Lattice polynomials in qualitative decision making

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Jointly with D. Dubois, J.-L. Marichal, H. Prade, T. Waldhauser, . . .

Decision making **MCDM**

Basic Problem: Model preference relations $\precsim$

Utility based approach: Given a scale X...

A preference $\precsim$ on $X_1 \times \cdots \times X_n$ is **represented** by $U\colon X_1 \times \cdots \times X_n \to X$

$$\mathbf{x} \precsim \mathbf{y} \iff U(\mathbf{x}) \leqslant U(\mathbf{y})$$

Limitation: The role of local preferences (criteria) is not explicit!

Example:

If **House A** is cheaper, bigger and better located than **House B**, then **House A** should be preferred to **House B**.

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Aggregation: $x_1, \ldots, x_n \quad \longrightarrow \quad y = A(x_1, \ldots, x_n)$

Let X be a scale (bounded chain)...

An **aggregation function** on X is a mapping $A\colon X^n \to X$ such that:

1. A is order-preserving: for every $\mathbf{x}, \mathbf{y} \in X^n$

$$\mathbf{x} \leqslant \mathbf{y} \implies A(\mathbf{x}) \leqslant A(\mathbf{y})$$

2. A preserves the boundaries:

$$\inf_{\mathbf{x} \in X^n} A(\mathbf{x}) = \inf X \quad \text{and} \quad \sup_{\mathbf{x} \in X^n} A(\mathbf{x}) = \sup X.$$

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Refined model **MCDM**

1. n attributes $X_1, \ldots, X_n$

2. Preference on X_i (criterion) is represented by a **local utility function**

$$\varphi_i \colon X_i \to X$$

NB: we assume that each φ_i is **order-preserving**.

3. Preference on $X_1 \times \cdots \times X_n$ is represented by an **overall utility function**

$$U(x_1, \ldots, x_n) := A(\varphi_1(x_1), \ldots, \varphi_n(x_n))$$

where $A \colon X^n \to X$ is an aggregation function.

Examples of aggregation functions:

Numerical setting: Let $X \subseteq \mathbb{R}_+$ and $[n] = \{1, \ldots, n\}$

1. Weighted arithmetic means: For $\mathbf{x} \in X^n$ and $\sum w_i = 1$,

$$WAM(\mathbf{x}) := \sum_{i \in [n]} w_i x_i$$

2. Choquet integrals: For a capacity $\mu\colon 2^{[n]} \to [0,1]$ and $\mathbf{x} \in X^n$

$$\mathcal{C}_\mu(\mathbf{x}) := \sum_{i \in [n]} (x_{\sigma(i)} - x_{\sigma(i-1)}) \cdot \mu(\{\sigma(i), \ldots, \sigma(n)\})$$

where $0 = x_{\sigma(0)} \leqslant x_{\sigma(1)} \leqslant \cdots \leqslant x_{\sigma(n)}$ (σ permutation)

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Sugeno integral

Let X be a chain with least and greatest elements 0 and 1, respectively.

An **order capacity** is a mapping $\nu\colon 2^{[n]} \to X$, such that

1. $\nu(I) \leqslant \nu(J)$ whenever $I \subseteq J$,
2. $\nu(\varnothing) = 0$ and $\nu([n]) = 1$.

For an order capacity $\nu\colon 2^{[n]} \to X$...

The **Sugeno integral** on X w.r.t. ν is defined by

$$\mathcal{S}_\nu(\mathbf{x}) := \bigvee_{i\in[n]} \nu(\{\sigma(i),\ldots,\sigma(n)\}) \wedge x_{\sigma(i)}$$

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A qualitative setting for decision making **MCDM**

Setting:

1. n attributes $X_1, \ldots, X_n$ and a scale X (finite chain)
2. Preference on X_i is represented by $\varphi_i \colon X_i \to X$

   NB: Each X_i can be thought of as a finite chain and φ_i's **order-preserving**
3. Preference on $X_1 \times \cdots \times X_n$ represented by a **Sugeno utility function**

$$U(x_1, \ldots, x_n) := A(\varphi_1(x_1), \ldots, \varphi_n(x_n))$$

   where $A \colon X^n \to X$ is a Sugeno integral.

Terminology: We refer to these preference relations as **Sugeno preferences**

Outline

1. Preliminaries: Sugeno integrals as lattice polynomial functions
2. Characterizations of lattice polynomial functions
3. Generalization of polynomial functions: Sugeno utility functions
4. Axiomatic approach to Sugeno preferences **MCDM**.
5. Final remarks: **MCDM** vs **DMU**
6. **If time allows:** Complete descriptions and factorizations

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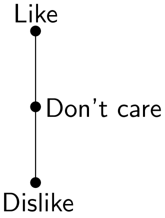
Preliminaries

Let X be a (finite) **distributive lattice** endowed with

1. the usual operations $\wedge$ and $\vee$, and
2. least and greatest elements 0 and 1, respectively.



