_{\mathcal{X}}(A_1 \cap A_2), \min(\mu_{\mathcal{X}}(A_1), \mu_{\mathcal{X}}(A_2))) = \min(\mu_{\mathcal{X}}(A_1), \mu_{\mathcal{X}}(A_2))$ (monotonicity).
- $\max(\mu_{\mathcal{X}}(A_1 \cap A_2), \mu_{\mathcal{Y}}(B_1), \mu_{\mathcal{Y}}(B_2))$
- $\max(\mu_{\mathcal{Y}}(B_1 \cup B_2), \min(\mu_{\mathcal{X}}(A_1), \mu_{\mathcal{X}}(A_2)))$
- $\max(\mu_{\mathcal{Y}}(B_1 \cup B_2), \mu_{\mathcal{Y}}(B_1), \mu_{\mathcal{Y}}(B_2)) = \mu_{\mathcal{Y}}(B_1 \cup B_2)$ (monotonicity).

It is clear that the conjunction of these terms absorbs the third one, and the first equality in Lemma 3 reduces to the equality $\min(\lambda, \max(\mu_{\mathcal{X}}(A_1 \cap A_2), \mu_{\mathcal{Y}}(B_1), \mu_{\mathcal{Y}}(B_2))) = \lambda$, where $\lambda = \min(\mu_{\mathcal{X}}(A_1), \mu_{\mathcal{X}}(A_2), \mu_{\mathcal{Y}}(B_1 \cup B_2))$, which is equivalent to the first inequality. The second inequality is proved likewise, exchanging A and B, $\mathcal{X}$ and $\mathcal{Y}$. □

The case of a union of more than two Cartesian products: It turns out that the two inequalities (6) and (7) imply similar ones for more than two pairs of sets (A_i, B_i), forming a union $R = \cup_{i=1}^n A_i \times B_i$ of Cartesian products, namely:

Lemma 5. (6) *and* (7) *imply:*

$$\max(\mu_{\mathcal{X}}(\cap_{i=1}^k A_i), \max_{j=1}^{\ell} \mu_{\mathcal{Y}}(B_j)) \geq \min(\min_{i=1}^{k} \mu_{\mathcal{X}}(A_i), \mu_{\mathcal{Y}}(\cup_{j=1}^{\ell} B_j))\,, \quad (8)$$

$$\max(\mu_{\mathcal{Y}}(\cap_{j=1}^{\ell} B_j), \max_{i=1}^{k} \mu_{\mathcal{X}}(A_i)) \geq \min(\min_{j=1}^{\ell} \mu_{\mathcal{Y}}(B_j), \mu_{\mathcal{X}}(\cup_{i=1}^{k} A_i))\,. \quad (9)$$

Proof: Inequality (8) holds for $k = \ell = 2$ (this is (6)). Suppose that inequality (8) holds for $i = 1, \dots k-1$ and $\ell = 2$. We can write, by assumption:

$$\max(\mu_{\mathcal{X}}(\cap_{i=1}^{k-1} A_i), \mu_{\mathcal{Y}}(B_1), \mu_{\mathcal{Y}}(B_2)) \geq \min(\min_{i=1}^{k-1} \mu_{\mathcal{X}}(A_i), \mu_{\mathcal{Y}}(B_1 \cup B_2))\,. \quad (10)$$

Moreover we can write (6) for $A = \cap_{i=1}^{k-1} A_i, A_k, B_1, B_2$. So we can write the inequality

$$\max(\mu_{\mathcal{X}}(\cap_{i=1}^{k} A_i), \mu_{\mathcal{Y}}(B_1), \mu_{\mathcal{Y}}(B_2)) \geq \min(\mu_{\mathcal{X}}(\cap_{i=1}^{k-1} A_i), \mu_{\mathcal{X}}(A_k), \mu_{\mathcal{Y}}(B_1 \cup B_2))\,. \quad (11)$$

Suppose $\mu_{\mathcal{X}}(\cap_{i=1}^{k-1} A_i) \geq \max(\mu_{\mathcal{Y}}(B_1), \mu_{\mathcal{Y}}(B_2))$. So the first inequality (10) reduces to

$$\mu_{\mathcal{X}}(\cap_{i=1}^{k-1} A_i) \ \geq \min(\min_{i=1}^{k-1} \mu_{\mathcal{X}}(A_i), \mu_{\mathcal{Y}}(B_1 \cup B_2))\,.$$

Then we can replace $\mu_{\mathcal{X}}(\cap_{i=1}^{k-1} A_i)$ by $\min(\min_{i=1}^{k-1} \mu_{\mathcal{X}}(A_i), \mu_{\mathcal{Y}}(B_1 \cup B_2))$ in the second inequality (11), and get (8).

Otherwise, $\mu_{\mathcal{X}}(\cap_{i=1}^{k-1} A_i) \leq \max(\mu_{\mathcal{Y}}(B_1), \mu_{\mathcal{Y}}(B_2))$, and the first inequality (10) reads

$$\max(\mu_{\mathcal{Y}}(B_1), \mu_{\mathcal{Y}}(B_2)) \geq \min(\min_{i=1}^{k-1} \mu_{\mathcal{X}}(A_i), \mu_{\mathcal{Y}}(B_1 \cup B_2))$$

so we have

$$\max(\mu_{\mathcal{X}}(\cap_{i=1}^{k} A_i), \mu_{\mathcal{Y}}(B_1), \mu_{\mathcal{Y}}(B_2)) \geq \min((\min_{i=1}^{k-1} \mu_{\mathcal{X}}(A_i)), \mu_{\mathcal{X}}(A_k), \mu_{\mathcal{Y}}(B_1 \cup B_2)),$$

which is (8) again. Proving that the inequality (8) holds for any (k, ℓ) is similar, knowing it holds for $(k, 2)$, assuming it holds until $(k, \ell - 1)$. The inequality (9) is proved in a similar way, exchanging A and B, X and Y. □

Lemma 6. *If $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ satisfy the two inequalities* (8) *and* (9), *then $S_{\mu_{\mathcal{X}}} \perp S_{\mu_{\mathcal{Y}}}$.*

Proof: First notice that the inequalities (8) and (9) can be written in the style of Lemma 3. The first one reads:

$$\min(\min_{i=1}^{k} \mu_{\mathcal{X}}(A_i), \mu_{\mathcal{Y}}(\cup_{j=1}^{\ell} B_j)) = \max \begin{cases} \min(\mu_{\mathcal{X}}(\cap_{i=1}^{k} A_i), \mu_{\mathcal{Y}}(\cup_{j=1}^{\ell} B_j)), \\ \min(\min_{i=1}^{k} \mu_{\mathcal{X}}(A_i), \max_{j=1}^{\ell} \mu_{\mathcal{Y}}(B_j)). \end{cases}$$

We first prove that

$$\max_{S \subseteq \mathcal{X}} \min(\mu_{\mathcal{X}}(S), \min_{x \in S} \mu_{\mathcal{Y}}(xR)) \geq \max_{T \subseteq \mathcal{Y}} \min(\mu_{\mathcal{Y}}(T), \min_{y \in T} \mu_{\mathcal{X}}(Ry)).$$

Consider the term $\min(\mu_{\mathcal{Y}}(T), \min_{y \in T} \mu_{\mathcal{X}}(Ry))$ that we identify with the left-hand side of the above equality (letting $T = \cup_{j=1}^{\ell} B_j$, $B_j = \{y_j\}$, $k = \ell = |T|$, $A_i = Ry_i$). This equality then reads:

$$\begin{aligned} &\min(\min_{y_i \in T} \mu_{\mathcal{X}}(Ry_i), \mu_{\mathcal{Y}}(T)) \\ &\quad = \max(\min(\mu_{\mathcal{X}}(\cap_{y_i \in T} Ry_i), \mu_{\mathcal{Y}}(T)), \min(\min_{y_i \in T} \mu_{\mathcal{X}}(Ry_i), \max_{y_j \in T} \mu_{\mathcal{Y}}(\{y_j\}))) \\ &\quad = \max(\min(\mu_{\mathcal{X}}(S_T), \mu_{\mathcal{Y}}(T)), \max_{t \in T} \min(\min_{y \in T} \mu_{\mathcal{X}}(Ry), \mu_{\mathcal{Y}}(\{t\}))). \end{aligned}$$

where $S_T = \cap_{y \in T} Ry$. Now we can prove that

- $\mu_{\mathcal{Y}}(T) \leq \min_{x \in S_T} \mu_{\mathcal{Y}}(xR)$. Indeed $S_T = \cap_{y \in T} Ry$ if and only if $S_T \times T \subseteq R$ if and only if $T = \cap_{x \in S_T} xR$. So, the term $\min(\mu_{\mathcal{X}}(S_T), \mu_{\mathcal{Y}}(T))$ is upper bounded by $\max_{S \subseteq \mathcal{X}} \min(\mu_{\mathcal{X}}(S), \min_{x \in S} \mu_{\mathcal{Y}}(xR))$.
- The same holds for the term $\min(\min_{y \in T} \mu_{\mathcal{X}}(Ry), \mu_{\mathcal{Y}}(\{t\}))$. Indeed
  - — as $t \in T$, $\min_{y \in T} \mu_{\mathcal{X}}(Ry) \leq \mu_{\mathcal{X}}(Rt)$, choosing $y = t$.
  - — Let $x \in Rt$. Then $\mu_{\mathcal{Y}}(\{t\}) \leq \mu_{\mathcal{Y}}(xR)$ since $t \in xR$ as well.

So, $\min(\min_{y\in T}\mu_{\mathcal{X}}(Ry),\mu_{\mathcal{Y}}(\{t\}))\leq \min(\mu_{\mathcal{X}}(Rt),\mu_{\mathcal{Y}}(xR)),\forall x\in Rt$. Hence,

$$\min(\min_{y\in T}\mu_{\mathcal{X}}(Ry),\mu_{\mathcal{Y}}(\{t\}))\leq \min(\mu_{\mathcal{X}}(Rt),\min_{x\in Rt}\mu_{\mathcal{Y}}(xR))$$

that is also upper bounded by $\max_{S\subseteq\mathcal{X}}\min(\mu_{\mathcal{X}}(S),\min_{x\in S}\mu_{\mathcal{Y}}(xR))$. We thus get $S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}(\mathbf{1}_R))\geq S_{\mu_{\mathcal{Y}}}(S_{\mu_{\mathcal{X}}}(\mathbf{1}_R))$.

The converse inequality $S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}(\mathbf{1}_R))\leq S_{\mu_{\mathcal{Y}}}(S_{\mu_{\mathcal{X}}}(\mathbf{1}_R))$ can be proved likewise, by symmetry, using (9). □

The above Lemmas yield a necessary and sufficient condition expressed in the following theorem for the commutation of two S-integrals.

Theorem 1. *Consider two capacities $\mu_{\mathcal{X}}$ on $\mathcal{X}$ and $\mu_{\mathcal{Y}}$ on $\mathcal{Y}$. $S_{\mu_{\mathcal{Y}}}\perp S_{\mu_{\mathcal{X}}}$ if and only if $\forall A_1,A_2\subseteq\mathcal{X},\forall B_1,B_2\subseteq\mathcal{Y}$,:*

$$\max(\mu_{\mathcal{X}}(A_1\cap A_2),\mu_{\mathcal{Y}}(B_1),\mu_{\mathcal{Y}}(B_2))\geq\min(\mu_{\mathcal{X}}(A_1),\mu_{\mathcal{X}}(A_2),\mu_{\mathcal{Y}}(B_1\cup B_2))\,,$$

$$\max(\mu_{\mathcal{Y}}(B_1\cap B_2),\mu_{\mathcal{X}}(A_1),\mu_{\mathcal{X}}(A_2))\geq\min(\mu_{\mathcal{Y}}(B_1),\mu_{\mathcal{Y}}(B_2),\mu_{\mathcal{X}}(A_1\cup A_2))\,.$$

Proof: Proposition 3 shows we can restrict to Boolean functions (relations R) on $\mathcal{X}\times\mathcal{Y}$ without loss of generality. Then we show that commutation is equivalent to a certain identity for relations R of the form $(A_1\times B_1)\cup(A_2\times B_2)$ (Lemma 2). We show this identity implies the two inequalities of the theorem (Lemmas 3 then 4). For the sufficiency , we have shown these inequalities can be extended to more than just pairs of sets (Lemma 5). Finally we show that these extended inequalities imply the commutation condition (Lemma 6). As any relation on a finite Cartesian product can be expressed as a finite union of Cartesian products, the two inequalities of the theorem are necessary and sufficient conditions for the commutation of Sugeno integrals. □

4.2. *Commuting capacities*

As an S-integral is entirely characterized by its underlying capacity, we will say that the two capacities commute when the corresponding S-integrals commute. Theorem 1 does not clearly explain what are the capacities that commute. We already know that two possibility measures, as well as two necessity measures commute, while a possibility measure does not commute with a necessity measure. In this subsection, we try to explicitly describe all pairs of commuting capacities.

