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Qualitative decision theory with preference relations and comparative uncertainty: An axiomatic approach ☆

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Abstract

This paper investigates a purely qualitative approach to decision making under uncertainty. Since the pioneering work of Savage, most models of decision under uncertainty rely on a numerical representation where utility and uncertainty are commensurate. Giving up this tradition, we relax this assumption and introduce an axiom of *ordinal invariance* requiring that the Decision Maker's preference between two acts only depends on the relative position of their consequences for each state. Within this qualitative framework, we determine the only possible form of the corresponding decision rule. Then assuming the transitivity of the strict preference, the underlying partial confidence relations are those at work in non-monotonic inference and thus satisfy one of the main properties of possibility theory. The satisfaction of additional postulates of unanimity and anonymity enforces the use of a necessity measure, unique up to a monotonic transformation, for encoding the relative likelihood of events.

Keywords: Decision under uncertainty; Ordering relations; Possibility theory; Social choice

1. Introduction

In the field of decision making under uncertainty (DMU), several important results have been published that justify the use of various criteria for the comparison of alternatives.

☆ This paper extends preliminary results of the two last authors [22].
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Each decision criterion is justified by a set of axioms putting constraints on the choice behaviour of the Decision Maker (DM). The most classical results are found in the seminal works of Von Neumann and Morgenstern [37] and Savage [41]. Such approaches rely on the use of a quantitative criterion for the comparison of alternatives. The justification of this criterion requires several strong assumptions. For instance, the axiom system usually implies that the subjective value attached to each consequence, as well as the degrees of confidence of the possible events, can be quantified. However, in practical applications, the elicitation of the information required by a quantitative model is often not an easy task. This is why some alternative models have been proposed in AI, relying on a more ordinal representation of preferences and uncertainty, e.g., the *qualitative utility function* proposed in [17] among others. In contrast to expected utility, qualitative utility relies on the use of ordinal information (preference order on consequences, relative confidence of events). Nevertheless, it shares a common feature with the expected utility criterion: *the commensurability of the preference scale and the uncertainty scale that are used in the model*. In other words, both scales are part of a bigger one, and degrees of uncertainty can be compared to degrees of preference (typically via the notion of certainty equivalent of a lottery).

Recent works in AI propose to escape this assumption in two different ways. The first idea is to compare two acts on the basis of their consequences in the *most plausible states of the world* [5,7,8,45,46], using more or less complex criteria. This class of approaches also aims at providing the user not only with a decision rule, but also with a high-level language (typically, a logic) for the description of preferences, e.g., a base of conditional preferences [5] or a base of defaults encoding either desires or beliefs [46]. The criteria used in these domains are generally qualitative, e.g., the maximin, minimax regret and competitive ratio criteria axiomatized in [7,8]. The second idea is to rely on a decision rule that compares the plausibility of the sets of states in which one act has a better consequence than the other as first suggested in [13]. A particular case of this rule (the so-called "Concordance rule") has been originally proposed in the framework of multiple-criteria decision making in the early seventies by Roy [40]. It can be easily adapted to DMU using general uncertainty relations, thus defining new decision models. The aim of the present paper is to provide a unified axiomatic framework for this kind of decision rule and to study its capability at describing the DM preferences. So, this paper intends to assess the potential and drawbacks of purely ordinal approaches from a theoretical point of view.

In the next section, we motivate our approach by discussing various quantitative and qualitative models for decision making under uncertainty and discuss their limitations especially with respect to making preference and uncertainty commensurate. In Section 3, an axiomatic framework for purely ordinal preference models is introduced: the structure of the resulting decision rules is characterized and the kind of confidence relation compatible with such rules is investigated. In Section 4 we characterize the qualitative (ordinal) decision rules and the uncertainty theory compatible with the transitivity of the strict preference on acts. Finally, in Section 5, we show that the satisfaction of a postulate of unanimity enforces the use of necessity measures as the unique way of comparing events. Adding a postulate of invariance under permutation of equally plausible states ensures the unicity of the ordering of events induced by the necessity measures. The conclusion emphasizes the high price in expressivity paid by adopting a purely ordinal approach and suggests lines of research that may lead to more realistic decision rules, while preserving ordinality to a large extent.

2. Motivations for an ordinal approach

This section presents the basic setting and discusses the merits and limitations of classical decision rules used in DMU, such as expected utility and the minmax Wald criterion [47] in the face of qualitative information regarding uncertainty and preference. These decision rules, most of which are based on preference functionals, assume a common scale for representing preference and uncertainty and they lead to a ranking of act which is sensitive to the encoding of utility and/or plausibility on this scale. In the tradition of social choice and multicriteria decision making, other decision rules are proposed that are more robust because they rely on ordinal pairwise comparison of consequences of acts and on the comparison of events separately.

2.1. Decision models based on a preference functional

Decision making under uncertainty implies a choice among a set of potential acts (decisions) the consequences of which are not perfectly known. From a formal point of view, such a decision problem is characterized by a set S of states representing the possible situations, a set X of possible consequences, and a set of acts viewed as elements of X^S: an act is a mapping $f: S \to X$ where $f(s)$ represents the consequence of act f for any state $s \in S$. In this paper, S and X are supposed to be finite. Thus, if $X = \{x_1, \ldots, x_m\}$ and $S = \{s_1, \ldots, s_n\}$, an act $f \in X^S$ is completely characterized by the vector of consequences $(f(s_1), \ldots, f(s_n))$.

The preference relation $\succsim$ on X^S is usually built through the use of a *decision rule* defining the preference $f \succsim g$ as a function of vectors $(f(s_1), \ldots, f(s_n))$ and $(g(s_1), \ldots, g(s_n))$. From $\succsim$ we can define an *indifference relation* $\sim$, and a *strict preference relation* $\succ$ by:

$$f \sim g \quad \Leftrightarrow \quad f \succsim g \text{ and } g \succsim f,$$
$$f \succ g \quad \Leftrightarrow \quad f \succsim g \text{ and } \operatorname{not}(g \succsim f).$$

Most of the rules used for decision making under uncertainty involve a real-valued function u on X encoding the utility or the relative attractiveness of the consequences and a numerical set-function μ on 2^S representing the confidence of the events (seen as sets of states). Such a confidence function on S (originally called a capacity [9], and sometimes called a fuzzy measure [44]) is a mapping μ defined from 2^S to $[0, 1]$ such that:

- $\mu(\emptyset) = 0$,
- $\mu(S) = 1$,
- $\forall A, B \subseteq S, A \subseteq B \Rightarrow \mu(A) \leqslant \mu(B)$.

The last property means that, if A implies B, then B cannot be less likely than A, i.e., a form of monotonicity w.r.t. implication. Important subclasses of confidence functions are:

- *Additive capacities (probabilities)* P, characterized by: $\forall A, B \subseteq S,\ A \cap B = \emptyset \Rightarrow P(A \cup B) = P(A) + P(B)$.
  For any additive capacity, there exists a distribution $p : S \mapsto [0,1]$ such that $\forall A \subseteq S, P(A) = \sum_{s \in A} p(s)$.
- *Possibility measures* Π, characterized by: $\forall A, B \subseteq S, \Pi(A \cup B) = \max(\Pi(A), \Pi(B))$.
  For any possibility measure Π on a finite set, there exists a possibility distribution $\pi : S \mapsto [0,1]$ such that $\forall A \subseteq S, \Pi(A) = \max_{s \in A} \pi(s)$.
- *Necessity measures* N, characterized by: $\forall A, B \subseteq S, N(A \cap B) = \min(N(A), N(B))$.
  For any necessity measure N on a finite set, there exists a possibility distribution $\pi : S \mapsto [0,1]$ such that $\forall A \subseteq S, N(A) = 1 - \max_{s \in \overline{A}} \pi(s)$. This is due to the fact that the dual of a possibility measure is a necessity measure (and conversely): $\forall A \subseteq S, N(A) = 1 - \Pi(\overline{A})$, where $\overline{A}$ is the complement of A.

These representations of uncertainty about the state of the world are simple because the set-function comparing the relative likelihood of events is completely determined by the knowledge of the degrees of plausibility of all states (encoded by a probability or a possibility distribution on S).

Most decision models are characterized by the definition of a preference functional $v : X^S \to Y$ where Y is an ordered set and $v(f)$ measures the subjective attractiveness of f for the DM. In such models, $v(f)$ is a function of the values $u(f(s_1)), \ldots, u(f(s_n))$ and of the set-function representing the Decision Maker's knowledge about the state of nature. The decisions are then ranked according to the values $v(f)$.

The most famous proposal in this family is obviously the expected utility model (EU) axiomatized by Savage [41]. It is based on a real-valued utility function u defined on X, measuring the subjective attractiveness of the consequences, and on a probability distribution p on S. Function u being unique up to a positive affine transformation in this model, we will assume here without loss of generality that $u : X \to [0,1]$. This defines a criterion $v(f)$ for any $f \in X^S$ by:

$$f \succsim g \quad \Leftrightarrow \quad v(f) \geqslant v(g) \quad \text{where } v(f) = \sum_{s \in S} p(s) u\big(f(s)\big). \tag{1}$$

The EU model has become a standard despite early criticisms formulated by Allais [1], Ellsberg [21] and later by Kahneman and Tversky [30]. However, such comments have motivated the emergence of non-EU models in the last decades. These models differ from the initial proposition either by adopting a preference functional which is not necessarily linear with respect to probabilities (see, e.g., the Rank-Dependent Utility model (RDU) proposed by Quiggin [38]) or by replacing the probabilistic representation of confidence of events by a non-additive one (see, e.g., the Choquet Expected Utility model (CEU) proposed by Schmeidler [42]). These two alternatives to the expected utility are based on a Choquet integral [9]. The Choquet integral is a powerful aggregation operator allowing positive and negative synergies between events (for instance, the confidence of $A \cup B$ can be greater or less than the sum of the individual confidences of A and B). When used with a probability measure, it boils down to expected utility.

Escaping the highly quantitative nature of these rules, more qualitative models have been proposed, that rely on ordinal evaluation scales. Such models are especially designed for preference modelling in the presence of poor, qualitative information. The most famous decision rule of this kind is the maximin rule of Wald [47], axiomatized by Arrow and Hurwicz [2]. It only presupposes that the set X of consequences is ranked in terms of merit by means of some utility function u valued on any ordinal scale. The Decision Maker is not assumed to know anything about the state of the world. Then acts are ranked according to the merit of their worst consequences, following a pessimistic attitude. The optimistic counterpart using the best consequences of acts has of course been proposed as well.

Some authors in qualitative decision theory have refined this criterion, assuming a plausibility ordering on states is available [5,8]. Then acts are ranked according to the merits of their worst consequences restricted to the most plausible states.

Another refinement of Wald criterion, the possibilistic qualitative criterion [17–19] is based on a utility function u on X and a possibility distribution π on S representing the relative plausibility of states, both mapping on the same totally ordered scale. A pessimistic criterion $v^-(f)$ is proposed of the form:

$$f \succsim g \quad \Leftrightarrow \quad v^-(f) \geqslant v^-(g) \quad \text{where } v^-(f) = \min_{s\in S} \max\bigl\{1 - \pi(s), u\bigl(f(s)\bigr)\bigr\}. \quad (2)$$

The value of $v^-(f)$ is small as soon as there exists a highly plausible state with low utility value. This criterion is actually a *weighted* extension of the Wald maximin criterion. The decisions are again made according to the merits of acts in their worst consequences restricted to the most plausible states. But the set of most plausible states now depends on the act itself. However, contrary to the other qualitative criteria, the possibilistic qualitative criterion presupposes that degrees of utility $u(f(s))$ and possibility $\pi(s)$ share the same scale and can be compared.

The optimistic counterpart of this criterion (see again [17–19]) is:

$$f \succsim g \quad \Leftrightarrow \quad v^+(f) \geqslant v^+(g) \quad \text{where } v^+(f) = \max_{s\in S} \min\bigl\{\pi(s), u\bigl(f(s)\bigr)\bigr\}. \quad (3)$$

Moreover, the optimistic and pessimistic possibilistic criteria are actually particular cases of a more general criterion based on the Sugeno integral [44], a qualitative counterpart of the Choquet integral (see [18,35]).

More recently, Giang and Shenoy [26,27] have considered an uncertainty-neutral possibilistic decision criterion, in the spirit of a conjoint use of Eqs. (2) and (3). Namely, the measurement scale is chosen as the set of possibility distributions on the set $\{0, 1\}$, 0 and 1 standing for the worst and the best consequences in X respectively. Each such possibility distribution is of the form of a qualitative lottery (α, β) with $\max(\alpha, \beta) = 1$, yielding 0 with possibility α and 1, with possibility β. This set is totally ordered in such a way that $(0, 1) > (1, 1) > (1, 0)$, decreasingly from the certainty of getting 1 to the certainty of getting 0. Their criterion subsumes each of the optimistic and pessimistic possibilistic criteria separately.

Lehmann [33] axiomatizes a refinement of the maximin criterion whereby ties between equivalent worst states are broken by considering their respective likelihoods. This decision rule takes the form of an expected utility criterion with qualitative (infinitesimal) utility levels. The axiomatization is carried out in the Von Neumann–Morgenstern style.

Compared to the expected utility model, the above qualitative schemes have some drawbacks. The pure maximin rule is not realistic, because it is overpessimistic, and has been given up accordingly. Its restriction to a set of most plausible states (weighted or not) is more realistic but still yields a very coarse ranking of acts. This kind of criterion does not use all the available information. Especially an act f can be ranked equal to another act g even if f is at least as good as g in all states and better in some states (including most plausible ones). This defect cannot be found with the expected utility model. It has actually been addressed by Cohen and Jaffray [10] who improve the maximin rule by comparing acts on the basis of their worst consequences of *distinct* merits. This refined rule always rates an act f better than another act g whenever f is at least as good as g in all states and better in some states. However, only a partial ordering of acts is then obtained. This last decision rule is actually no longer based on a preference functional.

2.2. The informational burden of the commensurability hypothesis

Let us now emphasize some difficulties resulting from implicit assumptions in decision models based on preference functionals, namely the commensurability (or comparability) between preference and uncertainty, and, for the EU model, the numerical nature of utilities. Suppose that the DM's preferences over X are known and are expressed by the weak-order $\succsim_X$ ($\succsim_X$ is precisely the preference relation encoded by the utility function). Suppose also that the subjective confidence of each event for the DM is known and is described by a binary uncertainty relation $\succsim_\Lambda$ on 2^S (induced for instance by a probability distribution or a possibility distribution on S).

The first remark is that, in the expected utility model, the way in which the weak-order $\succsim_X$ is numerically encoded will affect the preference relation on acts.

Example 1. Let us consider a decision problem with two states $\{s_1, s_2\}$ which are seen as *equally plausible* by the decision maker (i.e., $s_1 \succsim_\Lambda s_2$ and $s_2 \succsim_\Lambda s_1$) and a set of consequences $X = \{x_1, x_2, x_3, x_4\}$ such that $x_1 \succ_X x_2 \succ_X x_3 \succ_X x_4$. Let us consider two different utility functions u_1 and u_2 that could be used to encode the preference order $\succ_X$:

	x_1	x_2	x_3	x_4
u_1	0.9	0.6	0.4	0.3
u_2	0.7	0.6	0.4	0.1

Consider f such that $f(s_1) = x_1$ and $f(s_2) = x_4$, and g such that $g(s_1) = x_2$ and $g(s_2) = x_3$. If we want to use the expected utility model, the probability distribution on S must be $p(s_1) = p(s_2) = 0.5$. Using function u_1 in Eq. (1) we get the expected values $v_1(f) = 0.6$, $v_1(g) = 0.5$. Performing the same operations with u_2 leads to $v_2(f) = 0.4$, $v_2(g) = 0.5$. Therefore we get $f \succ g$ with u_1 and $g \succ f$ with u_2.

Hence, when using expected utility, the preference over acts depends on the particular utility function chosen to encode the preference order $\succ_X$. It shows that the model exploits some extra information not contained solely in the relations $\succsim_X$ and $\succsim_\Lambda$, namely the absolute order of magnitude of utility grades as compared with degrees of probabilities.

We now want to compare the same acts f and g as in the previous example, on the basis of their pessimistic qualitative utility. In this example, the possibility distribution π that describes the equal plausibility of events must be encoded as: $\pi(s_1) = \pi(s_2) = 1$. Then, the pessimistic possibilistic criterion reduces to the maxmin rule and gives: $v_1^-(f) = 0.3$ and $v_1^-(g) = 0.4$ from u_1 and therefore $g \succ f$, and we get exactly the same preference with function u_2 since $v_2^-(f) = 0.1$ and the $v_2^-(g) = 0.4$. Similarly, the optimistic criterion gives $v_1^+(f) = 0.9$ and $v_1^+(g) = 0.6$ from u_1, which yields $f \succ g$, and this preference remains unchanged when substituting u_2 to u_1. Thus, on this example where the initial information (represented by relations $\succsim_X$ and $\succsim_\Lambda$) is purely ordinal, using a qualitative criterion seems more natural than expected utility. Things are not so simple when the DM is more informed about the relative plausibilities of events, as shown in the following example.

Example 2. Suppose now that $s_2 \succ_\Lambda s_1$ in Example 1 and consider the following utility functions:

	x_1	x_2	x_3	x_4
u_1	0.4	0.3	0.2	0.1
u_2	0.9	0.8	0.7	0.6

Let us choose an arbitrary possibility distribution on S to represent the confidence weak-order on $\succsim_\Lambda$, e.g., $\pi(s_1) = 0.5$ and $\pi(s_2) = 1$. Now, using function u_1 in Eq. (3) we get the respective values for acts $v_1^+(f) = 0.4$, $v_1^+(g) = 0.3$. Performing the same calculations with function u_2 leads to $v_2^+(f) = 0.6$, $v_2^+(g) = 0.7$. Therefore we get $f \succ g$ with v_1^+ and $g \succ f$ with v_2^+.