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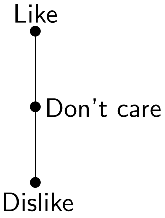

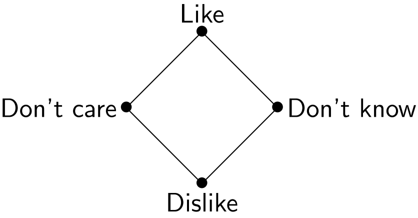

Lattice polynomial functions

A (**lattice**) **polynomial function** (on X) is any map $p : X^n \to X$, $n \geq 1$, obtainable by finitely many applications of the rules:

1. The **projections** $\mathbf{x} \mapsto x_i$, $i \in [n]$, and the **constant functions** $\mathbf{x} \mapsto c$, $c \in X$, are polynomial functions.
2. If $f : X^n \to X$ and $g : X^n \to X$ are polynomial functions, then $f \wedge g$ and $f \vee g$ are polynomial functions.

Example:

$\mathrm{median}(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee (x_2 \wedge x_3) \vee (x_3 \wedge x_1)$

NB: On chains the median is just outputs the "middle point".

Representations: Disjunctive Normal Form

A function $f\colon X^n \to X$ has a **disjunctive normal form (DNF)** if

$$f(\mathbf{x}) = \bigvee_{I \subseteq [n]} \Big(a_I \wedge \bigwedge_{i \in I} x_i\Big).$$

Proposition (Grätzer, Goodstein)

A function $p\colon X^n \to X$ is a polynomial function **iff** it has the **DNF**:

$$p(\mathbf{x}) = \bigvee_{I \subseteq [n]} \Big(p(\mathbf{1}_I) \wedge \bigwedge_{i \in I} x_i\Big)$$

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Sugeno integrals as lattice polynomial functions

The **Sugeno integral** on a chain X w.r.t. $v\colon 2^{[n]} \to X$ is defined by

$$\mathcal{S}_v(\mathbf{x}) := \bigvee_{i \in [n]} v(\{\sigma(i), \dots, \sigma(n)\}) \wedge x_{\sigma(i)}$$

for a permutation σ **s.t.** $x_{\sigma(1)} \leqslant \cdots \leqslant x_{\sigma(n)}$.

Theorem (Marichal): Let X be a distributive lattice...

A function $q : X^n \to X$ is the Sugeno integral $\mathcal{S}_v$ **iff**

$$q(\mathbf{x}) = \bigvee_{I \subseteq [n]} \Big(v(I) \wedge \bigwedge_{i \in I} x_i\Big).$$

Thus: Sugeno integrals are exactly those polynomials that are **idempotent**:

$$q(x, \dots, x) = x$$

Fact:

Every polynomial function (in part., Sugeno integral) is **order-preserving**.

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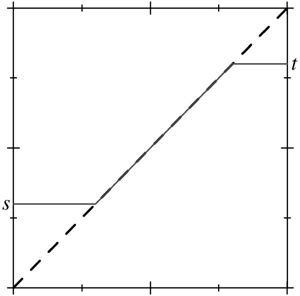
Median decomposability

For $c \in X$ and $\mathbf{x} \in X^n$, set $\mathbf{x}_i^c = (x_1, \ldots, x_{i-1}, c, x_{i+1}, \ldots, x_n)$.

Definition:

A function $f\colon X^n \to X$ is **median decomposable** if

$$f(\mathbf{x}) = \mathrm{median}\left(f(\mathbf{x}_i^0), x_i, f(\mathbf{x}_i^1)\right), \text{ for every } i \in [n] \text{ and } \mathbf{x} \in X^n$$



$t = f(\mathbf{x}_i^1)$

$s = f(\mathbf{x}_i^0)$

Characterization of polynomial functions

Fact

Every median decomposable function is order-preserving.

Theorem (Marichal)

A function $p\colon X^n \to X$ is

1. a polynomial function **iff** it is median decomposable.
2. a Sugeno integral **iff** it is idempotent and median decomposable.

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Extensions: pseudo-polynomial functions

Let $\mathbf{X} := X_1 \times \cdots \times X_n$, where X_i is a set with two distinct 0_i and 1_i.

Definition:

$f : \mathbf{X} \to X$ is a **pseudo-polynomial function** if it can be factorized as

$$f(\mathbf{x}) = p\big(\varphi_1(x_1), \ldots, \varphi_n(x_n)\big),$$

where $p \colon X^n \to X$ is polynomial function and each $\varphi_i \colon X_i \to X$ satisfies

$$\varphi_i(0_i) \leq \varphi_i(x_i) \leq \varphi_i(1_i). \tag{BC}$$

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Recall:

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Proposition (C. & Waldhauser)

Order-preserving pseudo-polynomials are exactly the Sugeno utility functions.