The case of Boolean capacities is of interest as it will be instrumental to address the general case:

Lemma 7. *If $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ are Boolean capacities and the two inequalities* (6) *and* (7) *hold for all $A_1,A_2\subseteq\mathcal{X}$ and for all $B_1,B_2\subseteq\mathcal{Y}$, then $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ are both necessity measures or possibility measures or one of them is a Dirac measure.*

Proof: Suppose $\mu_{\mathcal{X}}$ is not a necessity measure and $\mu_{\mathcal{Y}}$ is not a possibility measure. Then $\exists A_1, A_2 \subseteq \mathcal{X}, \mu_{\mathcal{X}}(A_1 \cap A_2) < \min(\mu_{\mathcal{X}}(A_1), \mu_{\mathcal{X}}(A_2))$, and $\exists B_1, B_2 \subseteq \mathcal{Y}, \mu_{\mathcal{Y}}(B_1 \cup B_2) > \max(\mu_{\mathcal{Y}}(B_1), \mu_{\mathcal{X}}(B_2))$. In the Boolean case it reads $\mu_{\mathcal{X}}(A_1 \cap A_2) = 0, \mu_{\mathcal{X}}(A_1) = \mu_{\mathcal{X}}(A_2) = 1, \mu_{\mathcal{Y}}(B_1 \cup B_2) = 1, \mu_{\mathcal{Y}}(B_1) = \mu_{\mathcal{Y}}(B_2) = 0$. Then inequality (6) is violated as $\max(\mu_{\mathcal{X}}(A_1 \cap A_2), \mu_{\mathcal{Y}}(B_1), \mu_{\mathcal{Y}}(B_2)) = 0$ and $\min(\mu_{\mathcal{X}}(A_1), \mu_{\mathcal{X}}(A_2), \mu_{\mathcal{Y}}(B_1 \cup B_2)) = 1$. The second inequality (7) is violated by choosing $A_1, A_2 \subseteq \mathcal{X}, B_1, B_2 \subseteq \mathcal{Y}$, such that $\mu_{\mathcal{Y}}(B_1 \cap B_2) = 0, \mu_{\mathcal{Y}}(B_1) = \mu_{\mathcal{Y}}(B_2) = 1, \mu_{\mathcal{X}}(A_1 \cup A_2) = 1, \mu_{\mathcal{X}}(A_1) = \mu_{\mathcal{X}}(A_2) = 0$, assuming $\mu_{\mathcal{Y}}$ is not a necessity measure and $\mu_{\mathcal{X}}$ is not a possibility measure. Obeying the two inequalities (6) and (7) enforces the following constraints in the Boolean case

$\mu_{\mathcal{Y}}$ possibility measure or $\mu_{\mathcal{X}}$ necessity measure

and

$\mu_{\mathcal{Y}}$ necessity measure or $\mu_{\mathcal{X}}$ possibility measure.

It enforces possibility measures on both sets $\mathcal{X}$ and $\mathcal{Y}$, or necessity measures (known cases where commuting occurs). Alternatively, if we enforce $\mu_{\mathcal{Y}}$ to be at the same time a possibility measure and a necessity measure, it is a Dirac function on $\mathcal{Y}$, and it can be any capacity on the other space. □

Corollary 2. *S-integrals with respect to Boolean capacities $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ commute if and only if they are both necessity measures or possibility measures or one of them is a Dirac measure.*

We can extend the result to the case when only one of the capacities is Boolean:

Proposition 5. *If one of $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ is Boolean, S-integrals commute if and only if they are both necessity measures or possibility measures or one of them is a Dirac measure.*

Proof: Suppose $\mu_{\mathcal{X}}$ is Boolean and is not a necessity measure and $\mu_{\mathcal{Y}}$ is not a possibility measure. Then $\exists A_1, A_2 \subseteq \mathcal{X}, \mu_{\mathcal{X}}(A_1 \cap A_2) < \min(\mu_{\mathcal{X}}(A_1), \mu_{\mathcal{X}}(A_2))$, and $\exists B_1, B_2 \subseteq \mathcal{Y}, \mu_{\mathcal{Y}}(B_1 \cup B_2) > \max(\mu_{\mathcal{Y}}(B_1), \mu_{\mathcal{X}}(B_2))$. For $\mu_{\mathcal{X}}$, it reads $\mu_{\mathcal{X}}(A_1 \cap A_2) = 0, \mu_{\mathcal{X}}(A_1) = \mu_{\mathcal{X}}(A_2) = 1$. Then the first inequality in Theorem 1 reduces to $\max(\mu_{\mathcal{Y}}(B_1), \mu_{\mathcal{X}}(B_2)) \geq \mu_{\mathcal{Y}}(B_1 \cup B_2)$, which implies that $\mu_{\mathcal{Y}}$ is a possibility measure. It contradicts the assumption on $\mu_{\mathcal{Y}}$. The rest of the reasoning is as above. □

Note that to violate (6) it is enough that neither $\mu_{\mathcal{X}}$ nor $\mu_{\mathcal{Y}}$ are possibility and necessity measures, and moreover for A_1, A_2, B_1, B_2 where, say $\mu_{\mathcal{X}}$ violates the axiom of necessities and $\mu_{\mathcal{Y}}$ violates the axiom of possibilities, i.e., we have $\mu_{\mathcal{X}}(A_1)$ and $\mu_{\mathcal{X}}(A_2)$ both greater than each of $\mu_{\mathcal{Y}}(B_1), \mu_{\mathcal{Y}}(B_2)$ and moreover $\mu_{\mathcal{Y}}(B_1 \cup B_2) > \mu_{\mathcal{Y}}(A_1 \cap A_2)$. Then the integrals will not commute.

In the non-Boolean case, we can give examples of commuting graded capacities that are neither only possibility measures, nor only necessity measures nor a Dirac function contrary to the Boolean case of Corollary 2.

Suppose $\mu_{\mathcal{Y}}$ is a possibility measure. Then inequality (6) trivially holds. Let $B_i = \{y_i\}, i = 1, 2$. In this case, the other inequality (7) reads

$$\forall y_1 \neq y_2 \in \mathcal{Y}, \max(\mu_{\mathcal{X}}(A_1), \mu_{\mathcal{X}}(A_2)) \geq \min(\mu_{\mathcal{Y}}(\{y_1\}), \mu_{\mathcal{Y}}(\{y_2\}), \mu_{\mathcal{X}}(A_1 \cup A_2)) .$$

as $\mu_{\mathcal{Y}}(B_1 \cap B_2) = 0$ in this case.[d] The most demanding case is when $\mu_{\mathcal{Y}}(y_1) = 1$ and $\mu_{\mathcal{Y}}(y_2)$ is the possibility degree π_2 of the second most plausible element in $\mu_{\mathcal{Y}}$. It is then equivalent to $\max(\mu_{\mathcal{X}}(A_1), \mu_{\mathcal{X}}(A_2)) \geq \min(\pi_2, \mu_{\mathcal{X}}(A_1 \cup A_2))$, which enforces a possibility measure for $\mu_{\mathcal{X}}$ only if $\pi_2 \geq \mu_{\mathcal{X}}(A)$ for all $A \subseteq \mathcal{X}$, namely $\pi_2 = 1$.

Example 2. (See also Halas *et al.*[15]) Let $\mathcal{X} = \{x_1, x_2\}; \mathcal{Y} = \{y_1, y_2\}$. Then let $\mu_{\mathcal{X}}(x_1) = \alpha, \mu_{\mathcal{X}}(x_2) = \alpha$, $\mu_{\mathcal{Y}}(y_1) = 1, \mu_{\mathcal{Y}}(y_2) = \alpha$, so a constant capacity $\mu_{\mathcal{X}}$ and a possibility measure $\mu_{\mathcal{Y}}$.

We have $\max(\mu_{\mathcal{X}}(A_1 \cap A_2), \mu_{\mathcal{Y}}(B_1), \mu_{\mathcal{Y}}(B_2)) \geq \min(\mu_{\mathcal{X}}(A_1), \mu_{\mathcal{X}}(A_2), \mu_{\mathcal{Y}}(B_1 \cup B_2))$ because the possible values are α or 1. The right-hand side is equal to 1 if and only if $A_1 = A_2 = \mathcal{X}$; in this case $\mu_{\mathcal{X}}(A_1 \cap A_2) = 1$.

We have $\max(\mu_{\mathcal{Y}}(B_1 \cap B_2), \mu_{\mathcal{X}}(A_1), \mu_{\mathcal{X}}(A_2)) \geq \min(\mu_{\mathcal{Y}}(B_1), \mu_{\mathcal{Y}}(B_2), \mu_{\mathcal{X}}(A_1 \cup A_2))$ because the possible values are α or 1. The right-hand side is equal to 1 if and only if $y_1 \in B_1$ and $y_2 \in B_2 = \mathcal{X}$; in this case $\mu_{\mathcal{Y}}(B_1 \cap B_2) = 1$. So $S_{\mu_{\mathcal{X}}} \perp S_{\mu_{\mathcal{Y}}}$. □

In the following, we lay bare the pairs of capacities that commute by applying the result of Corollary 2 to cuts of the capacities. We first prove that for Boolean functions on $\mathcal{X} \times \mathcal{Y}$, the double S-integrals are completely defined by the cuts of the involved capacities, thus generalizing Proposition 1 to double S-integrals. First, we show an inequality in the general case of L-valued functions.

Lemma 8. $\forall u, S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}(u)) \geq \max_{\lambda > 0} \min(\lambda, S_{\mu_{\mathcal{X}\lambda}}(S_{\mu_{\mathcal{Y}\lambda}}(u)))$.

Proof: For simplicity we denote $\mu_{\mathcal{X}}$ by μ and $\mu_{\mathcal{Y}}$ by ν

$$\begin{aligned} S_\mu(S_\nu(u)) &= \max_{A \subseteq \mathcal{X}} \min(\mu(A), \min_{x \in A} S_\nu(u(x, \cdot))) \\ &= \max_{A \subseteq \mathcal{X}} \min(\max_{\lambda > 0} \min(\lambda, \mu_\lambda(A)), \min_{x \in A} \max_{\alpha > 0} \min(\alpha, S_{\nu_\alpha}(u(x, \cdot)))) . \end{aligned}$$

Note that

$$\min_{x \in A} \max_{\alpha > 0} \min(\alpha, S_{\nu_\alpha}(u(x, \cdot))) \geq \max_{\alpha > 0} \min_{x \in A} \min(\alpha, S_{\nu_\alpha}(u(x, \cdot))) .$$

Then we have

$$\begin{aligned} S_\mu(S_\nu(u)) &\geq \max_{A \subseteq \mathcal{X}} \min(\max_{\lambda > 0} \min(\lambda, \mu_\lambda(A)), \max_{\alpha > 0} \min_{x \in A} \min(\alpha, S_{\nu_\alpha}(u(x, \cdot)))) \\ &= \max_{\lambda > 0} \max_{A \subseteq \mathcal{X}} \min(\min(\lambda, \mu_\lambda(A)), \max_{\alpha > 0} \min_{x \in A} \min(\alpha, S_{\nu_\alpha}(u(x, \cdot)))) \\ &= \max_{\lambda > 0} \min(\lambda, \max_{A \subseteq \mathcal{X}} \min(\mu_\lambda(A)), \max_{\alpha > 0} \min(\alpha, \min_{x \in A} S_{\nu_\alpha}(u(x, \cdot)))) \end{aligned}$$

[d]When $y_1 = y_2$ then Eq. (6) becomes $\max(\mu_{\mathcal{Y}}(\{y_1\}), \mu_{\mathcal{X}}(A_1), \mu_{\mathcal{X}}(A_2)) \geq \min(\mu_{\mathcal{Y}}(\{y_1\}), \mu_{\mathcal{X}}(A_1 \cup A_2))$ and this inequality is trivial.

$$= \max_{\lambda>0,\alpha>0} \min(\lambda, \alpha, \max_{A\subseteq\mathcal{X}} \min(\mu_\lambda(A), \min_{x\in A} S_{\nu_\alpha}(u(x,\cdot))))$$
$$= \max_{\lambda>0,\alpha>0} \min(\lambda, \alpha, S_{\mu_\lambda}(S_{\nu_\alpha}(u)) .$$

Suppose the maximum is attained for $\alpha^* \neq \lambda^*$ then since $\alpha \geq \beta \Rightarrow \nu_\alpha(A) \leq \nu_\beta(A)$, decreasing α to $\min(\alpha^*, \lambda^*)$ will increase $S_{\nu_\alpha}(u(x,\cdot))$, and decreasing λ to $\min(\alpha^*, \lambda^*)$ will increase $S_{\mu_\lambda}(S_{\nu_\alpha}(u))$. So we can assume $\alpha^* = \lambda^*$. □

Now we prove that this inequality is an equality for Boolean functions on $\mathcal{X}\times\mathcal{Y}$.

Proposition 6. $\forall R \subseteq \mathcal{X}\times\mathcal{Y}, S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R)) = \max_{\lambda>0}\min(\lambda, S_{\mu_{\mathcal{X}\lambda}}(S_{\mu_{\mathcal{Y}\lambda}}(\mathbf{1}_R)))$.

Proof: Let us restrict to the case when $u(x,y)$ in Lemma 8 is Boolean and is thus a relation $R \subseteq \mathcal{X}\times\mathcal{Y}$. In this case $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R)) = \max_{A\subseteq\mathcal{X}}\min(\mu_\mathcal{X}(A), \min_{x\in A}\mu_\mathcal{Y}(xR))$ and we are led to study conditions for the equality $\min_{x\in A}\max_{\alpha>0}\min(\alpha, \nu_\alpha(xR)) = \max_{\alpha>0}\min_{x\in A}\min(\alpha, \nu_\alpha(xR))$. Let $\alpha^*, \hat{x}$ be optima for $\min(\alpha, \nu_\alpha(xR))$ on the right hand side, that is,

$$\max_{\alpha>0}\min_{x\in A}\min(\alpha, \nu_\alpha(xR)) = \min(\alpha^*, \nu_\alpha(\hat{x}R)) .$$

It means that $\forall x \in A, \nu_{\alpha^*}(xR) = 1$, and due to monotonicity, $\forall x \in A, \forall\alpha \leq \alpha^*, \nu_\alpha(xR) = 1$. However, $\forall\alpha > \alpha^*, \exists x \in A, \nu_\alpha(xR) = 0$.

Hence $\min_{x\in A}\max_{\alpha>0}\min(\alpha, \nu_\alpha(xR)) = \min_{x\in A}\max_{\alpha^*\geq\alpha>0}\min(\alpha, \nu_\alpha(xR)) \leq \alpha^*$.