A similar preference inversion could be obtained using v^- in another example. This shows that the possibilistic utility criteria also exploit some extra information not contained in relations $\succsim_X$ and $\succsim_\Lambda$: this is again due to the particular way utility and possibility degrees are positioned on a common ordinal scale that is implicit in such models. Utility and possibility degrees are thus commensurate and ordered using a relation denoted $\succsim_{X\Lambda}$ defined on $S \cup X$ and respecting the following constraints:

$$s_i \succsim_{X\Lambda} s_j \quad \Leftrightarrow \quad \pi(s_i) \geqslant \pi(s_j), \qquad x_i \succsim_{X\Lambda} x_j \quad \Leftrightarrow \quad u(x_i) \geqslant u(x_j),$$
$$s_i \succsim_{X\Lambda} x_j \quad \Leftrightarrow \quad \pi(s_i) \geqslant u(x_j), \qquad x_i \succsim_{X\Lambda} s_j \quad \Leftrightarrow \quad u(x_i) \geqslant \pi(s_j).$$

In the above example, the use of u_1 presupposes the following order: $s_2 \succ_{X\Lambda} s_1 \succ_{X\Lambda} x_1 \succ_{X\Lambda} x_2 \succ_{X\Lambda} x_3 \succ_{X\Lambda} x_4$ whereas u_2 presupposes: $s_2 \succ_{X\Lambda} x_1 \succ_{X\Lambda} x_2 \succ_{X\Lambda} x_3 \succ_{X\Lambda} x_4 \succ_{X\Lambda} s_1$. This explains the diverging results in the comparison of f and g using criterion v^+. But, in contrast with expected utility, possibilistic criteria are robust with respect to any monotonic transformation of the joint utility/uncertainty scale. If the ordering on $S \cup X$ is respected, this change will not affect the resulting preference relation between acts.

More generally, all the models based on a preference functional use such a scale $\succ_{X\Lambda}$ built from $\succsim_X$ and $\succsim_\Lambda$, i.e., requires additional information relating uncertain events and their certainty-equivalents which must be elicited by questioning the DM. In the following subsection, we introduce simple models escaping this commensurability assumption and thus avoiding the use of relation $\succsim_{X\Lambda}$.

2.3. Decision models based on pairwise comparisons

An alternative approach to decision only assumes that the DM has preferences on consequences encoded as a weak-order $\succsim_X$ on X, and a confidence relation $\succsim_\Lambda$ describing the relative likelihood of events. The basic properties of confidence measures extend to partial relations:

Definition 1 (*Monotonic confidence relation*). A relation $\succsim_\Lambda$ on 2^S is a confidence relation iff:

- $\succsim_\Lambda$ is reflexive,
- $S \succ_\Lambda \emptyset$ (non-triviality),
- $\forall A$, $S \succsim_\Lambda A$ and $A \succsim_\Lambda \emptyset$ (consistency).

A relation $\succsim_\Lambda$ on 2^S is said to be monotonic if and only if:

- $\forall A, B, C \subseteq S$, $A \succsim_\Lambda B \Rightarrow A \cup C \succsim_\Lambda B$,
- $\forall A, B, C \subseteq S$, $A \succsim_\Lambda B \cup C \Rightarrow A \succsim_\Lambda B$.

To each confidence relation corresponds a dual confidence relation using set-complementation:

Definition 2 (*Duality, self-duality*). Let $\succsim_\Lambda$ on 2^S be a monotonic confidence relation.

The dual of $\succsim_\Lambda$, denoted $\succsim_\Lambda^\top$ is defined by: $\forall A, B \subseteq S$, $A \succsim_\Lambda^\top B \Leftrightarrow \overline{B} \succsim_\Lambda \overline{A}$.

The confidence relation $\succsim_\Lambda$ is self-dual iff: $\succsim_\Lambda = \succsim_\Lambda^\top$ that is to say iff $\forall A, B \subseteq S$, $A \succsim_\Lambda B \Leftrightarrow \overline{B} \succsim_\Lambda \overline{A}$.

It is easy to show that the dual of a monotonic confidence relation is a monotonic confidence relation. Any capacity obviously defines a monotonic confidence relation: $A \succsim_\Lambda B \Leftrightarrow \mu(A) \geqslant \mu(B)$, and the monotonic confidence relations satisfy the fundamental property of capacities. Moreover, the monotonicity of $\succsim_\Lambda$ implies the monotonicity of its strict part:

Proposition 1. *If $\succsim_\Lambda$ on 2^S is a monotonic confidence relation then*:

$$\begin{aligned} &\forall A, B \subseteq S, \quad A \subseteq B \quad \Rightarrow \quad B \succsim_\Lambda A, \\ &\forall A, B, C \subseteq S, \quad A \succ_\Lambda B \quad \Rightarrow \quad A \cup C \succ_\Lambda B \quad \textit{and} \\ &\qquad\qquad\qquad\qquad A \succ_\Lambda B \cup C \quad \Rightarrow \quad A \succ_\Lambda B. \end{aligned}$$

Possibility and necessity measures induce confidence relations that are dual of each other. Probability measures induce confidence relations that are self-dual and preadditive:

Definition 3. A relation $\succsim_\Lambda$ on 2^S is said to be preadditive if and only if:

$$\forall A, B, C \subseteq S, \quad A \cap (B \cup C) = \emptyset \quad \Rightarrow \quad (B \succsim_L C \quad \Leftrightarrow \quad A \cup B \succsim_L A \cup C).$$

Note that preadditive confidence relations are self-dual (but the converse is not true).

In the framework of this paper, we also use confidence relations that cannot be represented by a confidence function (e.g., because some events are not comparable). It may be so because the relation is not complete, just like the generalized qualitative probabilities axiomatized in [32].

The decision rule then relies on a comparison, for any pair (f, g) in X^S of the set of states where f performs as least as well as g, and the set of states where g performs as least as well as f. When the preferences over X are represented by the weak-order $\succsim_X$, the purpose of the first step is to collect the sets of states $[f \succsim_X g] = \{s \in S,\ f(s) \succsim_X g(s)\}$ and $[g \succsim_X f] = \{s \in S,\ g(s) \succsim_X f(s)\}$. These events are then compared by means of the confidence relation $\succsim_\Lambda$.

This yields the following general decision rule we call *Likely Dominance Rule*:

$$f \succsim g \quad \Leftrightarrow \quad [f \succsim_X g] \succsim_\Lambda [g \succsim_X f]. \tag{4}$$

Using this rule, one prefers act f to act g if and only if it is more likely to get a better consequence with f than with g. This decision rule has been proposed in [13,22] under the name *Lifting Principle*.

It can be seen as the counterpart of majority rules used in social choice theory (see, e.g., [4]) or concordance rules used in Electre Methods [39,40] for multicriteria decision making. In social choice theory, the set S is the set of voters. In multicriteria decision making it is the set of criteria, and the confidence relation reflects the relative weights of groups of voters or criteria.

For instance, the *Probability-Based Dominance Rule* consists in preferring f to g whenever the probability of getting a better consequence by f is as least as high as the probability of getting a better consequence by g: it assumes that $\succsim_\Lambda$ stems from a probability function. In social choice theory, this rule is called pairwise majority when a uniform probability distribution is used.

More qualitative variants of Eq. (4) can be defined, e.g., the *Possibility-Based Dominance Rule* based on a relation $\succsim_\Lambda$ induced by a possibility measure. An alternative is the *Necessity-Based Dominance Rule*, that requires that $\succsim_\Lambda$ be induced by a necessity measure.

It can easily be shown that possibility-based and necessity-based dominance rules yield a relation $\succsim$ whose asymmetric part is transitive but this is not the case for the probabilistic likely dominance rule (see Example 3, Section 4). Moreover, these three rules yield a complete relation $\succsim$, but this is not the case for all likely dominance rules. For example, one can consider a *family* of necessity measures, and state that f is preferred to g iff, for each measure in the family, the necessity of $[f \succsim_X g]$ is greater than the necessity of $[g \succsim_X f]$. Such a rule is an example of likely dominance rule yielding a partial preference structure over the acts.

Finally and most noticeably, remark that preference reversals such as those observed in Examples 1 and 2 cannot occur with a likely dominance rule because the preference over the acts only depends on the relations $\succsim_\Lambda$ and $\succsim_X$ and not on their numerical representation. In this respect, preference models based on a likely dominance rule can be seen as *purely qualitative* approaches.

Remark 1. The decision rule proposed by Cohen and Jaffray [10] is akin to the likely dominance rule because it involves a comparison between the set of states where f performs better than g, and the set of states where where g performs worse than f. However these sets are compared on the basis of the states having worst consequences in each of them, not in terms of the likelihood relation (which is not available in their model).

3. A qualitative version of Savage's framework

This section aims at providing an axiomatic framework for purely qualitative decision models which do not assume that utilities of consequences can be compared to plausibilities of states. A representation theorem is derived, which ensures that the underlying decision rule for choosing between acts is a likely dominance rule. Following the work of Savage, we start from a user-driven preference relation over the acts, and we introduce natural axioms making it is possible to construct a preference relation $\succsim_P$ on the consequences and a confidence relation $\succsim_L$ on the events, so as to recover the DMs preferences by a decision rule—here, precisely, the likely dominance rule. Let us first recall more of the setting and some of the postulates proposed by Savage [41] in order to characterize the expected utility model.

3.1. Savage's axioms

In the Savage setting, the set of potential acts, X^S, is identified with the set of possible acts itself since we need a comparison model allowing to decide whether f is at least as good as g, denoted $f \succsim g$, or not, *whatever* (f, g), i.e., a comparison model that is well-defined on the entire set X^S.

Axiom P1. $(X^S, \succsim)$ is a weakly ordered space ($\succsim$ is complete, reflexive, transitive).

Such an axiom in Savage's work is justified by the goal assigned to the theory. If acts are ranked according to expected utility then the preference is transitive, reflexive, and complete. In this paper, we do not want to require such a property: the DM Preferences still make sense without being complete or transitive.

Among acts in X^S there are *constant acts* such that: $\exists x \in X\colon \forall s \in S,\ f(s) = x$. It seems reasonable to identify X with the set of constant acts. Such an act will thus be denoted f_x, or simply x whenever it is not ambiguous. A preference relation $\succsim_P$ on X can be induced from $\succsim$ by:

$$\forall x, y \in X \quad (x \succsim_P y \quad \Leftrightarrow \quad f_x \succsim f_y). \tag{5}$$

Obviously, in Savage's axiomatics $\succsim_P$ inherits all the properties of $\succsim$: under P1, it must be reflexive, complete, transitive, etc. These properties can be lost if we relax P1 as such. So they may have to be explicitly postulated if P1 is not accepted *a priori*.

A consequence of working with the set of potential acts is that, for any pair (f, g) of acts and for any subset of states $A \subset S$, we can soundly construct a mixed act fAg the components of which are those of f on the elements of A, and those of g on the other states:

$$\forall s \in S \quad (fAg)(s) = \begin{cases} f(s) & \text{if } s \in A, \\ g(s) & \text{if } s \notin A. \end{cases}$$

More generally, $f_1 A_1 f_2 A_2 \ldots f_k A_k g$ is the act the consequences of which are those of f_i on the states of A_i, $i = 1, \ldots, k$, and those of g on the other states (not in $A_1 \cup A_2 \cup \ldots \cup A_k$). This construction is at work in the following basic Savage postulate, and in the notion of conditional preference:

Axiom P2 (*Sure-Thing Principle*). $\forall A \subseteq S, \forall f, g, h, h' \in X^S,\ (fAh \succsim gAh \Leftrightarrow fAh' \succsim gAh')$.

Definition 4 (*Conditional Preference*). f is said to be weakly preferred to g, conditioned on A if and only if $\forall h \in X^S,\ fAh \succsim gAh$. This is denoted by $(f \succsim g)_A$.

The Sure-Thing Principle means that the preference of f over g does not depend on states where the two acts have the same consequences. It enables $(f \succsim g)_A$ to hold as soon as $fAh \succsim gAh$ for some h only. Conditional preference $(f \succsim g)_A$ means that f is at least as good as g when the state space is restricted to A. Notice that, would the Sure-Thing Principle not hold, conditional preference might no longer be a complete relation. Any event $A \subseteq S$ which is unable to make a difference between any pair of acts is considered as null (e.g., irrelevant for the DM).

Definition 5 (*Null events*). An event A is said to be null if and only if: $\forall f, g, h \in X^S$, $fAh \sim gAh$.

Note that this definition of null events also makes sense when $\succsim$ is not complete.

Another hypothesis is that the preference order on consequences is unique and does not depend on the conditioning events considered. This is the third Savage postulate:

Axiom P3. $\forall A \subseteq S$, A not null, $(x \succsim y)_A \Leftrightarrow x \succsim_P y$.

The following proposition points out obvious consequences of P1 P2 and P3, that may also fail to hold if P1 is weakened:

Proposition 2. *The following properties hold*:

- Quasi-transitivity of $\succsim$: *if* P1 *holds then* $\succ$ *is transitive.*
- Transitivity of $\sim$: *if* P1 *holds then* $\sim$ *is transitive.*
- Weak Unanimity: *if* P1 *and* P2 *hold, then*: $\forall A, B \subseteq S$ *such that* $A \cap B = \emptyset, \forall f, g \in X^S$, $((f \succsim g)_A$ *and* $(f \succsim g)_B) \Rightarrow (f \succsim g)_{A \cup B}$.
- Monotonicity: *if* P1, P2 *and* P3 *hold, then*: $x \succ_P y$ *and* $y \succsim_P z \Rightarrow x \succsim_P z$
  if $\forall s \in A, f'(s) \succ_P f(s)$, *then* $f \succsim g \Rightarrow f'Af \succsim g$—Left Monotonicity (LM),
  if $\forall s \in A, g(s) \succ_P g'(s)$, *then* $f \succsim g \Rightarrow f \succsim g'Ag$—Right Monotonicity (RM).

An important particular case of weak unanimity is a classical condition which says that an act is not less preferred than another if the former is not less preferred than the latter both when event A occurs and when it does not. It is equivalent to the unanimity postulate Q3 proposed by Lehmann [32] in his axiomatization of generalized qualitative probabilities. Weak unanimity is a strong version of Grant et al.'s Weak Sure Thing Principle [28] (namely, these authors restrict unanimity when $B = \overline{A}$). The monotonicity condition requires that, if f is preferred to g, then enhancing f or degrading g will obviously not reverse the preference. This condition obviously makes sense only in the context of a preference on consequences satisfying $x \succ_P y$ and $y \succsim_P z$ imply $x \succsim_P z$, e.g., when $\succsim_P$ is complete and quasi-transitive. In Savage's framework, all these very natural properties follow from P1—again they may have to be explicitly postulated if P1 is not accepted a priori.

The preference on acts also induces a confidence relation on events: it is sufficient to consider the set of binary acts, of the form $f_x A f_y$, which can be denoted xAy, where $x \succ_P y$. Clearly for fixed $x \succ_P y$, the set of acts $\{x, y\}^S$ is isomorphic to the set of events 2^S. Since the restrictions of $(X^S, \succsim)$ to $\{x, y\}^S$ may be inconsistent with the restriction to $\{x', y'\}^S$ for other choices of consequences x', y' such that $x' \succ_P y'$, Savage has introduced a fourth postulate:

Axiom P4. $\forall A, B \subseteq S, \forall x, y, x', y' \in X$: $x \succ_P y$ and $x' \succ_P y'$, $xAy \succsim xBy \Leftrightarrow x'Ay' \succsim x'By'$.

Under P4, the choice of $x, y \in X$ with $x \succ_P y$ is not important when defining the ordering between events in terms of binary acts. Hence, the following confidence relation on events can be derived from $\succsim$:

$$A \succsim_L B \quad \Leftrightarrow \quad (\exists x, y \in X\colon\ x \succ_P y \text{ and } xAy \succsim xBy). \tag{6}$$

Lastly, Savage has assumed that the weakly ordered set $(X, \succsim_P)$ is not trivial:

Axiom P5. X contains at least two elements x, y such that $f_x \succ f_y$ (or $x \succ_P y$).

Under P1–P5, the relation $\succsim_L$ on events is a monotonic confidence relation that is preadditive. Preadditivity is indeed just the specialization of the Sure Thing Principle for events. Savage has shown that relation $\succsim_L$ is a comparative probability ordering (S being finite or not), i.e., it has properties of confidence relations induced by probability measures:

Definition 6 (*Comparative probability*). A relation $\succsim_L$ on 2^S is a comparative probability iff:

(i) $\succsim_\Lambda$ is a weak order (complete, reflexive, transitive),
(ii) $S \succ_L \emptyset$,
(iii) $\forall A,\ A \succsim_L \emptyset$,
(iv) it is preadditive.

Not all comparative probabilities can be represented by probability measures. Savage also introduces two other postulates that enforce the existence (and uniqueness) of a numerical probability measure on an infinite set S, that can represent the confidence relation $\succsim_L$. However, these axioms are omitted here because they are irrelevant in the case of a finite set of states and for qualitative decision theory.

3.2. The ordinal invariance postulate

We now propose a new axiom aiming at acknowledging the qualitative nature of a decision process and the non-commensurability of the confidence of events and the preference on consequences. We call this axiom *Ordinal Invariance*:

Definition 7. Two pairs of acts (f, g) and (f', g') are said to be ordinally equivalent, denoted $(f, g) \equiv (f', g)$ if and only if:

$$\forall s \in S, (f(s) \succsim_P g(s) \quad \Leftrightarrow \quad f'(s) \succsim_P g'(s)).$$

Ordinal equivalence means that, for each state of the nature, the preference pattern between acts f and g is the same as the preference between f' and g', respectively.