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Problems...

Consider $f\colon \mathbf{X} \to X$.

Problem 1: Determine whether f is pseudo-polynomial function.

Problem 2: Find all possible factorizations $f = p\left(\varphi_1, \ldots, \varphi_n\right)$.

Remark:

f is a pseudo-polynomial **iff** f is **pseudo-median decomposable** **i.e.**

$$\text{there are } \varphi_i\colon X_i \to X \textbf{ s.t. } \quad f(\mathbf{x}) = \mathrm{median}\left(f(\mathbf{x}_i^{0_i}), \varphi_i(x_i), f(\mathbf{x}_i^{1_i})\right)$$

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A particular **but** interesting case...

However: we can determine and effectively compute all possible factorizations...

C., Waldhauser. Pseudo-polynomial functions over finite distributive lattices, *Fuzzy Sets and Systems* **239** (2014) 21–34

Full description: Birkhoff-Priestley representation and other lattice th. tools

Corollary (C. & Waldhauser): For a finite chain X...

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NB: For pseudo-polynomials, just relax order-preservation to boundary conditions

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Main Problems

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Question: What is a preference relation?

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Preference relations

Let $\mathbf{X}$ be a nonempty set.

A **pre-order** on $\mathbf{X}$ is a relation $\precsim \subseteq \mathbf{X}^2$ that is:

1. reflexive: $\forall \mathbf{x} \in \mathbf{X} : \mathbf{x} \precsim \mathbf{x}$,
2. transitive: $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X} : \mathbf{x} \precsim \mathbf{y},\ \mathbf{y} \precsim \mathbf{z} \implies \mathbf{x} \precsim \mathbf{z}$, and
3. complete: $\forall \mathbf{x}, \mathbf{y} \in \mathbf{X} : \mathbf{x} \precsim \mathbf{y}$ or $\mathbf{y} \precsim \mathbf{x}$.

Note: Pre-orders are not necessarily antisymmetric:

$$\forall \mathbf{x}, \mathbf{y} \in \mathbf{X} : \mathbf{x} \precsim \mathbf{y},\ \mathbf{y} \precsim \mathbf{x} \implies \mathbf{x} = \mathbf{y} \qquad \text{(AS)}$$

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Indifference relation

The **indifference relation** $\sim$ associated with $\precsim$ is defined by:

$$\mathbf{x} \sim \mathbf{y} \text{ iff } \mathbf{y} \precsim \mathbf{x} \text{ and } \mathbf{x} \precsim \mathbf{y}.$$

Note that...

1. $\sim$ is an equivalence relation.
2. $\leq := \precsim / \sim$ satisfies (AS) and $\mathbf{X}/\sim$ is a (finite) chain.

Preference relations in **MCDM**

Let $\mathbf{X} := X_1 \times \cdots \times X_n$, where each X_i is endowed with an ordering (criterion).

A **preference relation** on $\mathbf{X}$ is a pre-order $\precsim$ that satisfies the **Pareto condition:**

(P) $\forall \mathbf{x}, \mathbf{y} \in \mathbf{X}$ **s.t.** $\forall i \in [n],\ x_i \precsim_i y_i \implies \mathbf{x} \precsim \mathbf{y}$.

How can we represent them?

The **rank function** $r\colon \mathbf{X} \to \mathbf{X}/\sim$ of $\precsim$ is order-preserving and

$$\mathbf{x} \precsim \mathbf{y} \iff r(\mathbf{x}) \leqslant r(\mathbf{y}).$$

Since they satisfy the Pareto condition...

Preference relations are exactly those representable by order-preserving functions.

Preference relations in **MCDM**

Let $\mathbf{X} := X_1 \times \cdots \times X_n$, where each X_i is endowed with an ordering (criterion).

A **preference relation** on $\mathbf{X}$ is a pre-order $\precsim$ that satisfies the **Pareto condition:**

(P) $\forall \mathbf{x}, \mathbf{y} \in \mathbf{X}$ **s.t.** $\forall i \in [n],\ x_i \precsim_i y_i \implies \mathbf{x} \precsim \mathbf{y}$.

How can we represent them?

The **rank function** $r \colon \mathbf{X} \to \mathbf{X}/\sim$ of $\precsim$ is order-preserving and

$$\mathbf{x} \precsim \mathbf{y} \iff r(\mathbf{x}) \leqslant r(\mathbf{y}).$$

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Preference relations are exactly those representable by order-preserving functions.

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Axiomatic approach to Sugeno preferences **MCDM**

Here: Preference relations that are represented by Sugeno utility functions

$$U(\mathbf{x}) = q\big(\varphi_1(x_1), \ldots, \varphi_n(x_n)\big),$$

where $q\colon X^n \to X$ is a Sugeno integral and each $\varphi_i\colon X_i \to X$ is order-preserving.