So $\min_{x\in A}\max_{\alpha>0}\min(\alpha, \nu_\alpha(xR)) \leq \max_{\alpha>0}\min_{x\in A}\min(\alpha, \nu_\alpha(xR))$, and we get the equality since we already have the converse inequality in the general case due to Lemma 8. □

We know that commutation between integrals holds for functions $u(x,y)$ if it holds for relations. The above result shows that commutation between capacities will hold if and only if it will hold for their cuts, to which we can apply Corollary 2.

Corollary 3. *Capacities $\mu_\mathcal{X}$ and $\mu_\mathcal{Y}$ commute if and only if their cuts $\mu_{\mathcal{X}\lambda}$ and $\mu_{\mathcal{Y}\lambda}$ commute for all $\lambda \in L$.*

Proof: Suppose $\mu_\mathcal{X}$ and $\mu_\mathcal{Y}$ commute. It means that $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R))$ and $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R))$ use the same 2D capacity κ on $\mathcal{X}\times\mathcal{Y}$, namely $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R)) = S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R)) = \kappa(R)$. It is then clear that using Proposition 2,:

$$\begin{aligned}
\kappa_\lambda(R) &= S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R))_\lambda = S_{\mu_\mathcal{Y}}(S_{\mu_\mathcal{X}}(\mathbf{1}_R))_\lambda \\
&= S_{\mu_{\mathcal{X}\lambda}}(\mathbf{1}_{[\mu_\mathcal{Y}(R(x_1,\cdot))\geq\lambda]}, \cdots, \mathbf{1}_{[\mu_\mathcal{Y}(R(x_n,\cdot))\geq\lambda]}) \\
&= S_{\mu_{\mathcal{Y}\lambda}}(\mathbf{1}_{[\mu_\mathcal{X}(R(\cdot,y_1))\geq\lambda]}, \cdots, \mathbf{1}_{[\mu_\mathcal{X}(R(\cdot,y_n))\geq\lambda]}) \\
&= S_{\mu_{\mathcal{X}\lambda}}(S_{\mu_{\mathcal{Y}\lambda}}(R(x_1,\cdot)), \ldots, S_{\mu_{\mathcal{Y}\lambda}}(R(x_n,\cdot))) \\
&= S_{\mu_{\mathcal{Y}\lambda}}(S_{\mu_{\mathcal{X}\lambda}}(R(\cdot,y_1)), \ldots, S_{\mu_{\mathcal{Y}\lambda}}(R(\cdot,y_n))) \\
&= S_{\mu_\mathcal{X}\lambda}(S_{\mu_{\mathcal{Y}\lambda}}(\mathbf{1}_R)) = S_{\mu_\mathcal{Y}\lambda}(S_{\mu_{\mathcal{X}\lambda}}(\mathbf{1}_R)) .
\end{aligned}$$

Conversely, using Proposition 6 if $S_{\mu_{\mathcal{X}\lambda}}(S_{\mu_{\mathcal{Y}\lambda}}(\mathbf{1}_R)) = S_{\mu_{\mathcal{Y}\lambda}}(S_{\mu_{\mathcal{X}\lambda}}(\mathbf{1}_R))$ for all $\lambda \in L$ and $R \subseteq \mathcal{X} \times \mathcal{Y}$ it implies $S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}(\mathbf{1}_R)) = S_{\mu_{\mathcal{Y}}}(S_{\mu_{\mathcal{X}}}(\mathbf{1}_R))$ for all $R \subseteq \mathcal{X} \times \mathcal{Y}$, which by Proposition 3, is equivalent to the commutation of S-integrals w.r.t. $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ for all 2-place functions u. □

For instance, in the above Example 2, the commutation is clear because when $\lambda > \alpha$ the λ-cut of $\mu_{\mathcal{X}}$ is a necessity measure (with focal set $\mathcal{X}$) and $\mu_{\mathcal{Y}}$ is a Dirac function focused on y_1. And when $0 < \lambda \leq \alpha$, the λ-cut of $\mu_{\mathcal{X}}$ is the vacuous possibility measure, and so is the λ-cut of $\mu_{\mathcal{Y}}$. More generally we can claim that

Corollary 4. *Capacities $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ commute if and only if their cuts $\mu_{\mathcal{X}\lambda}$ and $\mu_{\mathcal{Y}\lambda}$ are two possibility measures, two necessity measures, or one of them is a Dirac measure, for each $\lambda \in L$.*

It is useful to describe capacities whose cuts are Boolean possibility of necessity measures using the focal sets of the cuts of a capacity μ on $\mathcal{X}$. First, recall that:

Lemma 9. *If $\mathcal{F}(\mu)$ denotes the focal sets of μ, then the focal sets of μ_λ form the family*

$$\mathcal{F}(\mu_\lambda) = \min_{\subseteq}\{E \subseteq \mathcal{X} : \mu_{\#}(E) \geq \lambda\}$$

containing the least sets for inclusion in the family of focal sets of μ with weight at least λ.

Indeed the focal sets of a Boolean capacity form an antichain, that is, they are not nested, and if $\mu_{\#}(E) > \mu_{\#}(F) \geq \lambda$, while $F \subset E$, then E is not focal for μ_λ. In particular, a Boolean necessity measure has a single focal set, while focal sets of a (Boolean) possibility measure are singletons.

Lemma 10. *For any capacity μ on $\mathcal{X}$,*

(1) *μ_λ is a necessity measure if and only if there is a single focal set E with $\mu_{\#}(E) \geq \lambda$ such that for all focal sets F in $\mathcal{F}(\mu)$ with weights $\mu_{\#}(F) \geq \lambda$, we have $E \subset F$.*
(2) *μ_λ is a possibility measure if and only if there is a set S of singletons with $\mu_{\#}(x_i) \geq \lambda$ such that for all focal sets F in $\mathcal{F}(\mu)$ with weights $\mu_{\#}(F) \geq \lambda$, we have $S \cap F \neq \emptyset$.*
(3) *μ_λ is a Dirac measure if and only if there is a focal singleton $\{x\}$ with $\mu_{\#}(\{x\}) \geq \lambda$ such that for all focal sets F in $\mathcal{F}(\mu)$ with weights $\mu_{\#}(F) \geq \lambda$, we have $x \in F$.*

Proof:

(1) The condition does ensure that E is the only focal set of μ_λ hence it is a necessity measure. If the condition does not hold it is clear that μ_λ has more than one focal set, hence is a not a necessity measure.

(2) The condition does ensure that the focal sets of μ_λ are the singletons in S, hence it is a possibility measure. If the condition does not hold it is clear that μ_λ has a focal set that is not a singleton, hence is not a possibility measure.
(3) The condition implies that μ_λ is both a possibility and a necessity measure, hence a Dirac measure. If it is not satisfied, either μ_λ has more than one focal set or its focal set is not a singleton.

□

We call a possibility or a necessity measure *non-trivial* when they are not Dirac functions. Note that, when μ_λ is a possibility measure, and μ_α is a non-trivial necessity measure, then $\alpha > \lambda$:

Proposition 7. *If* $\forall\lambda > 0, \mu_\lambda$ *is a possibility measure or a necessity measure, and* $\exists\theta > 0$*,* μ_θ *is a possibility measure, then* $\forall\lambda \leq \theta, \mu_\lambda$ *is a possibility measure.*

Proof: Suppose μ_θ is a possibility measure whose focal sets are the singletons of a subset S. Take $\lambda < \theta$ and suppose μ_λ is a not a possibility measure. So there is a focal set E of μ with weight less than θ but at least equal to λ that is focal for μ_λ. Clearly the singletons of S are also focal sets of μ_λ, hence disjoint with E and μ_λ is not a necessity measure, which contradicts the assumption about μ. Hence μ_λ is a possibility measure when $\lambda < \theta$. □

We are then in a position to state the main result of this section as pictured on Fig. 2. Note that x^* must be contained in all focal sets of $\mu_\mathcal{X}$ with weight greater than θ_D, and y^* must be disjoint from all focal sets of $\mu_\mathcal{Y}$ with weight less than or equal to θ_D.

λ	$\mu^\lambda_\mathcal{X}$	$\mu^\lambda_\mathcal{Y}$
1 θ_N	necessity	necessity
θ_D	capacity	Dirac on y^*
θ_Π	Dirac on x^*	capacity
0	possibility	possibility

Fig. 2. Commuting capacities.

Theorem 2. *Two capacities* $\mu_\mathcal{X}$ *and* $\mu_\mathcal{Y}$ *commute if and only if there exist at most three thresholds* $\theta_N \geq \theta_D \geq \theta_\Pi \in L$ *such that, up to exchanging* $\mu_\mathcal{X}$ *and* $\mu_\mathcal{Y}$*,*

- *For* $1 \geq \lambda > \theta_N$*, the* λ*-cuts of* $\mu_\mathcal{X}$ *and* $\mu_\mathcal{Y}$ *are non-trivial necessity measures.*

- *For $\theta_N \geq \lambda > \theta_D$, the λ-cut of $\mu_\mathcal{Y}$ is a Dirac measure focused on y^* contained in the focal sets of higher weights, $\mu^\lambda_\mathcal{X}$ being any Boolean capacity.*
- *For $\theta_D \geq \lambda > \theta_\Pi$, the λ-cut of $\mu_\mathcal{X}$ is a Dirac measure focused on x^* contained in the focal sets of higher weights, $\mu^\lambda_\mathcal{Y}$ being any Boolean capacity with focal sets disjoint from y^*.*
- *For $\theta_\Pi \geq \lambda$, the λ-cuts of $\mu_\mathcal{X}$ and $\mu_\mathcal{Y}$ are non-trivial possibility measures.*

Proof: By construction, the if part of the proof is obvious. Now suppose that $\mu_\mathcal{X}$ and $\mu_\mathcal{Y}$ commute. We apply Corollary 4, and Proposition 7.

First suppose that for $\lambda = 1$ the λ-cuts of $\mu_\mathcal{X}$ and $\mu_\mathcal{Y}$ are non-trivial possibility measures. If the λ-cut of $\mu_\mathcal{X}$ is not a possibility measure for some $\lambda < 1$, it has another focal set E and this set is disjoint from the focal singletons of the 1-cut of $\mu_\mathcal{X}$. So this λ-cut of $\mu_\mathcal{X}$ is not commuting with the λ-cut of $\mu_\mathcal{Y}$ (which is not a Dirac function). By symmetry, $\mu_\mathcal{X}$ and $\mu_\mathcal{Y}$ are both non-trivial possibility measures. In this case $\theta_N = \theta_D = \theta_\Pi = 1$.

Now suppose that a threshold θ_Π exists such that for $\lambda > \theta_\Pi$, the λ-cuts of $\mu_\mathcal{Y}$ are a Dirac function while for $\lambda \leq \theta_\Pi$, the λ-cuts of $\mu_\mathcal{Y}$ are possibility measures. The commutation of $\mu_\mathcal{X}$ and $\mu_\mathcal{Y}$ enforces no constraint on the focal sets of $\mu_\mathcal{X}$ with weights greater than θ_Π. However the focal sets of $\mu_\mathcal{X}$ with weight θ_Π say $\mathcal{F}(\mu_{\mathcal{X}\theta_\Pi})$ are singletons such that $\forall E \in \mathcal{F}(\mu_\mathcal{X})$, s.t. $\mu_\mathcal{X}(E) > \theta_\Pi, \exists\{s\} \in \mathcal{F}(\mu_{\mathcal{X}\theta_\Pi}) : s \in E$. This makes it sure that for $\lambda \leq \theta_\Pi$, the λ-cuts of $\mu_\mathcal{X}$ and $\mu_\mathcal{Y}$ commute because they are possibility measures. In this case, $\theta_N = \theta_D = \theta_\Pi < 1$

Next, suppose that a threshold $\theta_D < 1$ exists such that for $\lambda > \theta_D$, the λ-cuts of $\mu_\mathcal{X}$ are any Boolean capacity. It enforces a Dirac function focused on some $y^* \in \mathcal{Y}$ for the λ-cuts of $\mu_\mathcal{Y}$. As $\mu_\mathcal{X}$ and $\mu_\mathcal{Y}$ commute, there are constraints on their λ-cuts for $\lambda \leq \theta_D$. If they are possibility measures we are back to the previous case. Suppose not, and that the λ-cuts of $\mu_\mathcal{Y}$ are general Boolean capacities for $\theta_D \geq \lambda > \theta_\Pi$. It enforces a Dirac function on the other side, focused on some $x^* \in \mathcal{X}$. There are additional constraints induced:

- First x^* must be contained in the focal sets of $\mu_\mathcal{X}$ with weights greater than θ_D (hence these focal sets must overlap) to make it sure the only focal set of the λ-cuts of $\mu_\mathcal{X}$ is $\{x^*\}$ for $\theta_D \geq \lambda > \theta_\Pi$.
- Moreover, as μ_Y is strictly monotonic on its focal sets, y^* is necessarily disjoint from the focal sets of $\mu_\mathcal{Y}$ with weights $\theta_D \geq \lambda > \theta_\Pi$.

In this case, $\theta_N = 1 > \theta_D > \theta_\Pi \geq 0$. If $\theta_\Pi = 0$, $\mu_\mathcal{X}$ and $\mu_\mathcal{Y}$ are two-tiered capacities the cut of one being a Dirac above θ_D the cut of the other being a Dirac below θ_D. Otherwise, the λ-cuts of $\mu_\mathcal{X}$ and $\mu_\mathcal{Y}$ are possibility measures for $\lambda \leq \theta_\Pi$.