Axiom OI. $\forall f, f', g, g' \in X^S, ((f, g) \equiv (f', g')) \Rightarrow (f \succsim g \Leftrightarrow f' \succsim g')$.

Axiom OI expresses that only the relative positions of the consequences of the two acts are important, but not the consequences themselves, nor the positions of the two acts relatively to other acts: it means that changing the consequences of f and g on any state in such a way that the preference pattern between their consequences is preserved does not change the relative preference between the acts. It does not require any commensurability or comparability between the relative merit of consequences (of constant acts) and the relative likelihood of events (= merit of binary acts). Ordinal invariance can actually be seen as the statement, in the framework of decision under uncertainty, of an "independence of irrelevant alternatives" condition used in social choice theory [43], completed with a neutrality condition making preferences independent of the labels of the acts considered. It can also be understood as the counterpart of a non-compensation axiom introduced by Fishburn in the framework of multicriteria analysis [24]. Note also that OI is a strong version of the two key-postulates of Savage theory, P2 and P4 as shown by the following:

Proposition 3. (OI + $\succsim$ *reflexive*) $\Rightarrow$ P2.

Proposition 4. (OI + $\succsim$ *reflexive*) $\Rightarrow$ P4.

To be non-trivial, Proposition 4 presupposes P5 holds. Let us now develop our axiomatization of the likely dominance rule. Following Savage, and starting from a relation $\succsim$, we can derive the relative attractiveness of the consequences using Eq. (5): $(x \succsim_P y \Leftrightarrow f_x \succsim f_y)$. Similarly, we can soundly build a confidence relation $\succsim_L$ from $\succsim$, using Eq. (6): $A \succsim_L B \Leftrightarrow (\exists x, y \in X\colon\ x \succ_P y$ and $xAy \succsim xBy))$, since P4 is implied by OI and the reflexivity of $\succsim$.

At this point, we only need two additional conditions (plus non-triviality) to recover the likely dominance rule from OI. The first one is the reflexivity of $\succsim$, which is not questionable, and the second one is the total comparability of constant acts (i.e., the completeness of $\succsim_P$) which is a natural property because there is no conflict between states, in the comparison of constant acts. These conditions form a significant weakening of P1, since at this point we do not assume the transitivity of $\succsim$ nor even its quasi-transitivity nor its completeness.

Our set of axioms is consistent. Indeed, these properties, and some other ones, hold for any likely dominance rule:

Proposition 5. *Let $\succsim_X$ be a reflexive and complete preference relation on X such that $\exists x, y \in X, x \succ_X y$ and $\succsim_\Lambda$ be a confidence relation on 2^S. Let us consider the preference relation $\succsim$ defined on X^S by $f \succsim g \Leftrightarrow [f \succsim_X g] \succsim_\Lambda [g \succsim_X f]$ and $\succsim_P$ its projection on X defined by Eq.* (5)*. We have*:

- $\succsim$ *is reflexive.*
- $\succsim$ *satisfies* OI.
- $\succsim$ *is complete on constant acts* (*i.e.*, $\succsim_P$ *is complete*).
- $\forall x, y \in X, (x \succsim_P y \Leftrightarrow x \succsim_X y)$.
- $\succsim$ *also satisfies* P2, P4, P5.
- *If* $\succsim_\Lambda$ *is monotonic, then* $\succsim$ *satisfies* P3.
- *If* $\succsim_X$ *is quasi-transitive and* $\succsim_\Lambda$ *is monotonic, then* $\succsim$ *satisfies* LM *and* RM.

Conversely, we have the following representation theorem characterizing the likely dominance decision rule:

Theorem 1. *If* $\succsim$ *is complete on constant acts, reflexive and satisfies* P5 *and* OI *then there exists a preference relation* $\succsim_P$ *on X defined by* (5) *and a reflexive, preadditive and non-trivial relation* $\succsim_L$ *on S defined by* (6) *such that, for any* $f, g \in X^S$, $f \succsim g \Leftrightarrow [f \succsim_P g] \succsim_L [g \succsim_P f]$.

Corollary 1. *If the* DM *preference* $\succsim$ *is complete on constant acts, reflexive on* X^S *and satisfies* P5, OI *holds if and only if* $\succsim$ *is representable by a likely dominance rule.*

The previous result provides an operationally testable characterization of the likely dominance rules. If OI is accepted as a norm for qualitative models, then the only available decision rules are likely dominance rules. This result holds even if $\succsim$ is not supposed to be complete, and thus concerns a wider class of decision rules than those given in the paper of Dubois et al. [12].

Nevertheless, this result does not induces fully well-behaved relations on X and 2^S. Of course, the relation $\succsim_P$ is obviously reflexive and complete, and, when $\succsim$ is explicitly built with a likely dominance rule, $\succsim_P$ coincides with $\succsim_X$. But it does not necessarily satisfies Savage's P3, which is not a consequence of OI. Indeed, consider an act h, two constant acts x and y such that $x \succ y$, and some event $A \neq S$. The two pairs of acts (x, y) and (xAh, yAh) are clearly not order-equivalent. Hence axiom OI cannot put any constraints relating the preference pattern between x and and y and the preference pattern relating xAh and yAh. Besides, our minimal axiomatization implies very little about $\succsim_L$, except that it is reflexive (due to the reflexivity of $\succsim$), preadditive (due to P2) and non-trivial (due to P5). But the confidence relation may fail to be monotonic, since the global relation $\succsim$ is not supposed to be such.

3.3. The monotonicity condition

The addition of the very natural requirement of monotonicity of $\succsim$ (that is compatible with the other axioms—see Proposition 5) results in a sounder structure for both $\succsim_P$ and $\succsim_L$.

Proposition 6. *If $\succsim$ is reflexive, complete on constant acts, and satisfies* P5, OI, LM *and* RM*, then*:

- *$\succsim_L$ is a monotonic confidence relation,*
- *$\forall A \subseteq S, \emptyset \succsim_L A \Leftrightarrow A$ is null,*
- *$\succsim$ satisfies* P3.

Theorem 2. *$\succsim$ is complete on constant acts, reflexive, satisfies* P5, OI, LM *and* RM *if and only if there exists a complete preference relation $\succsim_P$ on X and a preadditive and monotonic confidence relation $\succsim_L$ on S such that, for any $f, g \in X^S$, $f \succsim g \Leftrightarrow [f \succsim_P g] \succsim_L [g \succsim_P f]$.*

It should be emphasized that the necessary preadditivity of $\succsim_L$ does not mean that the use of a preadditive confidence relation is compulsory in the definition of a likely dominance rule. Indeed, starting from a relation $\succsim_X$ on X, and a non-preadditive confidence relation $\succsim_\Lambda$, and constructing a relation $\succsim$ on acts with them by the likely dominance principle, Theorem 1 only proves the *existence* of a preadditive $\succsim_L$ yielding a likely dominance rule representing $\succsim$: everything works *as if* $\succsim_L$ were used, but, unlike $\succsim_P$ and $\succsim_X$, $\succsim_L$ and $\succsim_\Lambda$ do not necessarily coincide.

The reason for this apparent paradox is the following. First note that $[f \succsim_X g] \cup [f \succsim_X g] = S$ because of the completeness of $\succsim_X$ (the same holds for $\succsim_P$ since $\succsim_X = \succsim_P$). So only the pairs (A, B) such that $A \cup B = S$ are compared in the likely dominance rule. Then, $\succsim_L$ and $\succsim_\Lambda$ are related as follows:

Proposition 7. $A \succsim_L B \Leftrightarrow A \cup \overline{B} \succsim_\Lambda B \cup \overline{A}$.

So $\succsim_L$ and $\succsim_\Lambda$ always coincide for some pairs of sets as per the following proposition:

Proposition 8. $A \succsim_L B \Leftrightarrow A \succsim_\Lambda B$ *whenever* $A \cup B = S$.

An immediate consequence of Propositions 7 and 8 is that the two relations actually coincide if and only if $\succsim_\Lambda$ is preadditive:

Proposition 9. *If $\succsim_\Lambda$ reflexive, then $\succsim_\Lambda = \succsim_L \Leftrightarrow \succsim_\Lambda$ is preadditive.*

In summary, $\succsim_\Lambda$ and $\succsim_L$ coincide on the useful part of $2^S \times 2^S$ and the non-preadditivity of $\succsim_\Lambda$ cannot be revealed by observing a decision maker which would use a likely dominance rule for choosing between acts. These results explain why the preadditivity of $\succsim_\Lambda$ need not be requested from the start.

4. Ordinal decision rules and uncertainty theories compatible with a quasi-transitive preference

The transitivity of preferences between acts resulting from a qualitative decision rule defined as in the previous section is not ensured, not even the transitivity of its strict part. The failure of transitivity can be simply observed using the following example:

Example 3. Consider three consequences $x, y, z \in X$ such that $x \succ_P y$, $y \succ_P z$ and $x \succ_P z$ and suppose $S = \{s_1, s_2, s_3\}$ with probabilities $p(s_1) = p(s_2) = p(s_3) = 1/3$. Using the probabilistic likely dominance rule to compare the acts:

$$f = x\{s_1\}y\{s_2\}z\{s_3\}, \quad g = y\{s_1\}z\{s_2\}x\{s_3\} \quad \text{and} \quad h = z\{s_1\}x\{s_2\}y\{s_3\}$$

we get $[f \succsim g] = \{s_1, s_2\}$, $[g \succsim h] = \{s_1, s_3\}$ and $[h \succsim f] = \{s_2, s_3\}$ and therefore $f \succ g$, $g \succ h$ and $h \succ f$, i.e., an intransitivity cycle. On the contrary if probabilities of states are $p'(s_1) = 0.6$, $p'(s_2) = 0.3$ and $p'(s_3) = 0.1$ we get $f \succ g$, $g \succ h$ and $f \succ h$, a linear order.

This example is actually a case of Condorcet effect as observed in Social choice (see, e.g., [43]). From a descriptive point of view, it shows the ability of qualitative decision rules to explain some cyclic preference structures and this is a really original point when compared to more classical models based on preference functionals. However, from a prescriptive point of view, the possibility of getting cyclic preferences is an important source of difficulty. For this reason, we investigate in this section the possibility for qualitative decision models to fulfill some minimal transitivity requirements. It turns out that these requirements have drastic consequences on the nature of the underlying confidence relation.

4.1. Confidence relations under quasi-transitive preference in the ordinal approach

Let us just assume the reflexivity and the *quasi-transitivity* of $\succsim$, that is, the transitivity of its strict part. Namely, we consider the following axiom that can be seen as a useful weakening of P1.

Axiom A1. $\succsim$ is reflexive and quasi-transitive, and its restriction to constant acts is complete.

This weak requirement allows a wide class of preference structures to be concerned, including those with an intransitive indifference (semi-orders, interval orders) and those allowing incomparability. Allowing incomparability in the two other canonical situations (preference, indifference) provides a better description of indecision situations: indifference between two acts f, g can be dedicated to the case of a strong similarity between the two vectors of consequences, whereas incomparability reflects the existence of a conflict in the comparison of f and g (e.g., $\exists A \subset S$, not null, whose complement is not null and such that $fAg = f_x$ and $gAf = f_y$, where x and y represent the best and the worse consequences of X respectively). Such conflicts do not exist in the comparison of constant acts and therefore, we require the total comparability of constant acts only (i.e., the completeness of $\succsim_P$). Axiom A1 is weaker than the one used in [12], where the preference relation $\succsim$ is supposed to be complete on the set of all acts.

The first noticeable consequence of requirement A1 concerns the comparison of consequences and appears when X contains at least 3 distinct consequences:[1]

Axiom A5. X contains at least three elements x, y, z such that $f_x \succ f_y$, $f_y \succ f_z$ and $f_x \succ f_z$.

Proposition 10. *If* A1, OI, LM, RM, A5 *hold and* $\succsim_P$ *is not transitive, then*: $\forall A, B \subseteq S$, *if* A *and* B *are not null then* $A \cap B$ *is not null.*

Corollary 2. *If* A1, OI, LM, RM, A5 *hold and* $\succsim_P$ *is not transitive, then*: $\exists$ *a non-null event* $O \subseteq S$ *such that* $\forall A$:

- A *not null* $\Leftrightarrow O \subseteq A$.
- $O \subseteq A \Rightarrow O \sim_L A$.

It means that, when the quasi-transitivity of $\succsim$ is required, the only confidence relations compatible with a *non*-transitive $\succsim_P$ (actually, a non-transitive $\sim_P$, since the quasi-transitivity of $\succsim$ means the transitivity of $\succ_P$) are those that encode trivial or counterintuitive situations. First, the case in which all states are null, except one, namely when O is a singleton. The latter case is when the actual state is precisely known, and there is no uncertainty. Then, all acts are equivalent to constant acts, since $f \succsim g$ if and only if $f(s) \succsim_P g(s)$ where the non-null state is s. The properties of $\succsim_P$ then exactly reflect the properties of $\succsim$ which is not assumed to be transitive. The second case is when there is a sure (hence non-null) event, all states realizing it being null; this situation is hard to interpret. To rule out these situations the following non-triviality assumption is made:

Axiom 2NN. There are at least two non-null states.

Then, the full transitivity of $\succsim_P$ is a consequence of A1:

Proposition 11. *If* A1, OI, LM, RM, 2NN, *and* P5 *hold then* $\succsim_P$ *is a weak order.*

The second important consequence of the requirement of quasi-transitivity directly affects the confidence relation (from Proposition 6, we already know it is monotonic):

[1] Hopefully, A1 and A5 are compatible with OI, LM and RM (see Proposition 13).

Proposition 12. *If* A1, A5 *and* OI *hold then relation* $\succ_L$ *satisfies the property of negligibility*:

$$\forall A, B, C \subseteq S \text{ such that } A \cap B = A \cap C = B \cap C = \emptyset,$$
$$A \cup C \succ_L B \quad \text{and} \quad A \cup B \succ_L C \quad \Rightarrow \quad A \succ_L B \cup C.$$

This property, introduced by Friedman and Halpern [25] and Dubois and Prade [20], is a very drastic one and is the one that enforces the qualitative nature of the confidence relation. It indeed implies that $\forall A, B, C$ such that $A \cap B = A \cap C = B \cap C = \emptyset$,

$$A \succ_L B \quad \text{and} \quad A \succ_L C \quad \Rightarrow \quad A \succ_L B \cup C.$$

This property is generally incompatible with a probabilistic representation of uncertainty, but as shown later on is fully compatible with possibility theory. The idea that some events may have negligible plausibility in front of others is also advocated by Lehmann [32] in a framework where both probabilities and plausibilities involving negligibility effects coexist.

4.2. The oligarchic nature of decision rules in the ordinal setting

The role of the set of most plausible states in our ordinal setting for decision making under uncertainty has a familiar flavor in social choice. In analogy with the latter domain, one can actually show that there is one and only one set of vetoer states which is also decisive, i.e., a predominant event. This can be formalized as follows:

Definition 8 (*Vetoers, Decisive set and Predominant event*). For any subset $S' \subseteq S$, the state $s \in S'$ is said to be a *vetoer in* S' iff:

$$\forall f, g \in X^S, \quad \big[f(s) \succ_P g(s) \quad \Rightarrow \quad not\big((g \succ f)_{S'}\big)\big].$$

$O \subseteq S'$ is said to be *decisive in* S' if and only if:

$$\forall f, g \in X^S, \quad \big[\big(\forall s \in O, \big(f(s) \succ_P g(s)\big)_{S'}\big) \quad \Rightarrow \quad (f \succ g)_{S'}\big].$$

$O \subseteq S'$ is said to be *predominant in* S' iff it is decisive in S' and contains only vetoer-states in S'. A predominant event in S is simply called a *predominant event*.

We can show that any decisive set of states is more likely than its complement in any context:

Lemma 1. *If* $\succsim$ *satisfies* A1, A5, LM, RM *and* OI, *then, for any predominant event* O *in a non-null subset* $S' \subseteq S$, $\forall A \subseteq S'$,

$$O \subseteq A \quad \Leftrightarrow \quad A \succ_L S' \setminus A.$$

Now the main result can be obtained, on the existence and unicity of a decisive set in any (non-null) context:

Theorem 3. *If* $\succsim$ *satisfies* A1, A5, LM, RM *and* OI, *then, for any non-null* $S' \subseteq S$, *there exists one and only one predominant event* O *in* $2^{S'}$. *Moreover,* O *is such that*:

- $\forall A, B \subseteq S', (A \cap B = \emptyset$ *and* $A \succ_L B) \Rightarrow O \cap B = \emptyset$.
- $\forall A, B \subseteq S', (A \cap B = \emptyset$ *and* $O \subseteq A) \Rightarrow A \succ_L B$.

The following result, a direct consequence of the above theorem, is more explicit:

Corollary 3. *If* $\succsim$ *satisfies* A1, A5, LM, RM *and* OI, *then there exists one and only one predominant event.*

The intersection of all the events A such that $A \succ_L \overline{A}$ forms the set of most plausible states. It is the predominant event O, and it is never empty. It is the one on the basis of which almost every decision is made. The corollary expresses that, when the DM's preferences are quasi-transitive and compatible with OI, they necessarily rely on a confidence relation admitting such a predominant event O which makes any other non-compatible event negligible. When comparing f and g, the states outside this set are not taken into account (unless each of the most plausible states is indifferent among the two acts ($f(s) \sim_P g(s)$)). First, $f \succ g$ holds as soon as f is preferred to g for all states belonging to O. However, whenever two states s and s' are conflicting within O concerning a pair f, g (i.e., $f(s) \succ_P g(s)$ and $g(s') \succ_P f(s')$), no strict preference can be stated between the two acts. In Social Choice Theory, such a preference structure is said to be *oligarchic* (see, e.g., [43]).

The larger O, the less decisive the procedure. In case of total ignorance, i.e., when no state of the world is more likely than another, we get $O = S$: the preference relation $\succsim$ reduces to the functional dominance on X^S induced by the monotony condition (i.e., $f \succsim g$ iff $f(s) \succsim_P g(s)$ for all states). In the other extreme case, O contains a single state and the DM preferences can be described by a "dictatorial" decision rule due to the existence of a decisive state s. Such a rule makes sense when the DM is almost certain than the "true state" is s and neglects the other states: all the other possible states are ignored, except when comparing acts that are indifferent with respect to the almost certain state, as seen now. It makes such a framework very close to the approaches that decide on the basis of the most plausible states only—with the slight difference that, here, the less plausible states can be considered for breaking ties.

Making a step further, our Theorem 3 indeed shows that the decision rule does not reduce to the existence of a mere oligarchy. Indeed, it also proves that the property of Negligibility of the confidence relation, and thus the existence of a predominant event, is stable under conditioning. In other words, the above results indeed hold when restricting to any context $S' \subseteq S$. In particular, the approach remains valid in a dynamic setting, when new information about the actual state of the word is acquired by the DM. Decisions are dynamically consistent.

The reason for this good behavior is that when two acts share the same consequences on a subset E of states, these states are neglected as irrelevant in the choice between the two acts (this is a consequence of the sure thing principle). So, *there is a set of predominant states in* $\overline{E}$ *that acts as a local oligarchy* and only this set can discriminate between the acts. It implies that there exists not just one oligarchy, but also a *hierarchy* of oligarchies, a result that had never been emphasized before:

Corollary 4. *If* $\succsim$ *satisfies* A1, A5, LM, RM *and* OI, *there exists a partition of* S *into events* $O_1, \ldots, O_k, O_{k+1}$ *such that* $\forall j \leqslant k,\ O_j$ *is predominant in* $S_j = S \setminus \bigcup_{i=1}^{j-1} O_i$.

So, if $f(s) \sim_P g(s)$ on the set E of states only, and O_j is the highest subset not contained in E in the hierarchy of oligarchies, then O_j is decisive for choosing between f and g, namely $f \succsim g$ if and only if $f(s) \succsim_P g(s) \forall s \in O_j$.

Now, assuming both completeness and transitivity of $\succsim$ we obtain a hierarchy of predominant *states* (see also [12] for a different proof):

Theorem 4. *For any preference relation on acts that satisfies* P1, A5 *and* OI, *we have*:

- $s_i \sim_L s_j \Leftrightarrow s_i$ *and* s_j *are null*;
- $\forall f, g \in X^S, \quad f \succ g \Leftrightarrow \exists s^* \in S$ *such that* $\begin{cases} s^* \text{ is not null}, \\ f(s^*) \succ_P g(s^*) \\ \forall s \in S,\ [g(s) \succ_P f(s) \Rightarrow s^* \succ_L s]. \end{cases}$

This theorem shows the lexicographic structure of the preference induced by P1. It demonstrates a phenomenon which was already illustrated in Example 3 and also explains that comparative probabilities (and not only classical probabilities) are for the most part incompatible with a qualitative approach of decision. The resulting lexicographic rule corresponds to a very particular structure of uncertainty in which all the non-impossible states must be linearly ordered. As a consequence, requiring P1 forbids the description of total ignorance (where all the states are equally plausible). Moreover, there must exist a predominant state s^*, more plausible than any other state, and even more plausible than the complementary event $S \setminus \{s^*\}$; P1 can be satisfied if and only if one is in a situation of quasi-certainty about the actual state. As soon as s^* gives a better consequence for an act than for another one, the first act is preferred. The other possible states of the world are so negligible compared to s^* that they are not considered, unless the quasi-certain state does not allow a discrimination between the acts. The likely dominance rule can then be described in terms of a necessity measure based on a linear possibility distribution of the form $\pi(s_1) > \pi(s_2) > \cdots > \pi(s_n)$ as $f \succsim g$ if and only if $N([f \succsim_P g]) \geqslant N([g \succsim_P f])$.

4.3. Related literature

In some sense, the results of this paper revisit Savage's approach to decision making under uncertainty using the tools of social choice theory [43]. In social choice, the traditional approach is ordinal. Given a set of voters and a set of candidates, each voter defines a preference relation on the set of candidates, thus indicating which candidates are preferred to others, according to this voter's opinion. The problem is then to merge these local preference relations into a global one representing the collective preference, using an appropriate voting rule. In our framework, states of nature stand for voters, consequences stand for candidates. As said above, the likely dominance rule accounts for several well-known schemes in social choice: unanimity (it is obtained in the quasitransitive framework, when the predominant set is S itself—total ignorance about the state); dictatorship (when one state is more likely than all others in the quasi-transitive framework); pairwise majority (counting how many states favor f against g and g against h) when the confidence relation derives from a uniform probability distribution as in Example 3. One difference between social choice and decision making under uncertainty is that in the case of voting procedures it is expected that voters be equally important while states won't be equally plausible, generally.

Results similar to the ones of the previous subsection can be found in the social choice literature. A weak form of Theorem 3 (for $S' = S$) is proved by Weymark [48], where the obtained collective preference structures are called *oligarchic*. However this author does not lay bare the hierarchy of oligarchies. In the case of a fully transitive preference on acts, Theorem 4 is closely related to Arrow's Theorem [3]. Arrow finds the existence of a dictator state but does not lay bare the underlying lexicographic structure exhibited by Theorem 4. This theorem is in fact a counterpart of Fishburn's theorem obtained in the context of multicriteria analysis [23], where criteria play the role of states. See [11] for a detailed comparison between multicriteria analysis and decision under uncertainty in the scope of qualitative decision theory, and the corresponding representation results.

The decision rule obtained from the postulates of ordinal decision under uncertainty is somewhat similar to the one proposed by Maynard-Reid and Lehmann [36] for aggregating so-called belief states coming from sources of various reliability (sources play the role of states and the aim is to build a relation on events rather than on acts). In their work, belief states are represented by transitive and "modular" relations between states of nature, capturing the idea that a state is strictly more believed than another. The modularity of this relation is equivalent to the transitivity of its complement, that is, the weak belief relation is transitive. This is an assumption that is not made here. Moreover, they do not assume the irreflexivity of the strict belief relation, for the sake of capturing ideas of conflicts between states, and not only indifference: a conflict arises when two states are strictly preferred to each other (so, by transitivity, the strict relation cannot be irreflexive). This is also a major difference between their framework and ours. In the framework of decision under uncertainty, giving up the irreflexivity of the strict preference between acts looks hard to defend. When the set of sources is a linear order, the fusion rule of [36] yielding the global relation on states is the same as the lexicographic scheme obtained here from the likely dominance rule proposed. However when sources have equal reliability, their combination rule concludes that one state is more believed than another whenever at least one source does. It may lead to a very conflicting belief state. On the contrary, in the absence of dominating state, the likely dominance rule concludes that an act is better than another only when the former is better than the latter for all states.

The proposed ordinal framework for decision under uncertainty also comes close to knowledge representation and reasoning, especially non-monotonic reasoning. The property of Negligibility of the monotonic confidence relation obtained from Proposition 12 in the quasi-transitive setting has been previously identified in [20] as capturing the idea of "Acceptance" and by Friedman and Halpern [25,29] under the name "Qualitativeness". It is characteristic of conditional likelihood relations that are compatible with deductive closure. Namely, the set of accepted beliefs A in context C such that $A \cap C \succ_L \overline{A} \cap C$, induced by the relation $\succ_L$, is deductively closed if and only if the strict part of the confidence relation obeys Negligibility and Monotonicity.

To see the close link with non-monotonic reasoning, one has to recall that the nonmonotonic inference of B from A under a strict confidence relation can be modelled by the property $A \cap B \succ_L A \cap \overline{B}$. It can be shown, following [13,15], that any monotonic set-relation that satisfies Negligibility underlies a consequence relationship that satisfies the properties laid bare by Kraus, Lehmann and Magidor (KLM) for their so-called preferential entailment [31], but for the reflexivity (whereby A follows from itself). The latter must be restricted to non empty events: here, nothing can be deduced from contradictory contexts. The connection between Savage axioms and the KLM postulates for non-monotonic reasoning is explored in greater details in a companion paper [12]. In that paper, the primitive relation on acts is the strict preference $\succ$ whose negation provides the weak preference relation. The corresponding framework is more restrictive than here, since this weak preference relation is then necessarily complete.

In some sense, our system of decision postulates is an operationally testable axiomatization of preferential entailment in terms of choices between acts, like Savage's axioms are an act-driven justification of Probability Theory.

Preferential entailment can also be described using a partial ordering of events induced by a family of possibility relations [16]. The high compatibility of preferential entailment with possibility theory suggests a possibilistic behavior of quasi-transitive likely dominance rules. In the context of a complete weak order, negligibility is indeed equivalent to the characteristic axiom of possibility relations, that is $A \succsim_L B \Rightarrow A \cup C \succsim_L B \cup C$. It suggests the search for a Savage-like justification of possibility theory in the purely ordinal setting as a legitimate purpose. This is the topic of the next section.

5. Characterization of possibilistic likely dominance rules

Theorem 4 can be understood as a negative result that shows the conflict between the postulate of ordinal invariance and axiom P1. On the other hand, we know that, if we relax P1, very important and desirable properties of the preference relation on acts are lost, especially unanimity and monotonicity properties pointed out in Proposition 2. Insofar as they are not controversial, they should be postulated explicitly if they are to be taken advantage of. Requiring the monotonicity of $\succsim$ or its quasi-transitivity leads to an uncertainty structure that is very close to possibility theory. Making a step further, we conclude here our axiomatization by adding a principle of extended unanimity and we show that the family of rational likely dominance rules then reduces to those based on necessity measures. The unicity of the underlying necessity relation is then enforced by adding a postulate of anonymity.

5.1. *The necessity-based weak dominance rule*

The necessity-based weak dominance rule [13] builds the preference on acts from a weak-ordered preference relation on consequences, a possibility distribution π on states, and a necessity measure N on events:

$$f \succsim g \quad \Leftrightarrow \quad N\big([f \succsim_X g]\big) \geqslant N\big([g \succsim_X f]\big).$$

It can equivalently be rewritten in terms of the possibility measure on disjoint events induced by strict preference between consequences as

$$f \succsim g \quad \Leftrightarrow \quad \Pi\big([f \succ_X g]\big) \geqslant \Pi\big([g \succ_X f]\big).$$

This rule provides a well-behaved preference relation $\succsim$ on acts (see also [12]):

Proposition 13. *Let $\succsim_X$ be a weak order on X and N be a necessity measure on 2^S. The relation $\succsim$ defined by $f \succsim g \Leftrightarrow N([f \succsim_X g]) \geqslant N([g \succsim_X f])$ has the following properties*:

- *$\succsim$ is reflexive, complete and quasi-transitive.*
- *$\succsim$ satisfies* P2, P3, P4.
- *$\succsim$ satisfies* OI, LM, RM.
- *If $\succsim_X$ satisfies* P5 (*respectively* A5), *so does $\succsim$.*

Notice that this proposition shows the consistency of the set of postulates A1, P2, P3, P4, OI, A5, LM, RM used in the previous section.

When considering the negative theorem of Section 4 (Theorem 4), the only postulate that has been relaxed is the transitivity of indifference.

Example 4. Consider a set S involving only two totally possible states s_1, s_2 (such that $\pi(s_i) = 1$, for $i = 1, 2$ and $\pi(s_i) < 1$ otherwise) and a weak order $\succsim_X$. We may have: $g(s_1) \succ_X f(s_1) \succ_X h(s_1)$, $f(s_2) \succ_X h(s_2) \succ_X g(s_2)$. So: $f \sim g$ (since $\Pi([f \succ_X g]) = \Pi([g \succ_X f]) = 1$, $g \sim h$ ($\Pi([g \succ_X h]) = \Pi([h \succ_X g]) = 1$), but $f \succ h$ (since $\Pi([h \succ_X f] < 1 = \Pi([f \succ_X h]))$. Thus $\sim$ is not transitive.

From a descriptive point of view, this lack of transitivity is rather satisfactory. Indeed, recall that indifference may appear in two cases: when the states are indifferent, or when they are conflicting. As shown in the previous example, two states of equal confidence can be in conflict for the comparison of (f, g), and the same two (and possibly others) equally likely states can be in conflict with respect to (g, h) without implying the indifference nor the conflict of these states for the comparison of f and h.

This example also shows the crucial role played by the most possible states: they form the first oligarchy. More generally, there exists a hierarchy of predominant events corresponding to the level sets of the possibility distribution.

Proposition 14. *Let $\succsim_X$ be a weak order on X, N be a necessity measure on 2^S derived from a possibility distribution π on a scale $L = \{\pi(s), s \in S\}$ and $\succsim$ the preference relation defined by $f \succsim g \Leftrightarrow N([f \succsim_X g]) \geqslant N([g \succsim_X f])$. Then,*

$$\forall \alpha_i \in L, \quad O_i = \big\{s, \pi(s) = \alpha_i\big\} \text{ is predominant in } S_i = S \setminus \big\{s, \pi(s) > \alpha_i\big\}.$$

Another property of the necessity-based rule is that null events form a class closed under union and intersection. In other words, there is a maximal null subset Z of S whose subsets are all null. It is easy to prove that null states are those which have zero possibility:

Proposition 15. *Under the necessity-based weak dominance rule, A is null if and only if $\Pi(A) = 0$, and the set of null states is $Z = \{s, \pi(s) = 0\}$.*

When there are two non-null states, the necessity and the possibility measures are said to be non-trivial.

5.2. Axiomatization of the necessity-based weak dominance rule with respect to a family of possibility distributions

Section 4 suggests the properties of confidence relations compatible with quasi-transitivity have a highly possibilistic flavor, since they are forced to satisfy the property of negligibility (see [16]). To recover the necessity-based dominance rule, we first need an additional postulate, namely a postulate of unanimity involving conditional preference:

Axiom EUN (*Extended Unanimity*). $\forall A, B \subseteq S, \forall f, g \in X^S$: $(f \succsim g)_A$ and $(f \succsim g)_B \Rightarrow (f \succsim g)_{A \cup B}$.

EUN states that, if f is preferred to g when one is sure that A occurs, or when one is sure that B occurs, then it is preferred to g when one is sure that either A or B, or both occur. This is exactly the postulate of "Closure under Union" proposed by Brafman and Tennenholtz [7,8] in their axiomatization of the maximin criterion. It is also a strong version of Lehmann's axiom of unanimity that only applies to complementary events (see Axiom Q3 in [32]). Projected to the set of events, and in the setting of other postulates, EUN implies the following property of the confidence relation $\succsim_L$:

If $A \succsim_L B$ and $C \succsim_L D$ then $A \cup C \succsim_L B \cup D$, whenever $(A \cup C) \cap (B \cup D) = \emptyset$.

Let us first check that EUN is satisfied by the necessity-based dominance principle:

Proposition 16. *Let $\succsim_X$ be a weak order on X and N be a necessity measure on 2^X. The relation $\succsim$ defined by $f \succsim g \Leftrightarrow N([f \succsim_X g]) \geqslant N([g \succsim_X f])$ satisfies* EUN.

Notice that EUN does not hold if necessity measures are changed into possibility measures in Proposition 16. Axiom EUN is thus useful to tell likely dominance rules based on necessity measures from those based on possibility measures. This axiom is crucial in getting the proper structure for null events.

Proposition 17. *If $\succsim$ satisfies* A1, A5, LM, RM, OI *and* EUN, *then $\exists Z \subseteq S$ such that A is null if and only if $A \subseteq Z$.*

The following result, a variant of which is in [12], then provides an axiomatization of the strict preference between acts in terms of a likely dominance rule for confidence relations represented by families of necessity measures:

Theorem 5. *If $\succsim$ satisfies* A1, A5, LM, RM, OI *and* EUN *then there exists a relation $\succsim_P$ on X and a family $\mathcal{N}$ of necessity measures such that for any $f, g \in X^S$, $f \succ g \Leftrightarrow \forall N \in \mathcal{N}, N([f \succsim_P g]) > N([g \succsim_P f])$.*

The proof only uses weak unanimity (EUN with disjoint sets A and B). It comes down to exploiting the fact that the confidence relation satisfies the negligibility property (Proposition 12), and since it is monotonic and the union of two null events is null (from EUN), the results of Friedman and Halpern [25] (and also [12,14]) show that the confidence relation generates a non-monotonic consequence relation in the style of Kraus Lehmann and Magidor [31]. Such relations can be represented in turn by families of possibility distributions [16].

Note that in [12], a strong form of A1 is used, assuming the completeness of $\succsim$ which can thus be fully represented. In the above theorem, only the strict part of the preference relation can be represented because of a lack of completeness of $\succsim$. Moreover, axioms LM and RM are not explicitly used in that paper. The Savage axiom P3 instead is employed. Moreover the so-called unanimity postulate of Lehmann [32] (which is EUN restricted to $B = \overline{A}$) is used instead of EUN. P3 and unanimity are instrumental in proving the monotonicity of $\succsim$.

5.3. Axiomatization of possibility theory under the likely dominance rule: the anonymity postulate

We are now in a position to provide a direct proof that, under assumptions of Theorem 5, if the confidence relation is transitive on states, then the only possible decision rule is the necessity-based weak dominance rule with respect to a unique (up to a monotonic transformation) possibility distribution. Let us assume the transitivity of the confidence relation, when restricted to singletons of the state space.

Axiom TS (*Transitivity of $\sim_L$ with respect to singletons*). $\forall\{s, s', s''\} \subseteq S$, $[\{s\} \sim_L \{s'\}$ and $\{s'\} \sim_L \{s''\}] \Rightarrow \{s\} \sim_L \{s''\}$.