Finally, suppose that for $\lambda = 1$, the λ-cuts of $\mu_\mathcal{X}$ and $\mu_\mathcal{Y}$ are non-trivial necessity measures. Let θ_N be the maximal value of λ such that one of the λ-cuts of $\mu_\mathcal{X}$ or $\mu_\mathcal{Y}$ is not a non-trivial necessity measure. Suppose, it is $\mu_\mathcal{X}$ whose θ_N-cut is not a non-trivial necessity measure. It can be

- A general capacity. Then the θ_N-cut of $\mu_\mathcal{Y}$ must be Dirac function.

- A Dirac function. Then the θ_N-cut of $\mu_{\mathcal{Y}}$ can be any capacity
- A non-trivial possibility measure. Then the θ_N-cut of $\mu_{\mathcal{Y}}$ must be the same.

Clearly, from θ_N down to 0, we are back to the previous situations. At most we have 4-tiered capacities when $1 \geq \theta_N > \theta_D > \theta_\Pi > 0$ as pictured on Figure 2. But we can have at the other extreme $\theta_N = \theta_D = \theta_\Pi = 0$ (then $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ are necessity measures). Other noticeable cases are when $\theta_N > \theta_D = \theta_\Pi > 0$ (then one of $\mu_{\mathcal{X}}, \mu_{\mathcal{Y}}$ is a necessity measure on top of a possibility measure), or yet $\theta_N > \theta_D > \theta_\Pi = 0$ (eliminating the bottom non-trivial possibility layer). □

Example 3. (See also Halas *et al.*[15]) We can apply Theorem 2 to find the condition for commutation on $\{x_1, x_2\} \times \{y_1, y_2\}$ where, without loss of generality, $\mu_{\mathcal{X}}(x_1) = \alpha_1 > \mu_{\mathcal{X}}(x_2) = \alpha_2$, $\mu_{\mathcal{Y}}(y_1) = \beta_1 > \mu_{\mathcal{Y}}(y_2) = \beta_2$. Note that cuts of capacity on two-element sets can only be Boolean possibility or necessity measures. So the capacities will commute except if there is $\lambda \in L$ such that the cut of $\mu_{\mathcal{X}}$ is a non-trivial possibility measure and the cut of $\mu_{\mathcal{Y}}$ is a non-trivial necessity measure. It is easy to check that, up to a permutation of $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$:

- $\theta_N = \max(\alpha_1, \beta_1)$ since for $\lambda > \max(\alpha_1, \beta_1)$ the λ-cuts of $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ are vacuous necessity measures
- $\theta_\Pi = \min(\alpha_2, \beta_2)$ since for $\lambda \leq \min(\alpha_2, \beta_2)$ the λ-cuts of $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ are vacuous possibility measures
- Suppose $\alpha_1 > \beta_1 > \alpha_2 > \beta_2$. Then $\theta_D = \alpha_2$ since for $\alpha_1 \geq \lambda > \alpha_2$ the λ-cuts of $\mu_{\mathcal{X}}$ are a Dirac function on x_1 and for $\alpha_2 \geq \lambda > \beta_2$ the λ-cuts of $\mu_{\mathcal{X}}$ are the vacuous possibility measure, while the λ-cuts of $\mu_{\mathcal{Y}}$ are a Dirac function on y_1
- Suppose $\alpha_1 > \alpha_2 > \beta_1 > \beta_2$. Then, for $\alpha_2 \geq \lambda > \beta_1$ the λ-cuts of $\mu_{\mathcal{X}}$ are the vacuous possibility measure, while the λ-cuts of $\mu_{\mathcal{Y}}$ are still the vacuous necessity measure, which prevents commutation.

It can thus be seen that the only cases when the two capacities do not commute are when $\max(\alpha_1, \alpha_2) < \min(\beta_1, \beta_2)$ or $\max(\beta_1, \beta_2) < \min(\alpha_1, \alpha_2)$ (take λ in the interval).

Note that this is the case in Example 1 since then $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = 0$.

However the commutation condition is clearly satisfied in Example 2, since it verifies the commutation condition $\max(\alpha_1, \alpha_2) \geq \min(\beta_1, \beta_2)$ and $\max(\beta_1, \beta_2) \geq \min(\alpha_1, \alpha_2)$ (the pairs (α_1, α_2) and (β_1, β_2) must be intertwined). □

We can express commuting capacities in closed form, in terms of underlying possibility, necessity and Dirac functions.

Corollary 5. *Two commuting capacities $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ can be put in the following form (up to a swap between $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$):*

$$\mu_{\mathcal{X}}(A) = \max(N_{\mathcal{X}}(A), \min(\theta_N, \kappa_{\mathcal{X}}(A)), \min(\theta_D, \delta_{\mathcal{X}}(A)), \min(\theta_\Pi, \Pi_{\mathcal{X}}(A)))\,, \quad (12)$$

$$\mu_{\mathcal{Y}}(B) = \max(N_{\mathcal{Y}}(A), \min(\theta_N, \delta_{\mathcal{Y}}(A)), \min(\theta_D, \kappa_{\mathcal{Y}}(A)), \min(\theta_\Pi, \Pi_{\mathcal{Y}}(A)))\,, \quad (13)$$

for suitable choices of thresholds $\theta_N \geq \theta_D \geq \theta_\Pi \in L$*, non-trivial necessity measures* $N_\mathcal{X}, N_\mathcal{Y}$ *and possibility measures* $\Pi_\mathcal{X}, \Pi_\mathcal{Y}$*, capacities* $\kappa_\mathcal{X}, \kappa_\mathcal{Y}$*, and Dirac functions* $\delta_\mathcal{X}, \delta_\mathcal{Y}$*.*

Proof: Suppose two capacities $\mu_\mathcal{X}$ and $\mu_\mathcal{Y}$ commute and let $\theta_N \geq \theta_D \geq \theta_\Pi \in L$ be the thresholds defined in the proof of Theorem 2, which ensures that there is $y^* \in \mathcal{Y}$ such that $\mu_\mathcal{Y}(\{y^*\}) = \theta_N$, $x^* \in \mathcal{Y}$ such that $\mu_\mathcal{X}(\{x^*\}) = \theta_D$. Define the following set functions

- $N_\mathcal{X}(A) = \begin{cases} \mu_\mathcal{X}(A) & \text{if } \mu_\mathcal{X}(A) > \theta_N \\ 0 & \text{otherwise} \end{cases}$;

  $N_\mathcal{Y}(B) = \begin{cases} \mu_\mathcal{X}(B) & \text{if } \mu_\mathcal{Y}(B) > \theta_N\,, \\ 0 & \text{otherwise.} \end{cases}$

  They are necessity measures by definition of θ_N
- $\kappa_\mathcal{X}(A) = \begin{cases} \mu_\mathcal{X}(A) & \text{if } \theta_N \geq \mu_\mathcal{X}(A) > \theta_D\,, \\ 1 & \text{if } A = \mathcal{X}\,, \\ 0 & \text{otherwise} \end{cases}$ and $\delta_\mathcal{Y}(B) = \begin{cases} 1 & \text{if } y^* \in B \\ 0 & \text{otherwise} \end{cases}$.

  $\delta_\mathcal{Y}$ is a Dirac function focused on y^*.
- $\delta_\mathcal{X}(A) = \begin{cases} 1 & \text{if } x^* \in A\,, \\ 0 & \text{otherwise} \end{cases}$ and $\kappa_\mathcal{Y}(B) = \begin{cases} \mu_\mathcal{Y}(B) & \text{if } \theta_D \geq \mu_\mathcal{Y}(B) > \theta_\Pi\,, \\ 1 & \text{if } B = \mathcal{Y}\,, \\ 0 & \text{otherwise.} \end{cases}$

  $\delta_\mathcal{X}$ is a Dirac function focused on x^*.
- $\pi_\mathcal{X}(x) = \begin{cases} \mu_\mathcal{X}(\{x\}) & \text{if } \mu_\mathcal{X}(\{x\}) \leq \theta_\Pi \\ 1 & \text{if } x = x^* \\ 0 & \text{otherwise} \end{cases}$;

  $\pi_\mathcal{Y}(y) = \begin{cases} \mu_\mathcal{Y}(\{y\}) & \text{if } \mu_\mathcal{Y}(\{y\} \leq \theta_\Pi\,, \\ 1 & \text{if } y = y^*\,, \\ 0 & \text{otherwise.} \end{cases}$

  Possibilitity measures $\Pi_\mathcal{X}, \Pi_\mathcal{Y}$ are those induced by possibility distributions $\pi_\mathcal{X}$ and $\pi_\mathcal{Y}$.

It is then easy to check that $\mu_\mathcal{X}$ and $\mu_\mathcal{Y}$ can be put in the form (12) and (13) □

These results make it easy to construct capacities that commute.

Example 4. Consider the commuting capacities in Example 3. Suppose $\alpha_1 > \beta_1 > \alpha_2 > \beta_2$. We can check that $\theta_N = \alpha_1, \theta_D = \alpha_2 \geq \theta_\Pi = \beta_2$. Necessity measures are vacuous ones $N^?_\mathcal{X}$ and $N^?_\mathcal{Y}$ with focal sets $\mathcal{X}$ and $\mathcal{Y}$ respectively. There is a Dirac function $\delta_{\mathcal{X}}^{x_1}$ focused on x_1 that can be used for the range of values $(\alpha_2, \alpha_1]$, and a Dirac function $\delta_Y^{y_1}$ focused on y_1 that can be used for the range of values $(\beta_2, \alpha_2]$. Consider the capacities $\kappa_{\mathcal{X}}$ which is the possibility measure with distribution $\pi(x_1) = 1, \pi(x_2) = \alpha_2$ and $\kappa_{\mathcal{Y}}$ which is the necessity measure with focals $\mathcal{Y}$ (weight 1) and $\{y_1\}$ with weight β_1. Finally let $\Pi_{\mathcal{X}}^?$ and $\Pi_{\mathcal{Y}}^?$ be vacuous necessity and possibility measures. Then it can be checked that

$$\mu_{\mathcal{X}}(A) = \max(N_{\mathcal{X}}^?(A), \min(\alpha_1, \delta_{\mathcal{X}}^{x_1}(A)), \min(\alpha_2, \kappa_{\mathcal{X}}(A)), \min(\beta_2, \Pi_{\mathcal{X}}^?(A)))$$

and

$$\mu_{\mathcal{Y}}(B) = \max(N_{\mathcal{Y}}^?(B), \min(\alpha_1, \kappa_{\mathcal{Y}}(B)), \min(\alpha_2, \delta_{\mathcal{Y}}^{y_1}(B)), \min(\beta_2, \Pi_{\mathcal{Y}}^?(B)))\,.$$

Finally in Example 2 we have $\mu_{\mathcal{X}}(x_1) = \alpha, \mu_{\mathcal{X}}(x_2) = \alpha$, and $\mu_{\mathcal{Y}}(y_1) = 1, \mu_{\mathcal{Y}}(y_2) = \alpha$. Then $\theta_N = \theta_\Pi = \alpha$, we can write $\mu_{\mathcal{X}}$ as $\max(N_{\mathcal{X}}^?, \min(\alpha, \Pi_{\mathcal{X}}^?))$, and $\mu_{\mathcal{Y}}$ as $\max(\delta_{\mathcal{Y}}^{y_1}, \min(\alpha, \Pi_{\mathcal{Y}}^?))$. □

Note that the corresponding Sugeno integrals are pessimistic on plausible events and optimistic on implausible ones. The pessimism and optimism of commuting set functions are controlled by the two parameters $\theta_N \geq \theta_\Pi \in L$, which may make qualitative decision theory more flexible. This way of controlling the pessimism and optimism of a decision criterion could be compared with an alternative proposal based on uninorms.[11]

5. 2D Capacities and Commutation

We have seen that a double Sugeno integral comes down to a Sugeno integral with respect to a 2D capacity obtained from two 1D capacities. A Sugeno integral based on a 2D capacity will be called a 2D S-integral.

It is interesting to compare the 2D S-integral based on a 2D capacity with the double S-integrals obtained using the projections of the 2D capacity and see if the commutation of these projections affects this issue. When the 1D capacities on $\mathcal{X}$ and $\mathcal{Y}$ commute, the 2D capacities on $\mathcal{X} \times \mathcal{Y}$ each of them induces are the same. The results in this section enable to express commuting double S-integrals in a symmetric form, as a 2D S-integral based on this 2D capacity.

A 2D fuzzy measure is simply a fuzzy measure μ on $\mathcal{X} \times \mathcal{Y}$. The definition of a (simple) 2D S-integral with respect to such a capacity is defined by

$$S_\mu(u) = \max_{T \subseteq \mathcal{X} \times \mathcal{Y}} \min(\mu(T), \min_{(i,j) \in T} u(x_i, y_j))\,.$$

Before exploring the properties of this type of integral in connection with the commutation problem, we need to study 2D capacities, and, in particular, conditions under which a 2D-measure can be reconstructed via its projections.

5.1. *Projections of 2D capacities*

We can define the projections of a 2D fuzzy measure μ on $\mathcal{Y}$ and on $\mathcal{X}$ as follows:

Definition 3. $\mu_{\mathcal{X}}^{\downarrow}(A) = \mu(A \times \mathcal{Y})$; $\mu_{\mathcal{Y}}^{\downarrow}(B) = \mu(\mathcal{X} \times B)$.

Without loss of generality, we focus on the projection on $\mathcal{Y}$. Let $R^{\downarrow}_{\mathcal{Y}} = \{y \in \mathcal{Y} : \exists x \in \mathcal{X}, \text{ s.t.} (x, y) \in R\}$ be the projection of $R \subseteq \mathcal{X} \times \mathcal{Y}$ on $\mathcal{Y}$. Note that $\mu^{\downarrow}_{\mathcal{Y}}(B) = \max_{R:B=R^{\downarrow}_{\mathcal{Y}}} \mu(R)$ because this maximum is reached for $R = \mathcal{X} \times B$. In other words, $\mu^{\downarrow}_{\mathcal{Y}}$ is the "shadow" of μ on $\mathcal{Y}$. This definition of projection is the same as for joint probability measures.