Lemma 2. *If $\succsim$ is complete and satisfies* OI, A1, A5, LM, RM, EUN *and* TS *then there exists a relation $\succsim_P$ on X and a necessity measure on 2^X such that*:

$$\forall f, g \in X^S \quad \bigl(f \succsim g \quad \Leftrightarrow \quad N\bigl([f \succsim_P g]\bigr) \geqslant N\bigl([g \succsim_P f]\bigr)\bigr).$$

TS is obviously a condition satisfied in Savage's framework. It is one more property lost with the relaxation of P1. From a descriptive point of view, not assuming the transitivity of $\sim$ for acts is desirable since the lack of comparability between acts due to conflicting states is generally not transitive—but this general lack of transitivity is not incompatible with axiom TS. Namely, the likely dominance rule induced by a single necessity measure is not transitive, the corresponding likelihood relation is not transitive, but it is on singletons.

We have seen in the previous example that there may be a conflict between f and g, a conflict between g and h, along with a functional dominance of f on h. This situation can still occur when comparing binary acts, e.g., the acts $f = x\{s_1, s_3\}y$, $g = x\{s_2\}y$, $h = x\{s_3\}y$: there is a conflict between s_2 and s_3 in the comparison of f vs. g and also in the comparison of g vs. h, but f dominates h. So, the entire $\sim_L$ relation is not transitive. Nevertheless, this situation of intransitivity cannot occur when comparing binary acts that represent *disjoint events*. Therefore, there is no reason to dispense with the transitivity of $\sim_L$ on states.

There is also a prescriptive argument in favor of TS: *ceteris paribus*, equally possible states should have an equal influence in the decision. This can be modelled by the following anonymity axiom requiring that the exchange of two equally possible states does not modify the preference:

Axiom ANO. If $s_1 \sim_L s_2$ then: $\forall f, g,\ \ f \succsim g \Leftrightarrow f_{s_1 \leftrightarrow s_2} \succsim g_{s_1 \leftrightarrow s_2}$ where $f_{s_1 \leftrightarrow s_2} = f(s_1)\{s_2\}f(s_2)\{s_1\}f$ and $g_{s_1 \leftrightarrow s_2} = g(s_1)\{s_2\}g(s_2)\{s_1\}g$.

$f_{s_1 \leftrightarrow s_2}$ is the same as act f except that the consequences of s_1 and s_2 are exchanged.

Proposition 18. *If* ($\succsim$ *is complete on constant acts, reflexive, satisfies* OI, P5, EUN, LM *and* RM), *then* (ANO $\Rightarrow$ TS).

Again, one can verify that this new axiom is encompassed by Savage's framework and is compatible with our set of axioms:

Proposition 19. (P1 + P2 + P3 + P4) $\Rightarrow$ ANO.

The merit of ANO as opposed to TS is to be expressed in terms of acts. The necessity-based weak dominance rule fulfills all the above introduced properties: Proposition 16 has shown that EUN is respected and TS is obviously satisfied for a single possibility distribution.

Proposition 20. *Let* $\succsim_X$ *be a weak order on* X *and* N *be a necessity measure on* 2^X. *The relation* $\succsim$ *defined by* $f \succsim g \Leftrightarrow N([f \succsim_X g]) \geqslant N([g \succsim_X f])$ *satisfies* ANO.

Putting together the above results, and especially Lemma 2, we are now in position to wrap up an ordinal possibilistic counterpart of Savage's axiomatics, assuming two non-null states:

Theorem 6. $\succsim$ *is complete and satisfies* (OI, A1, A5, LM, RM, EUN, ANO, 2NN) *if and only if there exists a weak order* $\succsim_P$ *on* X *and a non-trivial necessity measure such that for any* $f, g \in X^S$, $f \succsim g \Leftrightarrow N([f \succsim_P g]) \geqslant N([g \succsim_P f])$.

The possibility distribution laid bare in this representation theorem is qualitative in the sense that it is unique only up to a monotonic transformation. If there is only one non-null state, the transitivity of $\succsim_P$ is no longer ensured. But, since no state of possibility 1 is null when using the previous rule, the only case that escapes Theorem 6 is the deterministic one (a unique s^* exists such that $\pi(s^*) > 0$, where π is the possibility distribution associated to N). Then, the comparison of acts f and g exactly reflects the preferences between the consequences $f(s^*)$ and $g(s^*)$. Conversely, if N is deterministic, 2NN is obviously not satisfied. In this degenerate case, any complete and quasi-transitive comparison of consequences can be used.

Theorem 6 is also an ordinal Savage-like justification of possibility theory. Interestingly, it is not the only available one: [19] also provides a Savagian axiomatization of possibility theory, namely using a finite and totally ordered scale, where both possibility and necessity, as well as utility degrees lie. On the contrary, a unique comparative necessity relation is obtained here as representing the decision maker uncertainty, without assuming the existence of such a common value scale.

5.4. *The possibility-based weak dominance rule*

At this point, one should wonder whether the likely dominance rules based on the dual of a necessity relation, i.e., the possibility-based weak dominance rules, of the form:

$$f \succsim g \quad \Leftrightarrow \quad \Pi\big([f \succsim_X g]\big) \geqslant \Pi\big([g \succsim_X f]\big)$$

are as interesting as those based on necessity relations. First, note that they satisfy the following property:

$$\forall A, B \subseteq S \quad (f \succsim g)_A \text{ and } (f \succsim g)_B \quad \Rightarrow \quad (f \succsim g)_{A \cap B}$$

that looks like EUN (replacing $\cup$ by $\cap$) but that is much more difficult to justify. It is actually sufficient to replace EUN by this condition in the previous theorems to have a full characterization of the Possibility-based dominance rule. Nevertheless, this result is not really interesting, since the decision rule based on a possibility measure turns out to be highly drastic.

The relation on events induced by this rule is defined by $A \succsim_L B \Leftrightarrow N(A \cap \overline{B}) \geqslant N(B \cap \overline{A})$. It can be checked that the negligibility property holds for this relation. To see it note that if A, B, C are disjoint sets, $N(A \cup B) > N(C)$ and $N(A \cup C) > N(B)$, then both $A \cup B$ and $A \cup C$ contain the set $O = \{s, \pi(s) = 1\}$ (a characteristic property for having $N(A \cup B) > 0$ and $N(A \cup C) > 0$). So A contains O as well, and $N(A) > N(B \cup C) = 0$. But as soon as $\pi(s_1) = \pi(s_2) = 1$ for $s_1 \neq s_2$, all states are null, because $N(s) = 0$, $\forall s \in S$. However, not all events are null, but they only contain null states, which sounds strange. If there is only one maximally plausible state, only this state is not null.

It turns out that this decision rule only considers the states that receive a possibility degree of 1, and never takes the less plausible ones into account, even if they are not impossible (i.e., even if some graded plausibility is assumed, some states having intermediary possibility degrees). More precisely:

Proposition 21. *If $\succsim$ is defined from a complete relation $\succsim_X$ on X and a possibility measure Π by $f \succsim g \Leftrightarrow \Pi([f \succsim_X g]) \geqslant \Pi([g \succsim_X f])$ then $\exists s \in S$ such that $\Pi(\{s\}) = 1$ and*:

$$\forall f, g \in X^S, \quad f \succsim g \quad \Leftrightarrow \quad f(s) \succsim_X g(s).$$

Due to completeness, $f \succ g \Leftrightarrow \forall s \in S$ such that $\Pi(\{s\}) = 1,\ f(s) \succ_X g(s)$. This is a unanimity rule on the set $O = \{s, \pi(s) = 1\}$. By Proposition 21, the possibility-based dominance rule is very undecisive in general, since whenever an act f does not strongly dominate an act g for each of the most plausible states, f and g will be indifferent. In other words, the hierarchy of dominant sets laid bare in Corollary 4 reduces to two levels: O and its complement which, being null, never brings any further discrimination between acts.

Contrary to the necessity-based weak dominance rule, that may resort to states of lower plausibility when the act cannot be discriminated on the set of most plausible states, the possibility-based weak dominance rule makes it useless to introduce shades of plausibility in the representation of the decision maker knowledge. This decision rule coincides with the fusion rule in Maynard-Reid and Lehmann [36] when all states are equally plausible, but these authors do break ties using lower level coalitions of states (sources in their terminology). From the above discussion it is also patent that the possibility-based weak dominance rule induced by the possibility distribution π is the same as the (trivial) necessity-based weak dominance rule with respect to the possibility distribution π_* such that $\pi_*(s) = \pi(s)$ if $\pi(s) = 1$, and $\pi_*(s) = 0$, otherwise, that is with one decisive set $O = \{s, \pi(s) = 1\}$ and a null set $Z = S \setminus O$.

6. Conclusion

Regarding the question "Is it possible to make decisions on the basis of a rational and pure qualitative model that does not presuppose any commensurability assumption?", the results presented in this paper are at the same time positive and negative:

- On the one hand, our axiomatic study shows that yes, if ordinal invariance is accepted as a norm for qualitative models, then the likely dominance rules (and only them) are obtained and the postulate of quasi-transitivity identifies possibilistic structures as the only uncertainty structures compatible with such rules. The presence of a top oligarchy makes this approach close to those that decide on the basis of the most plausible states only [5,7,8,45,46].
  More generally, our system of postulates defines a underlying uncertainty theory which is the one precisely at work in non-monotonic logic. Our work can be understood as providing a Savage-like foundation of the KLM preferential entailment [12].
- On the other hand, the identification of hierarchies of oligarchies show that, no, such a model is not decisive enough, unless one is in a situation of quasi-certainty: as soon as for two equally plausible states the consequences of two acts yield conflicting preference profiles, the two acts are incomparable—so, the larger the oligarchies, the less decisive the rule. In other terms, acts cannot be completely ranked using a weak order, unless one is in a situation of quasi-certainty where the states of the world can be linearly ranked in terms of plausibility. In the latter case, basing almost all choices between acts on their consequence on the unique most plausible state may sound very adventurous.

Therefore, if pure qualitative decision making cannot be efficient except in very particular situations, is it possible to weaken our initial framework in order to get more useful decision rules? A first idea would be to partially relax the quasi-transitivity of $\succsim$ on X^S. In practical applications, most of the acts in X^S are purely imaginary and perhaps not even feasible. Hence, one can be tempted to require quasi-transitivity only on feasible acts. However, using Example 3, we could show that this does not provide new interesting decision rules as long as there exists a feasible subset of acts $X'^{S'}$ where $X' \subseteq X$

contains at least three distinct elements and $S' \subseteq S$ contains at least 3 non-null states. We could also consider a relaxation of quasi-transitivity into *acyclicity*, i.e., $\forall f_0, \ldots, f_k \in X^S$ $((\forall i = 1, \ldots, k,\ f_{i-1} \succ f_i) \Rightarrow \text{not}(f_k \succ f_0))$. Nevertheless, borrowing results obtained in Social Choice Theory [43], it can easily be shown that qualitative decision models compatible with acyclicity also have very special features. For example, they necessarily provide some state with an absolute veto, thus limiting the role of other states.

The last option is to relax axiom OI, which turns out to be a very demanding condition. This axiom includes a "qualitative" requirement ($f \succsim g$ only depends on the preference order for each state, and not on the values of the consequences i.e., preference is ordinal) *but also* an "independence-of-irrelevant-alternatives idea" ($f \succsim g$ only depends on consequences of type $f(s), g(s)$, $s \in S$, i.e., $\succsim$ is context-independent). The most natural weakening of OI is to cancel the "independence" side of the condition. For instance, the decision rules proposed by [8] cancel this condition without making any commensurablity hypothesis, since they compare acts on the basis of the consequences provided by the most plausible states only—using for instance a maximin criterion. Further research along this line may consider the use of less plausible states when the most plausible ones are indifferent.

Cancelling of the independence side of OI has also been proposed in the framework of multicriteria decision making in order to overcome difficulties due to Arrow-like theorems (see, e.g., [6]). It should be useful to follow the same path. As a first set of examples, one could consider the following relations:

$$f \Delta_h^+ g \quad \Leftrightarrow \quad [f \succsim h] \succsim_L [g \succsim h],$$
$$f \Delta_h^- g \quad \Leftrightarrow \quad [h \succsim g] \succsim_L [h \succsim f],$$

where $h \in X$ is a reference point within X^S used to specify the aspiration levels of the DM, for each state of the nature. The idea underlying these rules is to compare acts by evaluating their relative capabilities in satisfying or missing the DM aspirations. Another interesting set of examples is given by:

$$f \Delta^+ g \quad \Leftrightarrow \quad \forall x \in X,\ [f \succsim x] \succsim_L [g \succsim x],$$
$$f \Delta^- g \quad \Leftrightarrow \quad \forall x \in X,\ [x \succsim g] \succsim_L [x \succsim f].$$

Such rules can be seen as qualitative counterparts of the so-called *stochastic dominance* relation (see, e.g., [34]) used in decision under probabilistic uncertainty. A more systematic investigation of such decision rules, which determine the relative merits of acts by comparing them to prescribed reference profiles of consequences, seems to be of major interest.

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Appendix A

Proof of Proposition 1. Suppose A, B such as $A \subseteq B$. From the reflexivity of $\succsim_\Lambda$, we get $A \succsim_\Lambda A$. So, by monotonicity: $(\overline{A} \cap B) \cup A \succsim_\Lambda A$, i.e., since $A \subseteq B$, $B \succsim_\Lambda A$.

$A \succ_\Lambda B$ implies $A \cup C \succsim_\Lambda B$ by monotonicity. Since $A \cup C \sim_\Lambda B$ implies $B \succsim_\Lambda A$ by monotonicity, and thus contradicts $A \succ_\Lambda B$, we get $A \cup C \succ_\Lambda B$.

Similarly, $A \succ_\Lambda B \cup C$ implies $A \succsim_\Lambda B$ by monotonicity. Since $A \sim_\Lambda B$ implies $B \cup C \succsim_\Lambda A$ by monotonicity, and thus contradicts $A \succ_\Lambda B \cup C$, we get $A \succ_\Lambda B$. □

Proof of Proposition 2. *The transitivity of* $\succ$ (*respectively* $\sim$) is an obvious consequence of the transitivity of $\succsim$ and the definition of $\succ$ (respectively $\sim$).

Weak Unanimity: $(f \succsim g)_A$ and $(f \succsim g)_B$ means that, for any h: $fAh \succsim gAh$ and $fBh \succsim gBh$. Since it is assumed that $A \cap B = \emptyset$, P2 allows us to write $fAfBh \succsim gAfBh$ and $gAfBh \succsim gAgBh$. By transitivity, this implies: $fAfBh \succsim gAgBh$ for any h, i.e., $(f \succsim g)_{A\cup B}$.

Left and Right Monotonicity: P1 ensures that $\succsim_P$ is transitive, so monotonicity can be soundly defined. Consider $A \subseteq S$, $f, f' \in X^S$ such that $\forall s \in A f'(s) \succ_P f(s)$. By P3, we get $\forall s \in A, \forall h, f'\{s\}h \succsim f\{s\}h$. Since weak unanimity holds, it holds that $f'Ah \succsim fAh, \forall h$. So, by P2, $f'Af \succsim f$. When $f \succsim g$, we get $f'Af \succsim g$ by transitivity. Similarly, consider $g, g' \in X^S$ such that $\forall s \in A, (g \succ g')_{\{s\}}$. Since weak unanimity holds, we get $gAh \succsim g'Ah, \forall h$. So, by P2, $g \succsim g'Ag$. When $f \succsim g$, we get $f \succsim g \succsim g'$ by transitivity. □

Proof of Proposition 3. P2 writes: $\forall A \subseteq S, \forall f, g, h, h' \in X^S, (fAh \succsim gAh \Leftrightarrow fAh' \succsim gAh')$. Consider arbitrary $A \subseteq S$, $f, g, h, h' \in X^S$. Let us first prove that $(fAh, gAh) \equiv (fAh', gAh')$. Indeed:

- For any $s \in A$: $fAh(s) = fAh'(s)(= f(s))$ and $gAh = gAh'(= g(s))$. Hence, for any $s \in A$: $fAh(s) \succsim_P gAh(s) \Leftrightarrow fAh'(s) \succsim_P gAh'(s)$.
- For any $s \in \overline{A}$: $fAh(s) = gAh(s)(= h(s))$ and $fAh'(s) = gAh'(= h'(s))$. Since $\succsim$ is reflexive: both $fAh(s) \succsim_P gAh(s)$ and $fAh'(s) \succsim_P gAh'(s)$ hold. Hence, for any $s \in \overline{A}$, it holds that: $fAh(s) \succsim_P gAh(s) \Leftrightarrow fAh'(s) \succsim_P gAh'(s)$.

We get symmetrically $\forall s \in S, gAh(s) \succsim_P fAh(s) \Leftrightarrow gAh'(s) \succsim_P fAh'(s)$. Hence, $(fAh, gAh) \equiv (fAh', gAh')$ and OI can be applied—it implies that $(fAh \succsim gAh \Leftrightarrow fAh' \succsim gAh')$: we recognize the expression of P2. □

Proof of Proposition 4. P4 writes: $\forall A, B \in S, \forall x, y, x', y' \in X$ such that $x \succ_P y$ and $x' \succ_P y'$, $xAy \succsim xBy \Leftrightarrow x'Ay' \succsim x'By'$. Let us choose arbitrary $A, B \in S$ and $x, y, x', y' \in X$ such that $x \succ_P y$ and $x' \succ_P y'$. It presupposes that P5 holds, otherwise Proposition 4 is trivial.

First, notice that, $(xAy, xBy) \equiv (x'Ay', x'By')$. Indeed:

- For any $s \in A \cap B$: $xAy(s) = xBy(s) = x$ and $x'Ay'(s) = x'By'(s) = x'$. Since $\succsim$ is reflexive, so is also $\succsim_P$: $x \succsim_P x$ and $x' \succsim_P x'$. That is to say $xAy(s) \succsim_P xBy(s)$

and $x'Ay'(s) \succsim_P x'By'(s)$. For any $s \in \overline{A} \cap \overline{B}$: $xAy(s) = xBy(s) = y$ and $x'Ay'(s) = x'By'(s) = y'$. The same condition results.