One may try to compute the focal sets of the projections in terms of projections of the focal sets of the 2D capacity.

Proposition 8. *The focal sets of $\mu^{\downarrow}_{\mathcal{Y}}$ are among the projections on $\mathcal{Y}$ of the focal sets of μ: $\mathcal{F}(\mu^{\downarrow}_{\mathcal{Y}}) \subseteq \{R^{\downarrow}_{\mathcal{Y}} : R \in \mathcal{F}(\mu)\}$.*

Proof: Suppose $B \in \mathcal{F}(\mu^{\downarrow}_{\mathcal{Y}})$ and $\mu^{\downarrow}_{\mathcal{Y}}(B) = b > 0$. Then, $\forall C \subset B, \mu^{\downarrow}_{\mathcal{Y}}(C) < \mu^{\downarrow}_{\mathcal{Y}}(B)$. This is equivalent to $\mu(\mathcal{X} \times C) < b, \forall C \subset B$. So there is a focal set $R \subseteq \mathcal{X} \times B$ of μ with weight b, and none with weight greater than b. Clearly, $R^{\downarrow}_{\mathcal{Y}} \subseteq B$. Suppose there is a focal set of μ, say $S \subset \mathcal{X} \times B$ with weight b, and $S^{\downarrow}_{\mathcal{Y}} \subset B$. Then B would not be focal for $\mu^{\downarrow}_{\mathcal{Y}}$. So all focal sets of μ, the projections on $\mathcal{Y}$ of which are strict subsets of B have weights less than B. So there is some focal set of μ with weight b such that $R^{\downarrow}_{\mathcal{Y}} = B$. □

Note that the projection of a focal set of μ on $\mathcal{Y}$ is not always focal for $\mu^{\downarrow}_{\mathcal{Y}}$.

Example 5. For instance, suppose μ has only two focal sets, say $R = \{(x_2, y_1), (x_2, y_2)\}$ and $T = \{(x_1, y_1), (x_2, y_1)\}$, with $\mu(T) = 1 > \mu(R)$. R and T are not nested but $T^{\downarrow}_{\mathcal{Y}} = \{y_1\} \subset R^{\downarrow}_{\mathcal{Y}} = \{y_1, y_2\}$. So,

- $\mu^{\downarrow}_{\mathcal{Y}}(T^{\downarrow}_{\mathcal{Y}}) = \mu^{\downarrow}_{\mathcal{Y}}(\{y_1\}) = \mu(\mathcal{X} \times \{y_1\}) = \mu(\{(x_1, y_1), (x_2, y_1)\}) = \mu(T) = 1.$
- $\mu^{\downarrow}_{\mathcal{Y}}(R^{\downarrow}_{\mathcal{Y}}) = \mu^{\downarrow}_{\mathcal{Y}}(\mathcal{Y}) = 1.$

So, the projection of R on $\mathcal{Y}$ is not focal for $\mu^{\downarrow}_{\mathcal{Y}}$. □

In fact the projection of μ can be defined from the focal sets of μ as

$$\mu^{\downarrow}_{\mathcal{Y}}(B) = \mu(\mathcal{X} \times B) = \max_{R \in \mathcal{F}(\mu): R^{\downarrow}_{\mathcal{Y}} \subseteq B} \mu_{\#}(R)$$

Let us call *a sufficient fragment* of μ a weighted set $\{(R_i, \mu^*(R_i)), i = 1, \dots\}$ such that $\mu(R) = \max_{R_i \subseteq R} \mu^*(R_i)$. Clearly $\mathcal{F}(\mu)$ is (the smallest) sufficient fragment of μ and any family $\mathcal{S} \supset \mathcal{F}$, with $\mu^*(R_i) = \mu(R_i), R_i \in \mathcal{S}$ is sufficient. The projections $R^{\downarrow}_{\mathcal{Y}}$ of focal sets of μ form a sufficient fragment of $\mu^{\downarrow}_{\mathcal{Y}}$. One gets the focal sets of $\mu^{\downarrow}_{\mathcal{Y}}$ by deleting from this fragment the sets B' for which there is a subset $B \subsetneq B'$ such that $\mu(\mathcal{X} \times B) = \mu(\mathcal{X} \times B')$.

It is easy to check that the projections of the α-cuts of a 2D-capacity are equal to the α-cuts of their projections. Indeed, by definition, $(\mu_\alpha)^{\downarrow}_{\mathcal{Y}}(B) = \mu_\alpha(\mathcal{X} \times B) = (\mu^{\downarrow}_{\mathcal{Y}})_\alpha(B)$.

5.2. *Product 2D capacities*

In the other way around, consider building a joint capacity of $\mathcal{X} \times \mathcal{Y}$ from the knowledge of two local capacities, γ and δ on $\mathcal{X}$ and $\mathcal{Y}$ respectively. It is always possible to build a 2D capacity μ on $\mathcal{X} \times \mathcal{Y}$ from γ and δ as follows: for any $A \subseteq \mathcal{X}, B \subseteq \mathcal{Y}$, $\mu(A \times B) = \gamma(A) \otimes \delta(B)$, where it is expected to have (in view of the definition of projection), $0 \otimes 0 = 1 \otimes 0 = 0 \otimes 1 = 0, 1 \otimes 1$ and $\otimes$ is increasing for each argument. Clearly, $\otimes$ is a generalized conjunction. Because we work in the qualitative context of bounded chains (which motivates the use of Sugeno integrals), let $\otimes$ be the minimum. So, we define the *joint capacity* $\gamma \times \delta$ obtained from γ on $\mathcal{X}$ and δ on $\mathcal{Y}$ by the expression:

$$[\gamma \times \delta](R) = \max_{A,B: A \times B \subseteq R} \min(\gamma(A), \delta(B)) .$$

Note that $[\gamma \times \delta](A \times B) = \min(\gamma(A), \delta(B))$. In the sequel, we show that the focal sets of $\gamma \times \delta$ are Cartesian products, and they are among the Cartesian products $A \times B$ where A is a focal set of γ and B is a focal set of δ.

Proposition 9. *Any focal set of $\gamma \times \delta$ is the Cartesian product of a focal set of γ and a focal set of δ.*

Proof: If R is not a Cartesian product, it is clear that there exists A, B such that $[\gamma \times \delta](R) = \min(\gamma(A), \delta(B))$ where $A \times B \subset R$. So R is not focal. Consider a focal set $A \times B$ of $\gamma \times \delta$. Suppose A is not focal for γ, then $\gamma(A) = \gamma_{\#}(\hat{A})$ where $\hat{A} \subset A$. Clearly, $[\gamma \times \delta](A \times B) = [\gamma \times \delta](\hat{A} \times B)$. Hence $A \times B$ cannot be focal. So if $A \times B$ is focal for $\gamma \times \delta$ then, A is focal for γ and B for δ. □

But the Cartesian product of a focal set A of γ and a focal set B of δ is not necessarily a focal set of $\gamma \times \delta$. It can indeed contain the product of another focal set $A' \subset A$ of γ and B, such that $\min(\gamma(A'), \delta(B)) = \min(\gamma(A), \delta(B))$.

Example 6. Suppose γ and δ are necessity measures with focal sets $\{A, \mathcal{X}\}$ and $\{B, \mathcal{Y}\}$ respectively, with $\gamma(A) = a$ and $\delta(B) = b$ where $a > b$. $\gamma \times \delta$ is a necessity measure s.t. $[\gamma \times \delta](A \times B) = \min(a, b) = b$, $[\gamma \times \delta](A \times \mathcal{Y}) = \min(a, 1) = a$, $[\gamma \times \delta](\mathcal{X} \times B) = \min(1, b) = b$ and of course $[\gamma \times \delta](\mathcal{X} \times \mathcal{Y}) = \min(1, 1) = 1$. Then $\gamma \times \delta$ has focal sets $A \times B$ with weight b, $A \times \mathcal{Y}$ with weight a and $\mathcal{X} \times \mathcal{Y}$ with weight 1, while $\mathcal{X} \times B$ is not focal (it contains the focal set $A \times B$, which has the same weight b). □

Remark 1. An alternative definition could be found natural such as

$$[\gamma \odot \delta](R) = \min(\gamma(R^{\downarrow}_{\mathcal{X}}), \delta(R^{\downarrow}_{\mathcal{Y}})) .$$

But this would not be the same definition as above because $R \subset R^{\downarrow}_{\mathcal{X}} \times R^{\downarrow}_{\mathcal{Y}}$. If we let $\{A_i \times B_i : i = 1, \dots p\}$ be the maximal Cartesian products inside R, we may find that $[\gamma \times \delta](R) = \max_{i=1}^{p} \min(\gamma(A_i), \delta(B_j)) < \min(\gamma(R^{\downarrow}_{\mathcal{X}}), \delta(R^{\downarrow}_{\mathcal{Y}}))$.

The Cartesian products of the focal sets of γ and δ form a family $\{(A \times B, \min(\gamma(A), \delta(B)) : A \in \mathcal{F}(\gamma), B \in \mathcal{F}(\delta)\}$ which contains the focal sets of $\gamma \times \delta$, and maybe more sets — i.e., it is a sufficient fragment for $\gamma \times \delta$.

It is easy to check that the product is performed without any modification of the information contained in γ and δ

Proposition 10. *The projections of $\gamma \times \delta$ are γ and δ*

Proof: First let $\gamma(2^{\mathcal{X}}) = \{1 > a_1 \cdots > a_p\}$ and $\gamma(2^{\mathcal{Y}}) = \{1 > b_1 \cdots > b_q\}$. The weights of the joint capacity are $\{\min(a_i, b_j) : i = 1, \dots p, j = 1, \dots q\} = \gamma(2^{\mathcal{X}}) \cup \gamma(2^{\mathcal{Y}})$ since both sets contain 1. No information about weights is lost. Indeed, it is easy to see that $([\gamma \times \delta])^{\downarrow}_{\mathcal{Y}}(B_j) = [\gamma \times \delta](\mathcal{X} \times B_j) = \min(1, b_j) = b_j$, and so on. So we recover $\delta_{\#}$ on $\mathcal{Y}$ and it is a sufficient fragment of the projection (due to Proposition 8). □

5.3. *Decomposable 2D capacities*

Now we consider the opposite problem: When does a capacity contain the same amount of information as its projections? This issue is important to consider, since, as we shall see, the 2D capacity induced by a commuting double S-integral is decomposable in the following sense:

Definition 4. A 2D capacity μ is said to be decomposable if it is the product of its projections, namely, $\mu = \mu^{\downarrow}_{\mathcal{X}} \times \mu^{\downarrow}_{\mathcal{Y}}$.

Note that in the case of possibility measures decomposability is also called non-interactivity by Zadeh.[8,25] Let us try to find decomposability conditions. There is a direct consequence of Proposition 9 that shows that the focal sets of a joint capacity are Cartesian products:

Lemma 11. *If a 2D capacity μ is decomposable then all its focal sets are Cartesian products.*

The converse of Corollary 11 is false. The fact that the focal sets of μ are Cartesian products (and even disjoint products) does not guarantee the decomposability of μ

Example 7. Suppose $\mathcal{F}(\mu)$ has just three focal sets $A \times B$ (with weight 1), $A' \times B$ (with weight $a < 1$), $A \times B'$ (with weight $b < a$), where $A \neq A'$ and $B \neq B'$ with no inclusion. All are Cartesian products. It is clear that the focal sets of $\mu^{\downarrow}_{\mathcal{X}}$ are A (with weight 1) and A' (with weight a) and those of $\mu^{\downarrow}_{\mathcal{Y}}$ are B (with weight 1) and B' (with weight b). But $\mu^{\downarrow}_{\mathcal{X}} \times \mu^{\downarrow}_{\mathcal{Y}}$ has four focal sets, namely, the ones of μ, i.e., $A \times B$ (with weight 1), $A' \times B$ (with weight $a < 1$), $A \times B'$ (with weight $b < a$), plus $A' \times B'$ with weight b. The latter is focal as it contains none of the former. So $[\mu^{\downarrow}_{\mathcal{X}} \times \mu^{\downarrow}_{\mathcal{Y}}](A' \times B') > \mu(A' \times B') = 0$. □

It is also clear that

Lemma 12. *If a 2D capacity μ is decomposable then $\mu(A \times B) = \min(\mu^{\downarrow}_{\mathcal{X}}(A), \mu^{\downarrow}_{\mathcal{Y}}(B))$ for all focal sets $A \times B \in \mathcal{X} \times \mathcal{Y}$ of μ.*

But the converse is also false, since all 2D necessity measures satisfy $N(A \times B) = \min(N^{\downarrow}_{\mathcal{X}}(A), N^{\downarrow}_{\mathcal{Y}}(B))$, but not all of them are decomposable.

Example 8. Suppose that μ has two focals $R = \{(x_1, y_1), (x_1, y_2), (x_2, y_2)\}$ and $\mathcal{X} \times \mathcal{Y}$, with $\mu_{\#}(R) < 1$. Hence $R^{\downarrow}_{\mathcal{X}} = \{x_1, x_2\}$ and $R^{\downarrow}_{\mathcal{Y}} = \{y_1, y_2\}$. Then μ is a necessity measure (of course $\mu(\{x_1, x_2\} \times \{y_1, y_2\}) = 1$). Like for all necessity measures,

$$\mu(A \times B) = \mu((A \times \mathcal{Y}) \cap (\mathcal{X} \times B)) = \min(\mu^{\downarrow}_{\mathcal{X}}(A), \mu^{\downarrow}_{\mathcal{Y}}(B))$$

for all Cartesian products $A \times B \in \mathcal{X} \times \mathcal{Y}$ of μ.

But μ is not decomposable. It is clear that the only focal set of $\mu^{\downarrow}_{\mathcal{X}}$ is $\mathcal{X}$ (with weight 1) and the only focal set of $\mu^{\downarrow}_{\mathcal{Y}}$ is $\mathcal{Y}$ (with weight 1). Since one of its focal sets, R is not a Cartesian product, it is not focal for $\mu^{\downarrow}_{\mathcal{X}} \times \mu^{\downarrow}_{\mathcal{Y}}$, the joint capacity obtained from its projections. The latter is not equal to μ as its only focal set is $\mathcal{X} \times \mathcal{Y}$. In particular, $[\mu^{\downarrow}_{\mathcal{X}} \times \mu^{\downarrow}_{\mathcal{Y}}](R) = 0$. □

However, if none of the (necessary) conditions above is alone sufficient for defining decomposable capacities, their conjunction does characterizes capacities that are the joint of their projection, i.e., decomposable capacities:

Theorem 3. *A 2D capacity μ is decomposable if and only if all its focal sets are Cartesian products and, for all focal sets $A \times B \in \mathcal{X} \times \mathcal{Y}$ of μ, $\mu(A \times B) = \min(\mu^{\downarrow}_{\mathcal{X}}(A), \mu^{\downarrow}_{\mathcal{Y}}(B))$.*

Proof: We know that the projections of the focal sets are sufficient fragments of the projections of μ. So we can write $[\mu^{\downarrow}_{\mathcal{X}} \times \mu^{\downarrow}_{\mathcal{Y}}](R) = \max_{A \times B \in \mathcal{F}(\mu): A \times B \subseteq R} \min(\mu^{\downarrow}_{\mathcal{X}}(A), \mu^{\downarrow}_{\mathcal{Y}}(B)) = \max_{A \times B \in \mathcal{F}(\mu): A \times B \subseteq R} \mu(A \times B) = \mu(R)$ since all focal sets of μ are Cartesian products. □

As a special case it is clear that if μ is a possibility measure the condition in Proposition 3 reduces to the usual non-interactivity condition $\pi(x, y) = \min(\pi^{\downarrow}_{\mathcal{X}}(x), \pi^{\downarrow}_{\mathcal{Y}}(y))$ since all focal sets of μ are singletons $\{(x, y)\}$, i.e., special Cartesian products $\{x\} \times \{y\}$. In the case of a necessity measure N, the condition $N(A \times B) = \min(N^{\downarrow}_{\mathcal{X}}(A), N^{\downarrow}_{\mathcal{Y}}(B))$ is always true, so all we need for decomposability is that the (nested) focal sets of N are Cartesian products.

It is useful to describe the constraints on weights of focal sets of μ that are characteristic of its decomposability. We can first do it in the special case of Boolean capacities.

Proposition 11. *A Boolean 2D capacity μ is decomposable if and only if its focal sets are all Cartesian products of the form $\{A_i \times B_j : i = 1 \ldots p, j = 1 \ldots q\}$ such that both families $\{A_i : i = 1 \ldots p\}$ and $\{B_j : j = 1 \ldots q\}$ are antichains.*

Proof: If the requested conditions are required it is clear that $\{A_i \times B_j : i = 1 \ldots p, j = 1 \ldots q\}$ form an anti-chain as well, so they are indeed focal sets for μ. The focals of $\mu^{\downarrow}_{\mathcal{X}}$ are clearly all of $\{A_i : i = 1 \ldots p\}$, and those of $\mu^{\downarrow}_{\mathcal{Y}}$ are clearly all of $\{B_j : j = 1 \ldots q\}$. Now, $[\mu^{\downarrow}_{\mathcal{X}} \times \mu^{\downarrow}_{\mathcal{Y}}](A_i \times B_j) = \min(\mu^{\downarrow}_{\mathcal{X}}(A_i), \mu^{\downarrow}_{\mathcal{Y}}(B_j)) = 1$. The joint capacity has clearly no other focal sets than $\{A_i \times B_j : i = 1 \ldots p, j = 1 \ldots p\}$. So it is μ.
Conversely suppose $\{A_i : i = 1 \ldots p\}$ is not an antichain. Then there are i, k with $A_i \subset A_k$. Then A_i is not a focal for $\mu^{\downarrow}_{\mathcal{X}}$ and so there is no Cartesian product of the form $A_i \times B_j$ that is focal for $\mu^{\downarrow}_{\mathcal{X}} \times \mu^{\downarrow}_{\mathcal{Y}}$. So the joint of the projections of μ is not μ. □

The following result shows that decomposability is cut-worthy:

Proposition 12. *A 2D capacity is decomposable if and only if its α-cuts are decomposable.*

Proof: (i) Suppose μ is decomposable. Then $\mu(R) = \min(\mu^{\downarrow}_{\mathcal{X}}(A), \mu^{\downarrow}_{\mathcal{Y}}(B))$ for some $A \times B \subseteq R$, Hence $\mu_\alpha(R) = \min((\mu^{\downarrow}_{\mathcal{X}})_\alpha(A), (\mu^{\downarrow}_{\mathcal{Y}})_\alpha(B)) = \min((\mu_\alpha)^{\downarrow}_{\mathcal{X}})(A), (\mu_\alpha)^{\downarrow}_{\mathcal{Y}}(B))$.
(ii) Conversely suppose μ_α is decomposable for all $\alpha > 0$. Then, for some $A \times B \subseteq R$, $\mu(R) = \max_{\alpha>0} \min(\alpha, \mu_\alpha(R)) = \max_{\alpha>0} \min(\alpha, \min((\mu_\alpha)^{\downarrow}_{\mathcal{X}}(A), (\mu_\alpha)^{\downarrow}_{\mathcal{Y}}(B)))$. The maximum is attained for $\alpha = \mu(R)$, which implies $\mu^{\downarrow}_{\mathcal{X}}(A) \geq \mu(R)$ and $\mu^{\downarrow}_{\mathcal{X}}(B) \geq \mu(R)$. Moreover $\mu_\alpha(R) = 0$ for all $\alpha > \mu(R)$, hence $(\mu_\alpha)^{\downarrow}_{\mathcal{X}}(A) = 0$ or $(\mu_\alpha)^{\downarrow}_{\mathcal{Y}}(B) = 0$. In consequence $\mu^{\downarrow}_{\mathcal{X}}(A) = \mu(R)$ or $\mu^{\downarrow}_{\mathcal{X}}(B) = \mu(R)$. Hence $\mu(R) = \min(\mu^{\downarrow}_{\mathcal{X}}(A), \mu^{\downarrow}_{\mathcal{Y}}(B))$. □

Finally, one can see that the decomposability criterion for 2D capacities puts together the decomposability condition for necessity and for possibility measures:

Corollary 6. *A 2D capacity μ is decomposable if and only if the family of sets*

$$\mathcal{F}(\mu^{\downarrow}_{\mathcal{X}}) \times \mathcal{F}(\mu^{\downarrow}_{\mathcal{Y}}) = \{A \times B : A \in \mathcal{F}(\mu^{\downarrow}_{\mathcal{X}}), B \in \mathcal{F}(\mu^{\downarrow}_{\mathcal{Y}})\}$$

is a sufficient fragment for μ, and $\mu(A \times B) = \min(\mu^{\downarrow}_{\mathcal{X}}(A), \mu^{\downarrow}_{\mathcal{Y}}(B))$.

It is obvious that the double condition in the Corollary is sufficient for decomposability and the assumption that focal sets are Cartesian products is necessary. If we consider μ as a possibility distribution on $\mathcal{F}(\mu^{\downarrow}_{\mathcal{X}}) \times \mathcal{F}(\mu^{\downarrow}_{\mathcal{Y}})$, the second equality expresses that it is non-interactive in the sense of possibility theory.

5.4. *The symmetric form of commuting double S-integrals*

Let us now consider the problem of expressing 2D S-integrals as double Sugeno integrals $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(u))$ and $S_{\mu_\mathcal{Y}}(S_{\mu_\mathcal{X}}(u))$. In particular, we study to what extent a 2D S-integral can be viewed as a double Sugeno integral with respect to its projections.

We have seen that a double S-integral $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(u))$ is a 2D S-integral based on a capacity that maps each $R \subseteq \mathcal{X} \times \mathcal{Y}$ to L, defined by $\kappa_{\mathcal{X}\mathcal{Y}}(R) = S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R))$

$$S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(u)) = \max_{R \subseteq \mathcal{X} \times \mathcal{Y}} \min(\kappa_{\mathcal{X}\mathcal{Y}}(R), \min_{(x_i, y_j) \in R} u(x_i, y_j))$$

The following example shows that the 2D capacity $\kappa_{\mathcal{X}\mathcal{Y}}$ is not necessarily decomposable.

Example 9. (continued from Example 1): Let $\mathcal{Y} = \{y_1, y_2\}$, $\pi_1 = \pi_2 = 1$ on $\mathcal{X} = \{x_1, x_2\}$, there is a necessity measure N with focal $\mathcal{Y}$ on the other set. We can lay bare the 2D capacity involved in the definition of $U^{+min}_{ante}(\pi, w, u) = S_{N_\mathcal{Y}}(S_{\Pi_\mathcal{X}}(u))$. It is easy to see that $S_{N_\mathcal{Y}}(S_{\Pi_\mathcal{X}}(\mathbf{1}_R)) = 0$ whenever $R = \{(x_i, y_j)\}, i, j \in \{1, 2\}$ (singletons), but $S_{N_\mathcal{Y}}(S_{\Pi_\mathcal{X}}(\mathbf{1}_R)) = 1$ for $R = \{x_i\} \times \mathcal{Y}, i = 1, 2$ and for $R = \{(x_1, y_1), (x_2, y_2)\}, \{(x_1, y_2), (x_2, y_1)\}$ but is 0 for the two other two-elements subsets of $\mathcal{X} \times \mathcal{Y}$. So, the 2D capacity underlying $S_{N_\mathcal{Y}}(S_{\Pi_\mathcal{X}}(\mathbf{1}_R))$ has four focal sets some of which are not Cartesian products. On the other hand, the joint capacity $\Pi \times N$ has only two focal sets $R = \{x_i\} \times \mathcal{Y}, i = 1, 2$. Hence, the 2D capacity underlying $S_{N_\mathcal{Y}}(S_{\Pi_\mathcal{X}}(\mathbf{1}_R))$ is not equal to $\Pi_\mathcal{X} \times N_\mathcal{Y}$ and is not decomposable. □

Can 2D S-integrals also be captured by double S-integrals ? The answer is no in the general case. It is indeed easy to find 2D capacities μ underlying S-integrals which cannot be represented by a double S-integral, i.e., $\exists \mu$ such that $\nexists \mu_\mathcal{X}, \mu_\mathcal{Y} : S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(u)) = S_\mu(u), \forall u$.

Example 10. Consider the 2D capacity with only three focal sets, $\{(x_1, y_1)\}$, $\{(x_1, y_2)\}$ and $\{(x_2, y_1)\}$ which all receive the degree 1, and suppose that there exist $\mu_\mathcal{X}$ and $\mu_\mathcal{Y}$ such as $\forall u, S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(u)) = S_\mu(u)$.
Consider first the utility function $u(x, y) = 1$ if $x = x_2$ and $y = y_2$ and $u(x, y) = 0$ otherwise (the characteristic function of $\{(x_2, y_2)\}$. $S_\mu(u) = \mu(\{(x_2, y_2)\}) = 0$ because $\{(x_2, y_2)\}$ is not focal. But
$S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(u)) = \max(\min(\mu_\mathcal{X}(x_1), 0), \min(\mu_\mathcal{X}(\{x_2\}), 1), \min(\mu_\mathcal{X}(\{x_1, x_2\}, 0))) = \mu_\mathcal{X}(x_2)$. So, $\mu_\mathcal{X}(\{x_2\}) = 0$ is needed for the equality with the 2D integral.
Let now function $u'(x, y) = 1$ if $x = x_2$ and $y = y_1$ and $u(x', y) = 0$ otherwise. $S_\mu(u') = \mu(\{(x_2, y_1)\}) = 1$ because $\{(x_2, y_1)\}$ is a focal set of weight 1. Finally, $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(u')) = \max(\min(\mu_\mathcal{X}(x_1), 0), \min(\mu_\mathcal{X}(\{x_2\}), 1), \min(\mu_\mathcal{X}(\{x_1, x_2\}, 0))) = \mu_\mathcal{X}(x_2)$ again. So, we need $\mu_\mathcal{X}(\{x_2\}) = 1$, which contradicts $\mu_\mathcal{X}(\{x_2\}) = 0$. □

Of course, for some μ, the representation by a double S-integral is possible, e.g., when μ is the product of two possibility measures. In this case, μ is decomposable. But it is not always the case — Example 9 presents a non-decomposable 2D

S-integral that can be represented by a double S-integral. We can nevertheless show that:

Proposition 13. *If there exists $\mu_{\mathcal{X}}$ on $\mathcal{X}$ and $\mu_{\mathcal{Y}}$ on $\mathcal{Y}$ such that $\forall u, S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}(u)) = S_{\mu}(u)$, then $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ are the projections of μ on $\mathcal{X}$ and $\mathcal{Y}$, respectively, and $\forall A, B, \mu(A \times B) = \min(\mu_{\mathcal{X}}(A), \mu_{\mathcal{Y}}(B))$.*

Proof: Suppose that there exist $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ such as $\forall u, S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}(u)) = S_{\mu}(u)$. Consider any pair of subsets A of $\mathcal{X}$ and B of $\mathcal{Y}$ and let u be the characteristic function of $A \times B$, i.e., $u(x, y) = 1$ if $(x, y) \in A \times B$, $u(x, y) = 0$ otherwise.