- For any $s \in A \cap \overline{B}$: $xAy(s) = x$, $xBy(s) = y$, $x'Ay'(s) = x'$, $x'By'(s) = y'$. Since $x \succ_P y$ and $x' \succ_P y'$, we get $xAy(s) \succ_P xBy(s)$ and $x'Ay'(s) \succ_P x'By'(s)$.
- For any $s \in \overline{A} \cap B$: $xAy(s) = y$, $xBy(s) = x$, $x'Ay'(s) = y'$, $x'By'(s) = x'$. Since $x \succ_P y$ and $x' \succ_P y'$, we get $xBy(s) \succ_P xAy(s)$ and $x'By'(s) \succ_P x'Ay'(s)$.

So, whatever s, $[xAy(s) \succsim_P xBy(s) \Leftrightarrow x'Ay'(s) \succsim_P x'By'(s))$ and $(xBy(s) \succsim_P xAy(s) \Leftrightarrow x'By'(s) \succsim_P x'Ay'(s))]$ holds. So, $(xAy, xBy) \equiv (x'Ay', x'By')$. We can apply OI and get P4: $(xAy \succsim xBy \Leftrightarrow x'Ay' \succsim x'By')$. □

Proof of Proposition 5. Since $\succsim_\Lambda$ is a confidence relation, it is reflexive and it holds that $S \succ_\Lambda \emptyset$ and that $\forall A \subseteq S$, $S \succsim_\Lambda A$ (Definition 1 and Proposition 1).

Proof of the reflexivity of $\succsim$: the likely dominance rule gives $(f \succsim f \Leftrightarrow [f \succsim_X f] \succsim_\Lambda [f \succsim_X f]$. Hence, the reflexivity of $\succsim_\Lambda$ implies the reflexivity of $\succsim$.

Proof of the equivalence of $\succsim_P$ *and* $\succsim_X$: let f_a, f_b be two constant acts, corresponding to consequences a and b respectively. The likely dominance rule gives $f_a \succsim f_b \Leftrightarrow [a \succsim_X b] \succsim_\Lambda [b \succsim_X a]$. Suppose first that $a \succsim_X b$: it follows that $[a \succsim_X b] = S$. So for the confidence relation $\succsim_\Lambda$, it holds that $[a \succsim_X b] \succsim_\Lambda [b \succsim_X a]$. Hence we get: $f_a \succsim f_b$, i.e., $a \succsim_P b$. Now, suppose that $a \succsim_X b$ does not hold. Since $\succsim_X$ is assumed to be complete, this means that $b \succ_X a$, i.e., $[b \succsim_X a] = S$ and $[a \succsim_X b] = \emptyset$. Since, by definition, $S \succ_\Lambda \emptyset$, it follows that $f_b \succ f_a$, i.e., $b \succ_P a$

The *completeness of* $\succsim$ for constant acts directly derives from (i) the equivalence of $\succsim_P$ and $\succsim_X$, and (ii) of the completeness of $\succsim_X$.

Proof of OI: consider four acts $f, g, f', g' \in X^S$ such that $(f, g) \equiv (f', g')$. It holds that $[f \succsim_X g] = [f' \succsim_X g']$ and $[g \succsim_X f] = [g' \succsim_X f']$. Then, using the likely dominance rule, $(f \succsim g \Leftrightarrow [f \succsim_X g] \succsim_\Lambda [g \succsim_X f])$ and $(f' \succsim g' \Leftrightarrow [f \succsim_X g] \succsim_\Lambda [g \succsim_X f])$. Thus $(f \succsim g \Leftrightarrow f' \succsim g')$ which proves that OI is satisfied by the likely dominance rule .

Proof of LM: consider A, f, f', g such that $\forall s \in A$, $f'(s) \succ_P f(s)$ and $f \succsim g$. Since $\succsim_P$ and $\succsim_X$ coincide, we get $\forall s \in A$, $f'(s) \succ_X f(s)$

(i) For any $s \in \overline{A}$, $f'Af(s) = f(s)$. So, $[f'Af \succsim_X g] \cap \overline{A} = [f \succsim_X g] \cap \overline{A}$.

(ii) For any $s \in A$, $f'(s) \succ_X f(s)$, so, assuming the quasi-transitivity of $\succsim_X$, for any $s \in A$ such that $f(s) \succsim_X g(s)$, $f'(s) \succsim_X g(s)$ holds. Hence, $[f \succsim_X g] \cap A \subseteq [f'Af \succsim_X g] \cap A$.

From (ii) and (i), we get: $[f \succsim_X g] \subseteq [f'Af \succsim_X g]$ which also reads $\overline{[f'Af \succsim_X g]} \subseteq \overline{[f \succsim_X g]}$. Since $\succsim_X$ is complete, the latter inclusion writes $[g \succ_X f'Af] \subseteq [g \succ_X f]$, which implies that $[g \succsim_X f'Af] \subseteq [g \succsim_X f]$. On the other hand, using the likely dominance rule, $f \succsim g \Leftrightarrow [f \succsim_X g] \succsim_\Lambda [g \succsim_X f]$. By monotonicity of $\succsim_\Lambda$, we get first $[f \succsim_X g] \succsim_\Lambda [g \succsim_X f'Af]$ (since $[g \succsim_X f'Af] \subseteq [g \succsim_X f]$), and by monotonicity again, we get $[f'Af \succsim_X g] \succsim_\Lambda [g \succsim_X f'Af]$ (since $[f \succsim_X g] \subseteq [f'Af \succsim_X g]$). Since the likely dominance rule is used, this means that $f'Af \succsim g$, which proves LM.

The *proof of* RM is very similar: one shows that $[f \succsim_X g'Ag] \cap \overline{A} = [f \succsim_X g] \cap \overline{A}$ and, by quasi-transitivity of $\succsim_X$, $[f \succsim_X g] \cap A \subseteq [f \succsim_X g'Ag] \cap A$. This implies $[f \succsim_X g] \subseteq [f \succsim_X g'Ag]$ and by completeness of $\succsim_X$ $[g'Ag \succsim_X f] \subseteq [g \succsim_X f]$. So, by monotonicity of $\succsim_\Lambda$, $[f \succsim_X g] \succsim_\Lambda [g \succsim_X f]$ implies $[f \succsim_X g'Ag] \succsim_\Lambda [g'Ag \succsim_X f]$, which by the likely dominance principle, means that $f \succsim g'Ag$.

Proof of P2 *and* P4: the likely dominance rule builds a reflexive preference that satisfies OI. So, by Propositions 3 and 4 P2 and P4 hold.

Proof of P3: consider two consequences a and b, and a non-null subset A of S. Assume that $a \succsim_P b$, i.e., by coincidence of $\succsim_X$ and $\succsim_P$, $a \succsim_X b$. The likely dominance rule implies that, whatever h, $aAh \succsim bAh \Leftrightarrow S \succsim_\Lambda B$ for some $B \supseteq \overline{A}$. $S \succsim_\Lambda B$ obviously holds ($\succsim_\Lambda$ is a confidence relation). So, $aAh \succsim bAh$ whatever h, i.e., $(a \succsim b)_A$. Thus, $a \succsim_P b \Rightarrow (a \succsim b)_A$.

Conversely, suppose that $aAh \succsim bAh$ for any h and that $b \succ_P a$, i.e., by coincidence, $b \succ_X a$. Using the likely dominance rule, $aAh \succsim bAh$ and $b \succ_X a$ imply that $\overline{A} \succsim_\Lambda S$. Consider now two acts f and g and let us apply as follows the monotonicity of $\succsim_\Lambda$: $\overline{A} \succsim_\Lambda S$ implies that $(\overline{A} \cup (A \cap [f \succsim_P g])) \succsim_\Lambda S$, and thus that $(\overline{A} \cup (A \cap [f \succsim_P g])) \succsim_\Lambda (\overline{A} \cup (A \cap [g \succsim_P f]))$. Hence, using the likely dominance rule ($fAh \succsim gAh$) whatever h, i.e., $(f \succsim g)_A$. Since we did not make any restriction on f and g, this means that A is null, which contradicts our hypothesis. Hence, when A is not null, $(a \succsim b)_A$ implies not$(b \succ_P a)$. Since $\succsim_P$ is complete, this implies $a \succsim_P b$.

Proof of P5: P5 is a direct consequence of the non-triviality of $\succsim_X$ ($\exists x, y \in X, x \succ_X y$ and of the equivalence of $\succsim_X$ and $\succsim_P$: we deduce $x \succ_P y$, which implies P5. $\square$

Proof of Theorem 1. *Main Result:* First, notice that the definition of $\succsim_P$ from $\succsim$ (Eq. (5)) is self-sufficient. $\succsim_L$ can also be soundly defined from $\succsim$ using Eq. (6) since P4 holds due to Proposition 4.

Suppose first that P5 holds and consider two consequences x and y such that $x \succ_P y$. To get the main result, it is sufficient to show that, whatever s, $f(s) \succsim_P g(s) \Leftrightarrow (x[f \succsim_P g]y)(s) \succsim_P (x[g \succsim_P f]y)(s)$. Indeed, if it is so, $(f, g) \equiv (x[f \succsim_P g]y), x[g \succsim_P f]y$ (recall that $\succsim_P$ is complete) and OI enables the deduction of $f \succsim g \Leftrightarrow x[f \succsim_P g]y \succsim x[g \succsim_P f]y$. By definition of $\succsim_L$, $[f \succsim_P g] \succsim_L [g \succsim_P f] \Leftrightarrow x[f \succsim_P g]y \succsim x[g \succsim_P f]y$. Hence, we would get $f \succsim g \Leftrightarrow [f \succsim_P g] \succsim_L [g \succsim_P f]$.

Let us show that $\forall s$, $f(s) \succsim_P g(s) \Leftrightarrow (x[f \succsim_P g]y)(s) \succsim_P (x[g \succsim_P f]y)(s)$.

- Any s such that $f(s) \sim_P g(s)$ belongs to $[f \succsim_P g]$ and to $[g \succsim_P f]$. Hence $x[f \succsim_P g]y(s) = x[g \succsim_P f]y(s) = x$. So, by reflexivity of $\succsim$ and thus of $\succsim_P$ we have: $x[f \succsim_P g]y(s) \sim_P x[g \succsim_P f]y(s)$.
- Any s such that $f(s) \succ_P g(s)$ belongs to $[f \succsim_P g]$ and does not belong to $[g \succsim_P f]$. Hence $x[f \succsim_P g]y(s) = x$ and $x[g \succsim_P f]y(s) = y$. Since $x \succ_P y$, $x[f \succsim_P g]y(s) \succ_P x[g \succsim_P f]y(s)$.
- Similarly, for s such that $g(s) \succ_P f(s)$, we get $x[g \succsim_P f]y(s) \succ_P x[f \succsim_P g]y(s)$.

Since $\succsim_P$ is assumed to be complete, these three items prove that $f(s) \succsim_P g(s) \Leftrightarrow x[f \succsim_P g]y(s) \succsim_P x[g \succsim_P f]y(s)$. So, we can derive $f \succsim g \Leftrightarrow [f \succsim_P g] \succsim_L [g \succsim_P f]$.

Finally notice that, when P5 does not hold, but $\succsim_P$ is complete, it holds that: $\forall f, g \in X^S, \forall s \in S$, $f(s) \sim_P g(s)$. Thus $\forall f, g, [f \succsim_P g] = [g \succsim_P f] = S$. Thus $\forall f, g, [f \succsim_P g] \succsim_L [g \succsim_P f]$ (reflexivity of $\succsim$ and thus of $\succsim_L$). Moreover, by OI and reflexivity of

$\succsim$ we get $f \succsim g, \forall f, g$. From "$\forall f, g, [f \succsim_P g] \succsim_L [g \succsim_P f]$ and $\forall f, g, f \succsim g$", we can derive "$\forall f, g, [f \succsim_P g] \succsim_L [g \succsim_P f] \Leftrightarrow f \succsim g$".

Reflexivity of $\succsim_L$: $\succsim$ is reflexive implies that, $\forall A, xAy \succsim xAy$, i.e., that $A \succsim_L A$.

Non-triviality of $\succsim_L$: by P5, $x \succ_P y$. So, $x \succ y$, i.e., $xSy \succ x\emptyset y$: we get $S \succ \emptyset$.

Preadditivity of $\succsim_L$. Consider 3 events A, B, C such that $A \cap (B \cup C) = \emptyset$. $B \succsim_L C \Leftrightarrow xBy \succsim xCy$. Since $A \cap (B \cup C) = \emptyset$ and P2 holds, we have $xBy \succsim xCy \Leftrightarrow xBxAy \succsim xCxAy$, which means $B \succsim_L C \Leftrightarrow A \cup B \succsim_L A \cup C$. □

Proof of Corollary 1. Theorem 1 proves the if part. The proof of OI in Proposition 5 does not use any hypothesis on $\succsim_A$ and $\succsim_X$, so, it proves the only if part. □

Proof of Proposition 6. Under P5, $\exists x, y \in X$ such as $x \succ_P y$. So, $A \succsim_L B \Leftrightarrow xAy \succsim_L xBy$. Moreover, OI and $\succsim$ complete on constant acts imply P2 (Proposition 3) and $f \succsim y \Leftrightarrow [f \succsim_P g] \succsim_L [g \succsim_P f]$ (Theorem 1).

Monotonicity of $\succsim_L$. Suppose that $A \succsim_L B$, i.e., $xAy \succsim xBy$. By LM, we get $xAx(\overline{A} \cap C)y \succsim xBy$, i.e., $A \cup C \succsim_L B$. Suppose that $A \succsim_L B \cup C$, i.e., $xAy \succsim x(B \cup C)y$. By RM, we get $xAy \succsim xBy$, i.e., $A \succsim_L B$.

Null events. A is null $\Rightarrow \emptyset \succsim_L A$ holds in any case. Indeed, suppose that A is null. By definition,$\forall f, g \in X^S, (f \succsim g)_A$, i.e., $fAh \succsim gAh$ whatever f, g, h; a particular case is: $yAy \succsim xAy$, i.e., $\emptyset \succsim_L A$. Conversely, suppose that $\emptyset \succsim_L A$. By monotonicity of $\succsim_L$, we get $\emptyset \succsim_L A \cap B$, whatever B, and by monotonicity again, $A \cap C \succsim_L A \cap B$, whatever B, C. By additivity of $\succsim_L$, this implies $(A \cap C) \cup \overline{A} \succsim_L (A \cap B) \cup \overline{A}$. Assume that $C \cup B = A$, and $C \cap B = \emptyset$. Then, since $\succsim_P$ is complete, letting $C \cup \overline{A} = [fAh \succsim_P gAh]$ and $B \cup \overline{A} = [gAh \succsim_P fAh]$, we get by the likely dominance rule $fAh \succsim gAh$. Since we did not make any restriction on f, g, h, this implies that A is null.

Proof of P3. Suppose that $(a \succsim b)_A$ and that not$(a \succsim b)$. Since $\succsim$ is complete on constant acts, this implies that $b \succ a$. By definition, $(a \succsim b)_A$ means that, whatever h, $aAh \succsim bAh$. With $h = a$, we get $a \succsim bAa$. By P4, it entails $\emptyset \succsim_L A$. Thus, A is null (see the last proof).

We have shown that $(a \succsim b)_A$ and not$(a \succsim b)$ implies A null, i.e., that $(a \succsim b)_A$ and A not null implies $a \succsim b$

Conversely, let us show that $a \succsim b$ implies $(a \succsim b)_A$. Two cases are to be considered.

First case: $a \succ b$. Since it holds that, for any A, $A \succsim_L \emptyset$, we get by P4: $aAb \succsim b$ and by P2: $aAh \succsim bAh$ for any h, i.e., $(a \succsim b)_A$.

Second case: $a \sim b$. So, whatever A and h, $[aAh \succsim_P bAh] = [bAh \succsim_P aAh] = S$. By reflexivity of $\succsim_L$ and the likely dominance rule, we get $aAh \succsim bAh$ whatever h, i.e., $(a \succsim b)_A$. □

Proof of Theorem 2.

($\Rightarrow$) We know by Theorem 1 that the relation $\succsim_P$ and $\succsim_L$ defined by Eqs. (5) and (6) are such that: $f \succsim y \Leftrightarrow [f \succsim_P g] \succsim_L [g \succsim_P f]$. Moreover, we know by Proposition 6 that $\succsim_L$ is an additive and monotonic confidence relation.

($\Leftarrow$) We know by Proposition 5 that the preference on X^S defined by a likely dominance rule from a monotonic confidence relation and a complete preference on X is reflexive, satisfies OI, LM, RM. We also know that $\succsim_X$ and $\succsim_P$ coincide, so the restriction to $\succsim$ to constant acts is complete. □

Proof of Propositions 7, 9 and 8. By P5, $\exists\, x, y \in X$: $x \succ_P y$. So, by Eq. (6), $A \succsim_L B \Leftrightarrow (xAy) \succsim (xBy)$. Since a likely dominance rule is used, $(xAy) \succsim (xBy)$ iff $[xAy \succsim_X xBy] \succsim_\Lambda [xBy \succsim_X xAy]$. Finally, since we know that $\succsim_X$ and $\succsim_P$ coincide (Proposition 5), $A \succsim_L B \Leftrightarrow [xAy \succsim_P xBy] \succsim_\Lambda [xBy \succsim_P xAy] \Leftrightarrow A \cup \overline{B} \succsim_\Lambda B \cup \overline{A}$ (indeed, $[xAy \succsim_P xBy] = A \cup (\overline{A} \cap \overline{B})$ and $[xBy \succsim_P xAy] = B \cup (\overline{A} \cap \overline{B})$).