First of all, $S_{\mu}(u) = \mu(A \times B)$. From Proposition 4 $S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}(u)) = \min(\mu_{\mathcal{X}}(A), \mu_{\mathcal{Y}}(B)))$. So, $S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}(u)) = S_{\mu}(u)$ implies $\mu(A \times B) = \min(\mu_{\mathcal{X}}(A), \mu_{\mathcal{Y}}(B))$.

Setting $A = \mathcal{X}$ we get $\mu(A \times \mathcal{Y}) = \min(\mu_{\mathcal{X}}(A), \mu_{\mathcal{Y}}(\mathcal{Y})) = \mu_{\mathcal{X}}(A)$, i.e., $\mu_{\mathcal{X}}$ is the projection of μ on $\mathcal{X}$. Setting $B = \mathcal{Y}$ we get $\mu(\mathcal{X} \times B) = \min(\mu_{\mathcal{X}}(\mathcal{X}), \mu_{\mathcal{Y}}(B)) = \mu_{\mathcal{Y}}(B)$, i.e., $\mu_{\mathcal{Y}}$ is the projection of μ on $\mathcal{Y}$. □

The above proposition does not say that the equality $\forall u, S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}(u)) = S_{\mu}(u)$ implies that μ is decomposable. However it is so if $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ commute, as shown in the following major result that puts commuting double S-integrals in a symmetric format:

Theorem 4. *If $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ commute, then there exists a decomposable 2D capacity μ such that $\forall u, S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}(u)) = S_{\mu_{\mathcal{Y}}}(S_{\mu_{\mathcal{X}}}(u)) = S_{\mu}(u)$, and $\mu = \mu_{\mathcal{X}} \times \mu_{\mathcal{Y}}$.*

Proof: The composition of two Sugeno integrals is a Sugeno integral so: $S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}(u)) = S_{\kappa_{\mathcal{X}\mathcal{Y}}}(u)$ and $S_{\mu_{\mathcal{Y}}}(S_{\mu_{\mathcal{X}}}(u)) = S_{\kappa_{\mathcal{Y}\mathcal{X}}}(u)$. Since we assume commutation, it implies $\kappa_{\mathcal{X}\mathcal{Y}} = \kappa_{\mathcal{Y}\mathcal{X}} = \mu$ since the double S-integrals viewed as 2D ones are characterized by a single 2D capacity. This is the case of commuting capacities.

Now, using Corollary 4, we know that cuts $\mu_{\mathcal{X}\lambda}$ and $\mu_{\mathcal{Y}\lambda}$ are two possibility measures, two necessity measures, or one of them is a Dirac measure, for each $\lambda \in L$. From Proposition 13, $\mu(A \times B) = \min(\mu^{\downarrow}_{\mathcal{X}}(A), \mu^{\downarrow}_{\mathcal{Y}}(B))$. From Theorem 3, it remains to show that the focals of μ are Cartesian products:

- If $\mu^{\lambda}_{\mathcal{X}}$ and $\mu^{\lambda}_{\mathcal{Y}}$ are two necessities then $\mu^{\lambda}(A \times B) = 1$: $A \times B$ is a focal element of μ^{λ}.
- If $\mu^{\lambda}_{\mathcal{X}}$ is a Dirac function on $\{a\}$ and $\mu^{\lambda}_{\mathcal{Y}}$ is a capacity then
$$S_{\mu^{\lambda}_{\mathcal{X}}}(S_{\mu^{\lambda}_{\mathcal{Y}}})(R) = S_{\mu^{\lambda}_{\mathcal{Y}}}(aR) = \begin{cases} 1 & \text{if } \exists B \in \mathcal{F}(\mu^{\lambda}_{\mathcal{Y}}), B \subseteq aR\,, \\ 0 & \text{otherwise.} \end{cases} = S_{\mu^{\lambda}}(R)$$
where μ^{λ} has focal sets $\{a\} \times B, B \in \mathcal{F}(\mu^{\lambda}_{\mathcal{Y}})$. It is clearly decomposable.
- If $\mu^{\lambda}_{\mathcal{X}}$ and $\mu^{\lambda}_{\mathcal{Y}}$ are two possibility functions then the focal sets are of the form $\{(x_i, y_j)\}$ for focal sets $\{x_i\}$ of $\mu^{\lambda}_{\mathcal{X}}$ and $\{y_j\}$ focal sets of $\mu^{\lambda}_{\mathcal{Y}}$. So it is decomposable.

As the cuts of μ are decomposable, so is μ, according to Proposition 12. Moreover, according to Proposition 13, $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ are the projections of μ, so $\mu = \mu_{\mathcal{X}} \times \mu_{\mathcal{Y}}$. □

Finally, we are in a position to provide the explicit form of two commuting double S-integrals in terms of an underlying 2D S-integral. First note that Sugeno integrals with respect to commuting capacities simplify, namely, it is easy to see that, using Corollary 5:

Lemma 13. *If* $\mu_{\mathcal{Y}} = \max(N_{\mathcal{Y}}, \min(\theta_N, \delta_{\mathcal{Y}}), \min(\theta_D, \kappa_{\mathcal{Y}}), \min(\theta_\Pi, \Pi_{\mathcal{Y}}))$, *then* $S_{\mu_{\mathcal{Y}}}(f) = \max(S_{N_{\mathcal{Y}}}(f), \min(\theta_N, f(y^*)) \min(\theta_D, S_{\kappa_{\mathcal{Y}}}(f)), \min(\theta_\Pi, S_{\Pi_{\mathcal{Y}}}(f)))$.

Proof: The focal sets of $\mu_{\mathcal{Y}}$ can be shared in three groups: $\mathcal{F}_N$ (nested sets with weights above threshold θ_N), a singleton $\{y^*\}$ with weight θ_N, $\mathcal{F}_D$ (focal sets with weights between θ_D and θ_Π), $\mathcal{F}_\Pi$ (focal singletons with weights not greater than θ_Π) — see Corollary 5. So we can write

$$S_{\mu_{\mathcal{Y}}}(f) = \max \begin{cases} \max_{E \in \mathcal{F}_N}, \min(\mu_{\mathcal{Y}\#}(E), \min_{y_i \in E} f(y_i)) \,, \\ \min(\mu_{\mathcal{Y}\#}(\{y^*\}), f(y^*)) \\ \max_{E \in \mathcal{F}_D \cup \{\mathcal{Y}\}}, \min(\mu_{\mathcal{Y}\#}(E), \min_{y_i \in E} f(y_i)) \,, \\ \max_{\{y_i\} \in \mathcal{F}_\Pi}, \min(\mu_{\mathcal{Y}\#}(\{y_i\}), f(y_i)) \,. \end{cases}$$

If $E \in \mathcal{F}_N, \mu_{\mathcal{Y}}(E) = N_{\mathcal{Y}}(E)$. If $E \in \mathcal{F}_D, N_{\mathcal{Y}\#}(E) = 0$, and $\mu_{\mathcal{Y}}(E) = \min(\theta_D, \kappa_{\mathcal{Y}}(E))$ since $\kappa_{\mathcal{Y}}(E) > \theta_\Pi$. Finally if $\{y_i\} \in \mathcal{F}_\Pi, \mu_{\mathcal{Y}}(\{y_i\}) = \min(\theta_\Pi, \pi_{\mathcal{Y}}(y_i)))$ since $N_{\mathcal{Y}\#}(\{y_i\})) = \kappa_{\mathcal{Y}\#}(\{y_i\})) = 0$.

$$\text{Hence } S_{\mu_{\mathcal{Y}}}(f) = \max \begin{cases} \max_{E \in \mathcal{F}_N}, \min(N_{\mathcal{Y}\#}(E), \min_{y_i \in E} f(y_i)), \\ \min(\theta_N, f(y^*)) \\ \max_{E \in \mathcal{F}_D \cup \{\mathcal{Y}\}}, \min(\theta_D, \kappa_{\mathcal{Y}\#}(E), \min_{y_i \in E} f(y_i)) \\ \max_{E \in \mathcal{F}_\Pi}, \min(\theta_\Pi, \pi_{\mathcal{Y}}(y_i), f(y_i)) \end{cases}$$

So, $S_{\mu_{\mathcal{Y}}}(f) = \max(S_{N_{\mathcal{Y}}}(f), \min(\theta_N, f(y^*)) \min(\theta_D, \kappa_{\mathcal{Y}}(f)), \min(\theta_\Pi, S_{\Pi_{\mathcal{Y}}}(f)))$. □

In order to extend this result to double integrals that commute, first notice that the set of focal sets of a capacity is the set of focal sets of its cuts:

Lemma 14. $\mathcal{F}(\mu) = \cup_{\lambda > 0} \mathcal{F}(\mu_\lambda)$

Proof: Suppose $E \in \mathcal{F}(\mu)$, then it is clear that $E \in \mathcal{F}(\mu_\lambda)$ for $\lambda = \mu(E)$. Conversely, if $E \in \mathcal{F}(\mu_\lambda)$, let $\lambda^* = \max\{\lambda : E \in \mathcal{F}(\mu_\lambda)\}$. Clearly, $\lambda^* = \mu(E)$ since if $\lambda > \lambda^*$ then $E \notin \mathcal{F}(\mu_\lambda)$ and $\mu(E) \geq \lambda^*$. Now suppose there is $F \subset E$ with $\mu(F) = \lambda^*$, then $E \notin \mathcal{F}(\mu_{\lambda^*})$ which is a contradiction. So, $E \in \mathcal{F}(\mu)$. □

Now we can obtain a simple symmetric form for the commuting double Sugeno integral:

Theorem 5. *If $\mu_\mathcal{X}$ and $\mu_\mathcal{Y}$ commute, the double integral of function $u : \mathcal{X} \times \mathcal{Y} \to L$ is of the form*

$$S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(u)) = S_{\mu_\mathcal{Y}}(S_{\mu_\mathcal{X}}(u)) =$$
$$\max(S_N(u), \min(\theta_N, S_{\kappa_Y}(u(\cdot, y^*))), \min(\theta_D, S_{\kappa_Y}(u(x^*, \cdot))), \min(\theta_\Pi, S_\Pi(u)))$$

with $N = N_X \times N_Y, \Pi = \Pi_X \times \Pi_Y$.

Proof: Under the assumption of commuting capacities, cuts of $\min(\mu_\mathcal{X}, \mu_\mathcal{Y})$ are

- Boolean necessity measures for $\lambda > \theta_N$ with focal sets $E_\lambda \times F_\lambda$ where E_λ (resp. F_λ) is the focal set of the λ-cut of $\mu_\mathcal{X}$ (resp. $\mu_\mathcal{Y}$).
- Capacities with focal sets $E_\lambda \times \{y^*\}$ where E_λ are all focal sets of the λ-cuts of $\mu_\mathcal{X}$ for $\theta_N \geq \lambda > \theta_D$.
- Capacities with focal sets $\{x^*\} \times F_\lambda$ where F_λ are all focal sets of the λ-cuts of $\mu_\mathcal{Y}$ for $\theta_D \geq \lambda > \theta_\Pi$.
- Boolean possibility measures for $\theta_\Pi \geq \lambda$ with focal singletons $\{x_i\} \times \{y_j\}$ where $\mu_\mathcal{X}(\{x_i\}) \geq \lambda$ and $\mu_\mathcal{Y}(\{y_j\}) \geq \lambda$.

All focal sets of $\min(\mu_\mathcal{X}, \mu_\mathcal{Y})$ are Cartesian products of this form due to Lemma 14. We can reconstruct this capacity and get its expression from its cuts, in the style of Corollary 5, as $\min(\mu_\mathcal{X}, \mu_\mathcal{Y}) = \max(N_\mathcal{X} \times N_\mathcal{Y}, \min(\theta_N, \kappa_X \times \delta_\mathcal{Y}^{y^*}), \min(\theta_D, \delta_\mathcal{X}^{x^*} \times \kappa_Y), \min(\theta_\Pi, \Pi_\mathcal{X} \times \Pi_\mathcal{Y})$.
Now we can apply Theorem 4 ($S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(u))$ is a 2D S-integral with respect to $\min(\mu_\mathcal{X}, \mu_\mathcal{Y})$) and Lemma 13 (applied to the 2D S-integral) and get the symmetric form of the double Sugeno integral in four terms. □

5.5. *Do decomposable 2D-capacities induce commuting 1D-capacities?*

Decomposability is not a sufficient condition for a 2D S-integral to be expressible as a double S-integral. If μ is a decomposable capacity we have that

$$S_\mu(u) = \max_{(A \times B) \in \mathcal{F}(\mu)} \min(\min(\mu_\mathcal{X}^\downarrow(A), \mu_\mathcal{Y}^\downarrow(B)), \min_{(x,y) \in A \times B} u(x, y)) .$$

The question is whether any 2D S-integral with respect to a decomposable capacity can be expressed in the form of a double S-integral. The following proposition and examples answer by the negative.

Proposition 14. *Let μ be a decomposable fuzzy measure on $\mathcal{X} \times \mathcal{Y}$. We have $S_\mu(u) \leq S_{\mu_\mathcal{X}^\downarrow}(S_{\mu_\mathcal{Y}^\downarrow}((u(x_1, \cdot)), \ldots, S_{\mu_\mathcal{Y}^\downarrow}(u(x_n, \cdot))))$ and $S_\mu(u) \leq S_{\mu_\mathcal{Y}^\downarrow}(S_{\mu_\mathcal{X}^\downarrow}(u(\cdot, y_1)), \cdots, S_{\mu_\mathcal{Y}^\downarrow}(u(\cdot, y_p)))$.*

Proof: Note that G is a focal set of $\mu^{\downarrow}_{\mathcal{Y}}$ and F a focal set of $\mu^{\downarrow}_{\mathcal{X}}$, if and only if $G \times F$ is a focal set of μ. We show the result for one inequality; the other is proved likewise.

$$\begin{aligned} S_\mu(u) &= \max\nolimits_{C\subseteq \mathcal{X}\times\mathcal{Y}} \min(\mu(C), \min\nolimits_{(x,y)\in C} u(x,y)) \\ &= \max\nolimits_{A,B} \min(\mu^{\downarrow}_{\mathcal{Y}}(A), \mu^{\downarrow}_{\mathcal{Y}}(B), \min\nolimits_{x\in A, y\in B} u(x,y)) \\ &= \max\nolimits_{A} \min(\mu^{\downarrow}_{\mathcal{X}}(A), \max\nolimits_B \min(\mu^{\downarrow}_{\mathcal{Y}}(B), \min\nolimits_{x\in A, y\in B} u(x,y))) \\ &= \max\nolimits_{A} \min(\mu^{\downarrow}_{\mathcal{X}}(A), \max\nolimits_B \min\nolimits_{x\in A} \min(\mu^{\downarrow}_{\mathcal{Y}}(B), \min\nolimits_{y\in B} u(x,y))) \\ &\leq \max\nolimits_{A} \min(\mu^{\downarrow}_{\mathcal{X}}(A), \min\nolimits_{x\in A} \max\nolimits_B \min(\mu^{\downarrow}_{\mathcal{Y}}(B), \min\nolimits_{y\in B} u(x,y))) \\ &= S_{\mu^{\downarrow}_{\mathcal{X}}}(S_{\mu^{\downarrow}_{\mathcal{Y}}}((u(x_1,\cdot)),\ldots,S_{\mu^{\downarrow}_{\mathcal{Y}}}(u(x_n,\cdot)))) \,. \end{aligned}$$

Note that we have turned $\max_B \min_{i\in A}$ into $\min_{i\in A}\max_B$, which induces the inequality. □

The inequalities in Proposition 14 may be strict, and this even if μ is decomposable. This should not be surprizing since commuting capacities have a very special form and their product yields only a subclass of decomposable capacities, with focal sets having a specific structure (nested Cartesian products with high weights, singletons with small weights and one-dimensional subsets of the form $\{x\} \times B$, for a fixed element $x \in \mathcal{X}$, with medium weights, as suggested by the proof of Theorem 4).

In the following example two double Sugeno integrals of a Boolean function on $\mathcal{X} \times \mathcal{Y}$ are equal for a class of functions but they are strictly greater than the corresponding 2D S-integral, using a decomposable 2D capacity.

Example 11. Consider $|\mathcal{X}| = |\mathcal{Y}| = 3$. Let $R = \{(x_1,y_3),(x_1,y_2),(x_2,y_2),(x_2,y_1),(x_3,y_1)\}$. Note that R is symmetric. Now consider the 2D Boolean capacity μ with the 4 focal sets $\{x_1,x_2\}\times\{y_1,y_2\}$, $\{x_1,x_2\}\times\{y_2,y_3\}$, $\{x_2,x_3\}\times\{y_2,y_3\}$, $\{x_2,x_3\}\times\{y_1,y_2\}$. Note that it is decomposable, and $\mu^{\downarrow}_{\mathcal{X}}$ has focals $\{x_1,x_2\}$ and $\{x_2,x_3\}$, $\mu^{\downarrow}_{\mathcal{Y}}$ has focals $\{y_1,y_2\}$ and $\{y_2,y_3\}$. They are the same capacity on the two sets $\mathcal{X}$ and $\mathcal{Y}$

It is clear that $\mu(R) = 0$ (R contains no focal sets). However, consider the double S-integrals $S_{\mu^{\downarrow}_{\mathcal{X}}}(S_{\mu^{\downarrow}_{\mathcal{Y}}}(\mathbf{1}_R))$ and $S_{\mu^{\downarrow}_{\mathcal{Y}}}(S_{\mu^{\downarrow}_{\mathcal{X}}}(\mathbf{1}_R))$. They are equal because they use the same capacity and R is symmetric.

Note that $Ry_1 = \{x_2,x_3\}$, $Ry_2 = \{x_1,x_2\}$, $Ry_3 = \{x_1\}$. We have

$$\begin{aligned} S_{\mu^{\downarrow}_{\mathcal{Y}}}(S_{\mu^{\downarrow}_{\mathcal{X}}}(\mathbf{1}_R)) &= \max_{B\in\mathcal{F}(\mathcal{Y})} \min(\mu^{\downarrow}_{\mathcal{Y}}(B), \min_{y_i\in B} \mu^{\downarrow}_{\mathcal{X}}(Ry_i)) \\ &= \max(\min(\mu^{\downarrow}_{\mathcal{Y}}(\{y_1,y_2\}), \min(\mu^{\downarrow}_{\mathcal{X}}(\{x_2,x_3\}), \mu^{\downarrow}_{\mathcal{X}}(\{x_1,x_2\})), \\ &\quad \min(\mu^{\downarrow}_{\mathcal{Y}}(\{y_2,y_3\}), \min(\mu^{\downarrow}_{\mathcal{X}}(\{x_1,x_2\}), \mu^{\downarrow}_{\mathcal{X}}(\{x_1\}))) \\ &= \max(\min(1,\min(1,1)), \min(1,\min(1,0))) = 1\,. \end{aligned}$$

□

So we have a case where $S_{\mu^{\downarrow}_{\mathcal{X}}}(S_{\mu^{\downarrow}_{\mathcal{Y}}}(u)) = S_{\mu^{\downarrow}_{\mathcal{Y}}}(S_{\mu^{\downarrow}_{\mathcal{X}}}(u)) = 1 > S_\mu(u) = 0$. for a class of symmetric functions on $\mathcal{X} \times \mathcal{Y}$. However, we do not claim that $S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}})(u) = S_{\mu_{\mathcal{Y}}}(S_{\mu_{\mathcal{X}}})(u)$ for all functions u. In particular, if $u(x_i, y_j) \neq u(x_j, y_i)$, the two double S-integrals may differ.

Let us exemplify the discrepancy between double S-integrals and 2D ones in the setting of possibility and necessity measures that first motivated this paper.

Consider the case of a 2D capacity on $\mathcal{Y} \times \mathcal{X}$ whose projection on $\mathcal{Y}$ is a necessity measure $N_{\mathcal{Y}}$ with possibility distribution w, and whose projection on $\mathcal{X}$ is a possibility measure $\Pi_{\mathcal{X}}$ with possibility distribution π.

Let the focal sets of $N_{\mathcal{Y}}$ be of the form $B_j = \{y_1, \ldots, y_j\} \subseteq \mathcal{Y}$ and $\alpha_j = N_{\mathcal{Y}\#}(B_j)$, i.e., $\alpha_j = 1 - w_{j+1}$, so that $N_{\mathcal{Y}}(B) = \max_{B_j \subseteq B} \alpha_j = \min_{y_j \notin B} 1 - w_j$ and the marginal S-integral on $\mathcal{Y}$ takes the two forms:

$$S_N(u(x_i, \cdot)) = \max_{B_j \subseteq B} \min(\alpha_j, \min_{y_k \in B_j} u(x_i, y_k)) = \min_{i=1,\ldots,n} \max(1 - w_j, u(x_i, y_j)).$$

The joint necessity-possibility function $\mu = \Pi_{\mathcal{X}} \times N_{\mathcal{Y}}$ is defined by the focal sets $\{x_i\} \times B_j$ and Möbius masses $\mu_\#(\{x_i\} \times B_j) = \min(\pi_i, \alpha_j), i, j = 1, \ldots, n$.
It can be checked that $\mu^{\downarrow}_{\mathcal{Y}} = N_{\mathcal{Y}}$ and $\mu^{\downarrow}_{\mathcal{X}} = \Pi_{\mathcal{X}}$, as prescribed by Proposition 10. Indeed, $\mu^{\downarrow}_{\mathcal{Y}}(B) = \mu(\mathcal{X} \times B) = \max_{B_i \subseteq B} \max_{x_i \in \mathcal{X}} \min(\pi_i, \alpha_j) = \max_{B_j \subseteq B} \alpha_j = N_{\mathcal{Y}}(B)$ and likewise for the other projection $\mu^{\downarrow}_{\mathcal{X}}(A) = \mu(A \times \mathcal{Y}) = \Pi_{\mathcal{X}}(A)$.

The 2D S-integral reads:

$$\begin{aligned} S_\mu(u) &= \max_{i,j=1,\ldots,n} \min(\min(\pi_i, \alpha_j), \min_{y_k \in B_j} u(x_i, y_k)) \\ &= \max_{j=1,\ldots,n} \min(\alpha_j, \max_{i=1,\ldots,n} \min(\pi_i, \min_{y_k \in B_j} u(x_i, y_k))) \\ &= \max_{i=1,\ldots,n} \min(\pi_i, \max_{j=1,\ldots,n} \min(\alpha_j, \min_{y_k \in A_j} u(x_i, y_k))) \\ &= S_{\mu^{\downarrow}_{\mathcal{X}}}(S_{\mu^{\downarrow}_{\mathcal{Y}}}(u)) = S_{\Pi_{\mathcal{X}}}(S_{N_{\mathcal{Y}}}(u)) \,. \end{aligned}$$

Thus, if a 2D capacity μ is the joint of a possibility measure on $\mathcal{X}$ and a necessity measure on $\mathcal{Y}$, then $S_\mu(u) = S_{\mu^{\downarrow}_{\mathcal{X}}}(S_{\mu^{\downarrow}_{\mathcal{Y}}}(u)), \forall u$, which is the value U^{+min}_{post} for u. But, as seen earlier, $S_{N_{\mathcal{X}}}(S_{\Pi_{\mathcal{Y}}}(u)) > S_{\Pi_{\mathcal{X}}}(S_{N_{\mathcal{Y}}}(u))$ in general, and the difference can be 1 as seen in Examples 1 and 9, continued below.

Example 12. $\mathcal{X} = \{x_1, x_2\}$, $\pi_i = 1$, and $w_i = 1, \forall i = 1, 2$, $\mathcal{Y} = \{y_1, y_2\}$, $u(x_1, y_1) = u(x_2, y_2) = 1$ and $u(x_2, y_1) = u(x_1, y_2) = 0$. The projections of the joint capacity, μ, are the vacuous necessity measure on $\mathcal{Y}$ and the vacuous possibility measure on $\mathcal{X}$. Note that on $\mathcal{Y}$ there is a single focal set $\mathcal{Y}$ while focal sets on $\mathcal{X}$ are all singletons. Hence focal sets of μ are of the form $\{x_i\} \times \mathcal{Y}, x_i \in \mathcal{X}$. Since u is the characteristic function of the set $\{(x_1, y_1), (x_2, y_2)\}$ that contains none of them, it yields $S_\mu(u) = \mu(\{(x_1, y_1), (x_2, y_2)\}) = 0$. We have seen that $U^{+min}_{post} = S_{\Pi_{\mathcal{Y}}}(S_{N_{\mathcal{X}}}(u)) = S_\mu(u) = 0$, which coincides $S_\mu(u)$. On the other hand, $U^{+min}_{ante}(\pi, w, u) = S_{N_{\mathcal{X}}}(S_{\Pi_{\mathcal{Y}}}(u)) = 1$ as already seen. So we have a very simple example where $S_\mu(u) = S_{\mu^{\downarrow}_{\mathcal{X}}}(S_{\mu^{\downarrow}_{\mathcal{Y}}}(u)) < S_{\mu^{\downarrow}_{\mathcal{Y}}}(S_{\mu^{\downarrow}_{\mathcal{X}}}(u))$. □

We have seen that when $\mu_{\mathcal{X}}$ is a possibility measure $\Pi_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ is a necessity measure $N_{\mathcal{X}}$, the double S-integral $S_{\Pi_{\mathcal{X}}}(S_{N_{\mathcal{Y}}}(u))$ is equal to the 2D S-integral $S_{\min(\Pi_{\mathcal{X}},N_{\mathcal{Y}})}$, while it differs from $S_{N_{\mathcal{Y}}}(S_{\Pi_{\mathcal{X}}}(u))$. The latter is also a 2D S-integral, albeit with respect to a 2D capacity that is not decomposable. This state of facts suggests that to evaluate a collective utility function, the functional $S_{\Pi_{\mathcal{X}}}(S_{N_{\mathcal{Y}}}(u))$ is better behaved since the 2D capacity associated to this double S-integral is built from $\Pi_{\mathcal{X}}$ and $N_{\mathcal{Y}}$ only. This line of thought needs further investigation.

6. Conclusion

In this paper, we have found necessary and sufficient conditions for two Sugeno integrals on distinct universes to commute. We have seen that even if we do not have to restrict to pairs of possibility or of necessity measures, the commuting capacities are essentially built from gluing possibility and necessity measures, except when the cut of one of them trivializes into a Dirac measure. This paper also offer insights into the counterpart of probabilistic independence and possibilistic non-interactivity for 2D capacities, we call decomposability. Characteristic criteria of decomposability have been proposed for 2D capacities, as well as some links between these criteria and the commutation of the projections of the 2D capacities.

An essential lesson is that, like with fuzzy sets under the max-min operations, we can reason about qualitative capacities using cuts, and that the double fuzzy integrals are cutworthy in the sense of De Baets and Kerre.[5] These results open the way to a generalization of the qualitative counterpart of Harsanyi[16] theorem for expected utility,[3] namely provide axioms over acts or possibilistic lotteries that justify a commuting double Sugeno integral to evaluate social utility. It is also interesting to apply this result to qualitative game theory since the commutation can be used to define a kind of qualitative counterpart to Nash equilibrium. Finally, it would be of interest to relate our results to the literature on commuting aggregation functions.[20]

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