If $\succsim_\Lambda$ is preadditive, $A \cup \overline{B} \succsim_\Lambda B \cup \overline{A} \Leftrightarrow A \succsim_\Lambda B$ (deleting $\overline{A} \cap \overline{B}$ on both sides of the inequality), so $A \succsim_L B \Leftrightarrow A \succsim_\Lambda B$. Conversely, if $\succsim_\Lambda = \succsim_L$, the preadditivity of $\succsim_\Lambda$ is an obvious consequence of the preadditivity of $\succsim_L$ (that itself holds since any likely dominance rule based on a reflexive $\succsim_\Lambda$ builds a reflexive $\succsim$ that also satisfies OI (proof of Proposition 5), and thus satisfies P2 (Proposition 3), that directly implies the preadditivity of $\succsim_L$ (see the proof of Proposition 6)).

If we now assume that $A \cup B = S$, i.e., $\overline{A} \cap \overline{B} = \emptyset$, $A \cup \overline{B} = A$ and $B \cup \overline{A} = B$. So, $A \succsim_L B \Leftrightarrow A \cup \overline{B} \succsim_\Lambda B \cup \overline{A}$ is the same as $A \succsim_L B \Leftrightarrow A \succsim_\Lambda B$. □

Proof of Proposition 10 . From $A5$, we know that $\exists x, y, z$ such that $x \succ_P y \succ_P z$. By Theorem 2, we know that the preference can be represented by a likely dominance rule So, $f \succsim g \Leftrightarrow [f \succsim_P g] \succsim_L [g \succsim_P f]$. Thanks to Proposition 6, $\succsim_L$ is preadditive and monotonic. In particular, the property of preadditivity implies that: $f \succsim g \Leftrightarrow [f \succ_P g] \succsim_L [g \succ_P f]$.

Suppose that $\succsim_P$ is not transitive, i.e., that there exist x_1, y_1, z_1 such as $x_1 \succ_P y_1$, $x_1 \sim_P z_1$ and $y_1 \sim_P z_1$. Let A and B be two non-null events. By Proposition 6, this implies that $A \succ_L \emptyset$ and $B \succ_L \emptyset$. Let f, g, h be the acts defined by: $f = x(A \cap B)x_1(A \cap \overline{B})z_1(\overline{A} \cap B)z$, $g = y(A \cap B)y_1(A \cap \overline{B})x_1(\overline{A} \cap B)z$, $h = z(A \cap B)z_1(A \cap \overline{B})y_1(\overline{A} \cap B)z$. Since $[f \succ_P g] = A$, $[g \succ_P f] = \emptyset$ and $A \succ_L \emptyset$, we get $f \succ g$. Similarly, since $[g \succ_P h] = B$, $[h \succ_P g] = \emptyset$ and $B \succ_L \emptyset$, we get $g \succ h$. The quasi-transitivity of $\succsim$ implies $f \succ h$: since $[f \succ_P h] = A \cap B$ and $[h \succ_P f] = \emptyset$ this implies $A \cap B \succ_P \emptyset$, i.e., $A \cap B$ not null. □

Proof of Corollary 2. Property 10 implies that $O = \bigcap\{B, B \text{ not null}\}$ is not null. Hence, $O \succ_L \emptyset$ (Proposition 6): O cannot be the empty set since $\emptyset \sim_L \emptyset$ by reflexivity.

If A is not null, then O is a subset of A by construction. Conversely, since $O \succ_L \emptyset$ and $O \subseteq A$, the monotonicity of $\succsim_L$ allows the derivation of $A \succ_L \emptyset$, i.e., by Proposition 6: A not null.

By monotonicity of $\succsim_L$, $O \subseteq A$ implies $A \succsim_L O$. Suppose that $A \succ_L O$: we get $A \cap \overline{O} \succ_L \emptyset$ by preadditivity of $\succsim_L$. So, $A \cap \overline{O}$ is not null: this yields a contradiction since any non-null event must contain O. □

Proof of Proposition 11. If there are two non-null states, then, there is no set $O \subseteq S$ such that A not null $\Leftrightarrow O \subseteq A$. Hence due to Corollary 2 the set of axioms A1, OI, LM, RM, A5, 2NN imply that $\sim_P$ is transitive. Since A1 also requires the completeness and the quasi-transitivity of $\succsim_P$, A1, OI, LM, RM, A5, 2NN implies that $\succsim_P$ is a weak order. □

Proof of Proposition 12. A5 requires that X contains at least three elements x, y, z such that $x \succ y$, $x \succ z$, $y \succ z$. So, by Definition 6, $A \succsim_L B \Leftrightarrow (xAy) \succsim (xBy)$.

Let $A, B, C \subseteq S$ be 3 events such that $A \cap B = A \cap C = B \cap C = \emptyset$, $A \cup C \succ_L B$, and $A \cup B \succ_L C$ and consider the six following acts: $ab = x(A \cup B)y$, $ac = x(A \cup C)y$,

$bc = x(B \cup C)y$, $a = xAy$, $b = xBy$, $c = xCy$. It holds that $A \cup C \succ_L B \Leftrightarrow ac \succ b$ and $A \cup B \succ_L C \Leftrightarrow ab \succ c$. Since A, B and C are disjoint, one can build the acts: $f = xAyBzCy$, $g = yAzBxCy$, $h = zAxByCy$. It is easy to check that $[(f, g) \equiv (ab, c)]$, $[(g, h) \equiv (ac, b)]$. Hence by OI we get $f \succ g$ and $g \succ h$. By transitivity of $\succ$, this implies that $f \succ h$. Since $[(f, h) \equiv (a, bc)]$, OI implies that $a \succ bc$, i.e., $xAy \succ x(B \cup C)y$. Thus, $A \succ_L B \cup C$. □

Proof of Lemma 1. Suppose that O is decisive in S' and for $x \succ_P y$ build the acts $xOy(S' \setminus O)h$ and $x(S' \setminus O)yOh$. Since O is decisive in S', we get $xOy(S' \setminus O)h \succ x(S' \setminus O)yOh, \forall h$. So, it holds that $xOy(S' \setminus O)y \succ x(S' \setminus O)yOy$, i.e., $O \succ_L S' \setminus O$.

Conversely, suppose that $O \succ_L S' \setminus O$ and consider f, g such as $\forall s \in O$, $f(s) \succsim_P g(s)$. Consider the acts $fS'h$ and $gS'h$. It holds that $O \subseteq [fS'h \succ_P gS'h]$ and $[gS'h \succ_P fS'h] \subseteq S' \setminus O$. By monotonicity of $\succsim_L$, $O \succ_L S' \setminus O$ gives $[fS'h \succ_P gS'h] \succ_L [gS'h \succ_P fS'h]$. By additivity of $\succ_L$, this is equivalent to $[fS'h \succsim_P gS'h] \succ_L [gS'h \succsim_P fS'h]$, i.e., $fS'h \succ gS'h$. Since we did not make any assumption on h, this means $(f \succ g)_{S'}$. □

Proof of Theorem 3.

Unicity of a predominant event. Assume $x \succ_P y$. Suppose S' contains at least two distinct predominant events O_1 and O_2. On the one hand we get $xO_1y(S' \setminus O_1)h \succ yO_1x(S' \setminus O_1)h, \forall h$ since O_1 is decisive in S'. On the other hand, since O_2 is predominant in S' and distinct from O_1, there exists at least one state $s \in O_2 \setminus O_1$ that is a vetoer. Thus, there is at least one state $s \in S' \setminus O_1$ that is a vetoer. This fact contradicts $xO_1y(S'\ O_1)h \succ yO_1x(S'\ O_1)h, \forall h$.

Existence of a predominant event. Recall that $\succsim_L$ is a monotonic negligibility relation. Let us first remark that if $A \subseteq S'$ is decisive in S' and $B \subseteq S'$ is decisive in S', then $A \cap B$ is non-empty and decisive in S'. Indeed, thanks to Lemma 1 the condition implies that $A \succ_L \overline{A} \cap S'$ and $B \succ_L \overline{B} \cap S'$. Write it as follows $(A \cap B) \cup (A \cap \overline{B}) \succ_L (\overline{A} \cap \overline{B} \cap S') \cup (\overline{A} \cap B)$ and $(A \cap B) \cup (\overline{A} \cap B) \succ_L (\overline{A} \cap \overline{B} \cap S') \cup (A \cap \overline{B})$. By monotonicity of $\succsim_L$, we get $(A \cap B) \cup (A \cap \overline{B}) \succ_L (\overline{A} \cap \overline{B} \cap S') \cup (\overline{A} \cap B)$ and $(A \cap B) \cup (\overline{A} \cap B) \cup (\overline{A} \cap \overline{B} \cap S') \succ_L (A \cap \overline{B})$. By negligibility, we get: $(A \cap B) \succ_L (\overline{A} \cap \overline{B} \cap S') \cup (\overline{A} \cap B) \cup (A \cap \overline{B})$, i.e., $(A \cap B) \succ_L S' \setminus (A \cap B)$. So, $(A \cap B)$ is decisive in S' due to Lemma 1. It cannot be empty: otherwise we would get $\emptyset \succsim_L S'$, which contradict the assumption that S' is not null (indeed, from Proposition 6 we know that $\emptyset \succsim_L S' \Leftrightarrow S'$ is null).

Since S' is not null, $S' \succ_L \emptyset$ (applying Proposition 6). From Lemma 1 we thus know that there exist at least one decisive event in S': S' itself. So, $O = \bigcap_{A, A \succ_L S' \setminus A}$, the intersection of all the sets decisive in S' is a non-empty set decisive in S'. Let us show that any state $s \in O$ is a vetoer. Consider f, g such that $f(s) \succ_P g(s)$. s belongs to $[fS'h \succ_P gS'h]$. Suppose that $gS'h \succ fS'h$, which is equivalent to $[gS'h \succ_P fS'h] \succ_L [fS'h \succ_P gS'h]$. Since $s \in [fS'h \succ_P gS'h]$ and $[gS'h \succ_P fS'h] \subseteq S' \setminus \{s\}$, the monotonicity of $\succsim_L$ implies $S' \setminus \{s\} \succ_L \{s\}$. So, s cannot belong to $O = \bigcap_{A, A \succ_L S' \setminus A}$. This yields a contradiction: s must be a vetoer.

$O = \bigcap_{A, A \succ_L S' \setminus A}$ is thus a predominant set in S'. We already know that it is unique.

Proof of the properties of O: let $A, B \subseteq S'$ be such that $A \cap B = \emptyset$ and $A \succ_L B$. By monotonicity of $\succsim_L$, $A \succ_L B$ implies $S' \setminus B \succ_L B$. Thus, by construction of O, O must be included in $S' \setminus B$ and thus disjoint of B.

Let $A, B \subseteq S'$ be such that $A \cap B = \emptyset$ and $O \subseteq A$. Since O is decisive, it holds that $O \succ_L S' \setminus O$. Since $A \cap B = \emptyset$, $O \subseteq A$ implies that B must be a subset of $S' \setminus O$. Therefore, $O \succ_L S' \setminus O$ and the monotonicity of $\succsim_L$ implies $A \succ_L B$. □

Proof of Corollary 4. The existence of the top predominant event O_1 is due to Theorem 3. Suppose that the property holds until level j. So, there is a O_j predominant in $S_j = S \setminus (O_1 \cup \cdots \cup O_{j-1})$. Then, two cases must be distinguished:

(i) $S_{j+1} = S_j \setminus O_j$ is a null event, then the hierarchy is: $O_1, \ldots, O_j$ and $O_{j+1} = S_{j+1}$.
(ii) Suppose that S_{j+1} is not null. Thanks to Theorem 3, there is a unique event O_{j+1} predominant in S_{j+1}, that also satisfies the three additional properties. This process can be iterated until the set of remaining states is a null event. □

Proof of Theorem 4. The result is derived from Corollary 4. The basic idea is to show that under completeness and transitivity of $\succsim$, any predominant event is a singleton. To this end, suppose there exists at least two states s and s' in the same predominant event. Consider three elements x, y, z of X such that $x \succ_P y \succ_P z$ (A5) and the three following acts: $f = y\{s\}z$, $g = z\{s\}x$, $h = x\{s\}y$. Since s is a vetoer, $f(s) \succ_P g(s)$ implies not($g \succ_P f$). $\succsim$ being complete, we get $f \succsim g$. Using similar arguments, $g(s') \succ_P h(s')$ implies $g \succsim h$. Hence we get $f \succsim h$ since $\succsim$ is transitive.

On the other hand, $[f \succsim_P h] = \emptyset$ and $[h \succsim_P f] = S$. Since $S \succ_L \emptyset$ (Proposition 6), we get $h \succ f$, which contradicts $f \succsim h$. □

Proof of Proposition 13. *Reflexivity and completeness of* $\succsim$ are direct consequences of the reflexivity and completeness of any measure, and of necessities in particular.

Quasi-transitivity of $\succsim$: let us consider three acts f, g, h. Since the relation $\succsim_P$ is a weak order, S can be partitioned as follows:

$$\begin{aligned}
&A = \{s, f(s) \succ_P g(s) \succ_P h(s)\} && B = \{s, f(s) \succ_P g(s) \sim_P h(s)\},\\
&C = \{s, f(s) \succ_P h(s) \succ_P g(s)\} && D = \{s, g(s) \succ_P f(s) \succ_P h(s)\},\\
&E = \{s, g(s) \succ_P f(s) \sim_P h(s)\} && F = \{s, g(s) \succ_P h(s) \succ_P f(s)\},\\
&G = \{s, h(s) \succ_P f(s) \succ_P g(s)\} && H = \{s, h(s) \succ_P f(s) \sim_P g(s)\},\\
&I = \{s, h(s) \succ_P g(s) \succ_P f(s)\} && J = \{s, f(s) \sim_P g(s) \succ_P h(s)\},\\
&K = \{s, f(s) \sim_P h(s) \succ_P f(s)\} && L = \{s, g(s) \sim_P h(s) \succ_P f(s)\},\\
&M = \{s, f(s) \sim_P g(s) \sim_P h(s)\}.
\end{aligned}$$

Suppose that $f \succ g$, $g \succ h$ and that not $f \succ h$, i.e., $h \succsim f$ due to the completeness of $\succsim$. Using the necessity-based dominance rule, this writes:

- $f \succ g$: $N([f \succsim g]) > N([g \succsim f])$, i.e., $\Pi([f \succ g]) > \Pi([g \succ f])$. Therefore: $\max(\Pi(A), \Pi(B), \Pi(C), \Pi(G), \Pi(K)) > \max(\Pi(D), \Pi(E), \Pi(F), \Pi(I), \Pi(L))$.
- $g \succ h$: $N([g \succsim h]) > N([h \succsim g])$, that is to say: $\max(\Pi(A), \Pi(D), \Pi(E), \Pi(F), \Pi(J)) > \max(\Pi(C), \Pi(G), \Pi(H), \Pi(I), \Pi(K))$.
- $h \succsim f$: $N([h \succsim f]) > N([f \succsim h])$, that is to say: $\max(\Pi(F), \Pi(G), \Pi(H), \Pi(I), \Pi(L)) \geqslant \max(\Pi(A), \Pi(B), \Pi(C), \Pi(D), \Pi(J))$.

We thus get a system of equations of the form:

$$\begin{cases} \max(a,b,c,g,k) > \max(d,e,f,i,l) \\ \max(a,d,e,f,j) > \max(c,g,h,i,k) \\ \max(f,g,h,i,l) \geqslant \max(a,b,c,d,j) \end{cases}$$

which is inconsistent.

Proof of P2, P3, P4, P5, OI, LM, RM: see the corresponding proofs in Proposition 5, that hold here since (i) the weak order $\succsim_X$ is obviously reflexive, quasi-transitive and complete and (ii) the relation $\succsim_\Lambda$: $\forall A, B \subseteq S$, $A \succsim_\Lambda B \Leftrightarrow N(A) \geqslant N(B)$ is a monotonic confidence relation. □

Proof of Proposition 14. Let us denote $L = \{\alpha, s \in S, \pi(s) = \alpha\}$ and rank the elements of L so that $\alpha_1 > \alpha_2 > \cdots > \alpha_j$.

Let us show that $O_i = \{s, \pi(s) = \alpha_i\}$ is a predominant event in $S_i = S \setminus \{s, \pi(s) > \alpha_i\}$, of any $i < j$. Obviously, $\Pi(O_i) = \alpha_i$ and $\Pi(S_i \setminus O_i) = \alpha_{i+1}$. Hence $\Pi(O_i) > \Pi(S_i \setminus O_i) = \alpha_{i+1}$, that is to say, since O_i and $S_i \setminus O_i$ are disjoint, $O_i \succ_L S_i \setminus O_i$. So, O_i is decisive in S_i. Let us finally show that any state s^* with possibility α_i is a vetoer in S_i. Consider f, g such that $f(s^*) \succsim_P g(s^*)$ and for any $s \in S \setminus S_i$, $f(s) = g(s)$. It holds that $\Pi([f \succ_P g]) = \alpha_i$ (since s^* belongs to $[f \succ_P g]$) and $\Pi([g \succ_P f]) \leqslant \alpha_i$ (since $\pi(s) > \alpha_i \Rightarrow s \in S \setminus S_i \Rightarrow f(s) = g(s)$). So, $\Pi([f \succ_P g]) \geqslant \Pi([g \succ_P f])$, i.e., $f \succsim g$: s^* is a vetoer. □

Proof of Proposition 15. A is null if and only if $\forall f, g, (f \succsim g)_A$. It writes $\Pi([f \succ_X g] \cap A) \geqslant \Pi([g \succ_X f] \cap A)$. Using f, g such that $g(s) \succ_X f(s)$ on A, it yields $\Pi(\emptyset) \geqslant \Pi(A)$, hence $\Pi(A) = 0$. So the maximal null set Z is the maximal set with $\Pi(Z) = 0$, that is $Z = \{s, \pi(s) = 0\}$. The converse is obvious. □

Proof of Proposition 16. $\forall A, (f \succsim g)_A \Leftrightarrow \Pi(A \cap [f \succ_P g]) \geqslant \Pi(A \cap [g \succ_P f])$. So, from $(f \succsim g)_A$ and $(f \succsim g)_B$ we get $\Pi(A \cap [f \succ_P g]) \geqslant \Pi(A \cap [g \succ_P f])$ and $\Pi(B \cap [f \succ_P g]) \geqslant \Pi(B \cap [g \succ_P f])$. So, $\max(\Pi(A \cap [f \succ_P g]), \Pi(B \cap [f \succ_P g])) \geqslant \max(\Pi(A \cap [g \succ_P f]), \Pi(B \cap [g \succ_P f]))$, i.e., $\Pi((A \cup B) \cap [f \succ_P g]) \geqslant \Pi((A \cup B) \cap [g \succ_P f])$, that is to say, $(f \succsim g)_{A \cup B}$. □

Proof of Proposition 17. If there is no null state then $\forall A \neq \emptyset$, $A \succ_L \emptyset$. So $Z = \emptyset$ and the result trivially holds. Now suppose there are null states, and let $Z = \{s, \{s\} \sim_L \emptyset\}$. If $s_1, s_2 \in Z$ then $\forall f, g, (f \succsim g)_{\{s_1\}}$ and $(f \succsim g)_{\{s_2\}}$. Using EUN, $(f \succsim g)_{\{s_1, s_2\}}$. Hence $\{s_1, s_2\}$ is null and $\{s_1, s_2\} \sim_L \emptyset$. Repeating this scheme, it follows that $Z \sim_L \emptyset$ and Z is null, like all of its subsets. Now when $A \subseteq S$ is not true, $\exists s \in A, s \succ_L \emptyset$. Since $\succsim$ is monotonic, $A \succ_L \emptyset$, and A is not null. □

Proof of Theorem 5. First assume there are no null states. Then, the confidence relation $\succ_L$ is such that $A \succ_L \emptyset, \forall A \neq \emptyset$, it is monotonic (since $\succsim_L$ is monotonic), and the negligibility property holds. As shown in [12,13], the associated inference relation (A implies $B \Leftrightarrow A \cap B \succ_L A \cap \overline{B}$) satisfies all the postulates of non-monotonic inference after Kraus et al. [31]. So, according to [16], it can be represented on disjoint sets by a family $\mathcal{F}$ of possibility distributions such that $A \succ_L B \Leftrightarrow \forall \pi \in \mathcal{F}, \Pi(A) > \Pi(B)$ (in [15] the direct link between such families and the negligibility property is provided). The equivalence $f \succ g \Leftrightarrow \forall N \in \mathcal{N}, N([f \succsim g]) > N([g \succsim f])$ is then deduced using Theorem 2.

Now suppose the set of null states is Z. Then it is easy to see that $(f \succsim g)_{\overline{Z}}$ implies (by EUN) $(f \succsim g)$ since $(f \succsim g)_Z$ always holds for a null set (Z is a null set again by EUN). Hence $(f \succ g)$ implies $(f \succ g)_{\overline{Z}}$, that is, by the likely dominance rule: $([f \succsim_P g] \cap \overline{Z}) \cup Z \succ_L ([g \succsim_P f] \cap \overline{Z}) \cup Z$. Since $\succ_L$ is preadditive, this is equivalent to $[f \succsim_P g] \cap \overline{Z} \succ_L [g \succsim_P f] \cap \overline{Z}$. Hence there is a family $\mathcal{F}$ of possibility distributions such that $\forall \pi \in \mathcal{F}, \Pi([f \succsim_P g] \cap \overline{Z}) > \Pi([g \succsim_P f] \cap \overline{Z})$ since there is no null set in $\overline{Z}$. □

Proof of Lemma 2. From the Representation theorems of Section 4, it only remains to show that if $\succsim_L$ is complete and (OI, A1, A5, LM, RM, EUN, and TS) hold then $\succsim_L$ can be encoded by a necessity measure such that $A \succsim_L B \Leftrightarrow N(A \cup \overline{B}) \geqslant N(\overline{A} \cup B)$.

TS + A1 + monotonicity of $\succsim_L$ requires that the projection of $\succsim_L$ on singletons is a weak order (indeed, it is reflexive, complete and transitive). So, it can be represented by a possibility distribution such that $\{s\} \succsim_L \{s'\} \Leftrightarrow \pi(s) \geqslant \pi(s')$. Let us now show that $A \succsim_L B \Leftrightarrow \Pi(A \cap \overline{B}) \geqslant \Pi(B \cap \overline{A})$, i.e., that $A \succsim_L B \Leftrightarrow \sup_{s \in A \cap \overline{B}} \pi(s) \succsim_L \sup_{s \in B \cap \overline{A}} \pi(s)$. In other terms, we must prove that $A \succsim_L B$ iff $\exists s^* \in A \cap \overline{B}$ such that, $\forall s \in B \cap \overline{A}$, $\{s^*\} \succsim_L \{s\}$.

Suppose that $\exists s^* \in A \cap \overline{B}$ such that, $\forall s \in B \cap \overline{A}$, $\{s^*\} \succsim_L \{s\}$. Then $\{s^*\} \succsim_L B$. Indeed, by EUN, applied to $\succsim_L$, $\{s^*\} \succsim_L \{s_1\}$ and $\{s^*\} \succsim_L \{s_2\}$ imply $\{s^*\} \succsim_L \{s_1, s_2\}$. One can iterate this procedure on all the elements of B until getting $\{s^*\} \succsim_L B$. Then the monotonicity of $\succsim_L$ allows the derivation of $A \succsim_L B$.

Conversely, suppose that the condition is false, i.e., that $\forall s_i \in A \cap \overline{B}$ $\exists s'_i \in B \cap \overline{A}$, $\{s'_i\} \succ_L \{s_i\}$. Hence $\forall s_i \in A \cap \overline{B}$, $B \cap \overline{A} \succ_L \{s_i\}$. Hence, by EUN, $B \cap \overline{A} \succ_L A \cap \overline{B}$. By additivity this means that $B \succ_L A$, i.e., not($A \succsim_L B$) (the relation is complete). This shows by contraposition that $A \succsim_L B$ implies $\exists s^* \in A \cap \overline{B}$ such that, $\forall s \in B \cap \overline{A}$, $\{s^*\} \succsim_L \{s\}$. □

Proof of Proposition 18. P5 requires that X contains at least two elements x, y such that $x \succ_P y$. Since P5 + OI + $\succsim$ reflexive are assumed, we know that whatever $A, B \subseteq S$ ($A \succsim_L B$ iff $xAy \succsim_L xBy$).

Moreover, since EUN holds we know that $xAy \sim xBy$ and $xAy \sim xCy$ imply $xAy \sim x(B \cup C)y$, i.e., that $A \sim_L B$ and $A \sim_L C$ imply $A \sim_L B \cup C$, for $A \cap (B \cup C) = \emptyset$.

Suppose that $\{s_1\} \sim_L \{s_2\}$ and that $\{s_1\} \sim_L \{s_3\}$. So $\{s_1\} \sim_L \{s_2, s_3\}$.

Consider the following acts: $f = x\{s_1\}y$, $g = x\{s_2, s_3\}y$. By definition, $f_{s_1 \leftrightarrow s_2} = x\{s_2\}y$ $g_{s_1 \leftrightarrow s_2} = x\{s_1, s_3\}y$. Since $\{s_1\} \sim_L \{s_2\}$, ANO implies that $f_{s_1 \leftrightarrow s_2}$ must be indifferent to $g_{s_1 \leftrightarrow s_2}$, i.e., $x\{s_2\}y \sim x\{s_1, s_3\}y$. Thus $\{s_2\} \sim_L \{s_1, s_3\}$. By monotonicity, this implies $\{s_2\} \succsim_L \{s_3\}$. Symmetrically, from the indifference between $\{s_1\}$ and $\{s_3\}$, we get that $\{s_3\} \succsim_L \{s_2\}$. Hence, it holds that $\{s_2\} \sim_L \{s_3\}$. □

Proof of Proposition 19. First of all, remark that the postulates of Savage imply that, if $x \sim_P y$, then $\forall s, f, x\{s\}f \sim y\{s\}f$. Indeed, by S3, we get $x\{s\}f \sim y\{s\}f$ if $\{s\}$ is not null, and, if $\{s\}$ is null, $x\{s\}f \sim y\{s\}f$ holds by definition.

Consider two states s_1 and s_2 such as $s_1 \sim_L s_2$:

- If $f(s_1) \succ_P f(s_2)$, $s_1 \sim_L s_2 \Leftrightarrow f(s_1)\{s_1\}f(s_2) \sim f(s_1)\{s_2\}f(s_2)$ (by P4). By P2, we get $f(s_1)\{s_1\}f(s_2)\{s_2\}f \sim f(s_1)\{s_2\}f(s_2)\{s_1\}f$, i.e., $f \sim f_{s_1\leftrightarrow s_2}$.
- If $f(s_2) \succ_P f(s_1)$, the same reasoning proves $f \sim f_{s_1\leftrightarrow s_2}$.
- If $f(s_1) \sim_P f(s_2)$, we know that the two consequences can be exchanged, i.e.: $f(s_1)\{s_1\}f(s_2)\{s_2\}f \sim f(s_1)\{s_1\}\, f(s_1)\{s_2\}f \sim f(s_2)\{s_1\}f(s_1)\{s_2\}f$. By transitivity, this means $f \sim f_{s_1\leftrightarrow s_2}$.

So, in any case, $f \sim f_{s_1\leftrightarrow s_2}$. One can prove in the same way that $g \sim g_{s_1\leftrightarrow s_2}$. So, $f \succsim g$ implies by transitivity of $\succsim$ that $f_{s_1\leftrightarrow s_2} \succsim g_{s_1\leftrightarrow s_2}$ (and conversely). □

Proof of Proposition 20. First, let $s, s' \in S$ be two states and $x^*, y^* \in X$ be two consequences such that $x^* \succ_P y^*$. When the necessity-based dominance rule is used, $\{s\} \succsim_L \{s'\} \Leftrightarrow \pi(s) \geqslant \pi(s')$. Indeed, $\{s\} \succsim_L \{s'\} \Leftrightarrow x^*\{s\}y^* \succsim_L x^*\{s'\}y^* \Leftrightarrow N(\overline{\{s'\}}) \geqslant N(\overline{\{s\}}) \Leftrightarrow \pi(s) \geqslant \pi(s')$.

Let s_1 and s_2 be two states such that $s_1 \sim_L s_2$. We have proved that this is equivalent to $\pi(s_1) = \pi(s_2)$. Let us use the following notations: $\alpha = \pi(\{s_1\}) = \pi(\{s_2\})$, $A = [f \succ_P g] \cap (S - \{s_1, s_2\})$ and $B = [g \succ_P f] \cap (S - \{s_1, s_2\})$.

Let us first suppose that $f(s_1) \succ_P g(s_1)$:

(i) Suppose that $f(s_2) \succ_P g(s_2)$. In this case $[f \succ_P g] = [f_{s_1\leftrightarrow s_2} \succ_P g_{s_1\leftrightarrow s_2}]$ and $[g \succ_P f] = [g_{s_1\leftrightarrow s_2} \succ_P f_{s_1\leftrightarrow s_2}]$. Hence $\Pi([f \succ_P g]) \geqslant \Pi([g \succ_P f]) \Leftrightarrow \Pi([f_{s_1\leftrightarrow s_2} \succ_P g_{s_1\leftrightarrow s_2}]) \geqslant \Pi([g_{s_1\leftrightarrow s_2} \succ_P f_{s_1\leftrightarrow s_2}])$: $f \succsim g \Leftrightarrow f_{s_1\leftrightarrow s_2} \succsim g_{s_1\leftrightarrow s_2}$.

(ii) Suppose that $g(s_2) \succ_P f(s_2)$. In this case: $[f \succ_P g] = \{s_1\} \cup A$, $[g \succ_P f] = \{s_2\} \cup B$, $[f_{s_1\leftrightarrow s_2} \succ_P g_{s_1\leftrightarrow s_2}] = \{s_2\} \cup A$, $[g_{s_1\leftrightarrow s_2} \succ_P f_{s_1\leftrightarrow s_2}] = \{s_1\} \cup B$. On the one hand $f \succsim g \Leftrightarrow \Pi(\{s_1\} \cup A) \geqslant \Pi(\{s_2\} \cup B)$. In other terms, $f \succsim g \Leftrightarrow \max(\alpha, \Pi(A)) \geqslant \max(\alpha, \Pi(B))$. On the other hand, $f_{s_1\leftrightarrow s_2} \succsim g_{s_1\leftrightarrow s_2} \Leftrightarrow \Pi(\{s_2\} \cup A) \geqslant \Pi(\{s_1\} \cup B)$, i.e., $f_{s_1\leftrightarrow s_2} \succsim g_{s_1\leftrightarrow s_2} \Leftrightarrow \max(\alpha, \Pi(A)) \geqslant \max(\alpha, \Pi(B))$. Hence $f \succsim g$ iff $f_{s_1\leftrightarrow s_2} \succsim g_{s_1\leftrightarrow s_2}$.

(iii) Suppose that $g(s_2)$ and $f(s_2)$ are indifferent. In this case: $[f \succ_P g] = \{s_1\} \cup A$, $[g \succ_P f] = B$, $[f_{s_1\leftrightarrow s_2} \succ_P g_{s_1\leftrightarrow s_2}] = \{s_2\} \cup A$, $[g_{s_1\leftrightarrow s_2} \succ_P f_{s_1\leftrightarrow s_2}] = B$. Hence $f \succsim g$ iff $\Pi(\{s_1\} \cup A) \geqslant \Pi(B)$. In other terms, $f \succsim g$ iff $\max(\alpha, \Pi(A)) \geqslant \Pi(B)$. Moreover, $f_{s_1\leftrightarrow s_2} \succsim g_{s_1\leftrightarrow s_2} \Leftrightarrow \Pi(\{s_2\} \cup A) \geqslant \Pi(B)$, that also writes $f_{s_1\leftrightarrow s_2} \succsim g_{s_1\leftrightarrow s_2} \Leftrightarrow \max(\alpha, \Pi(A)) \geqslant \Pi(B)$. Hence $f \succsim g \Leftrightarrow f_{s_1\leftrightarrow s_2} \succsim g_{s_1\leftrightarrow s_2}$.

So, in any case, $f(s_1) \succ_P g(s_1) \Rightarrow f \succsim g \Leftrightarrow f_{s_1\leftrightarrow s_2} \succsim g_{s_1\leftrightarrow s_2}$.

By symmetry, when $f(s_2) \succ_P g(s_2)$ or $g(s_1) \succ_P f(s_1)$ or $g(s_2) \succ_P f(s_2)$, we also get $f \succsim g$ iff $f_{s_1\leftrightarrow s_2} \succsim g_{s_1\leftrightarrow s_2}$. Finally, when $g(s_1)$ and $f(s_1)$ are indifferent and $g(s_2)$ and $f(s_2)$ are indifferent, $[f \succ_P g] = [f_{s_1\leftrightarrow s_2} \succ_P g_{s_1\leftrightarrow s_2}] = A$ and $[g \succ_P f] = [g_{s_1\leftrightarrow s_2} \succ_P f_{s_1\leftrightarrow s_2}] = B$. Hence in any case $f \succsim g \Leftrightarrow f_{s_1\leftrightarrow s_2} \succsim g_{s_1\leftrightarrow s_2}$. □

Proof of Theorem 6.

($\Rightarrow$) Since Lemma 2 has already been proved, we only have to show that $\succsim_P$ is a weak order and that the necessity measure is not trivial. The first condition is due to Proposition 11 and the existence of two non-null states. Let us prove it for the necessity measure N: we know that no event of the form $\overline{\{s\}}$ is null (since there are two non-null states). So, $\Pi(\overline{\{s\}}) > 0$. If none of the states receive a positive degree, this condition cannot hold. If only one state s^* receives a positive degree, then $\Pi(\overline{\{s^*\}}) = 0$. So, there are at least two states such that $\pi(s) > 0$. The necessity measure is not trivial.

($\Leftarrow$) From Proposition 13, we know that OI, A1, A5, LM, RM hold. ANO and EUN are due to Proposition 20. Let us prove there are two non-null states. 2NN here writes $\forall s, \Pi(\overline{\{s\}}) > 0$: This holds since N is non-trivial. $\square$

Proof of Proposition 21. The possibility-based dominance rule writes: $\forall f, g \in X^S$, $f \succsim g \Leftrightarrow \Pi([f \succsim_X g]) \geqslant \Pi([g \succsim_X f])$. Consider any pair of acts f, g. Suppose that $\exists s^* \in S$ such that $\Pi(\{s^*\}) = 1$ and $f(s^*) \succsim_X g(s^*)$. Then obviously $\Pi([f \succsim_X g]) = \max_{s\in[f\succsim_X g]} \Pi(\{s\}) = 1$. Thus, whatever $\Pi([g \succsim_X f])$, $\Pi([f \succsim_X g]) \geqslant \Pi([g \succsim_X f])$, i.e., $f \succsim g$. Conversely, suppose that $\forall s \in S$ such that $\Pi(\{s\}) = 1$, *not*$(f(s) \succsim_X g(s))$, i.e., since $\succsim_X$ is assumed to be complete, $\forall s \in S$ such that $\Pi(\{s\}) = 1$, $g(s) \succ_X f(s)$. Thus, $\Pi([f \succsim_X g]) < 1$ and $\Pi([g \succsim_X f]) = 1$: we get $g \succ f$. Thus, by contraposition, we have shown that $\exists s \in S$ such that $\Pi(\{s\}) = 1$ and $f(s) \succsim_X g(s)$. $\square$

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