Theorem (C. & Dubois & Prade & Waldhauser)

A **preference relation** $\precsim$ on $\mathbf{X}$ is representable by a Sugeno utility function **iff**

$$\forall \mathbf{x}, \mathbf{y} \in \mathbf{X} : \; \mathbf{x}_i^0 \prec \mathbf{x}_i^a \text{ and } \mathbf{y}_i^a \prec \mathbf{y}_i^1 \implies \mathbf{x}_i^a \precsim \mathbf{y}_i^a$$

Intuition:

Alternative axiomatization: Greco *et al.* and Bouyssou *et al.*

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Proof

Recall: For a finite chain X...

$U \colon \mathbf{X} \to X$ is a Sugeno utility function **iff** it is order-preserving and

$$U(\mathbf{x}_i^{0_i}) < U(\mathbf{x}_i^{a_i}) \text{ and } U(\mathbf{y}_i^{a_i}) < U(\mathbf{y}_i^{1_i}) \implies U(\mathbf{x}_i^{a_i}) \leqslant U(\mathbf{y}_i^{a_i}) \qquad (*)$$

If $\precsim$ is a preference relation satisfying:

$$\forall \mathbf{x}, \mathbf{y} \in \mathbf{X} : \; \mathbf{x}_i^0 \prec \mathbf{x}_i^a \text{ and } \mathbf{y}_i^a \prec \mathbf{y}_i^1 \implies \mathbf{x}_i^a \precsim \mathbf{y}_i^a,$$

then r verifies $(*)$ and thus it is a Sugeno utility function representing $\precsim$.

Conversely...

Recall: For a finite chain X...

$U\colon \mathbf{X} \to X$ is a Sugeno utility function **iff** it is order-preserving and

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Suppose $\precsim$ is represented by a Sugeno utility function U. **WLOG** U surjective.

Then $r = \alpha \circ U$ for some order-isomorphism α.

Since U satisfies $(*)$, r satisfies $(*)$, and **thus:**

$$\forall \mathbf{x}, \mathbf{y} \in \mathbf{X} : \ \mathbf{x}_i^0 \prec \mathbf{x}_i^a \text{ and } \mathbf{y}_i^a \prec \mathbf{y}_i^1 \implies \mathbf{x}_i^a \precsim \mathbf{y}_i^a. \qquad \square$$

Final remarks: **MCDM** vs **DMU**

Particular case: **DMU** (under uncertainty)

1. single attribute $Y = X_1 = X_2 = \cdots = X_n$ (n states),
2. single utility function $\varphi \colon Y \to X$ for each $i \in [n]$,
3. **state independent** Sugeno utility functions $U \colon \mathbf{X} \to X$,

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Computational approach: Chateauneuf & Grabisch & Labreuche & Rico

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Consider a preference relation $\precsim$ on $\mathbf{X} = Y^n$...

Theorem (Dubois & Prade & Sabbadin)

$\precsim$ is representable by a state-independent Sugeno utility function **iff**

1. **Restricted disj. dominance:** $\forall \mathbf{x}, \mathbf{y}, \mathbf{c} \in \mathbf{X}: \ \mathbf{y} \prec \mathbf{x},\ \mathbf{c} \prec \mathbf{x} \implies \mathbf{y} \vee \mathbf{c} \prec \mathbf{x}$,
2. **Restricted conj. dominance:** $\forall \mathbf{x}, \mathbf{y}, \mathbf{c} \in \mathbf{X}: \ \mathbf{x} \prec \mathbf{y},\ \mathbf{x} \prec \mathbf{c} \implies \mathbf{x} \prec \mathbf{y} \wedge \mathbf{c}$

where $\mathbf{c}$ is a constant tuple.

Question: Is **MCDM** more expressive than this **DMU** setting?

In other words: Can **state-dependent** Sugeno utility functions $U\colon \mathbf{X} \to X$

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Question: Yes!

Example: Let $Y = \{1,2,3\} = X$ with the natural ordering, and consider $\precsim$:

$$[(3,3)] = \{(3,3),(2,3)\},$$
$$[(3,2)] = \{(3,2),(3,1),(1,3),(2,2),(2,1)\},$$
$$[(1,2)] = \{(1,2),(1,1)\}.$$

This relation does not satisfy **RDD**:

$$(2,2) \prec (2,3) \text{ and } (1,3) \prec (2,3) \text{ but } (2,2) \vee (1,3) \sim (2,3)$$

Hence: not representable by a state-independent Sugeno utility function.

However: Taking $q(x_1, x_2) = (2 \wedge x_1) \vee (2 \wedge x_2) \vee (3 \wedge x_1 \wedge x_2)$ and

$$\varphi_1(3) = 3 = \varphi_1(2),\ \varphi_1(1) = 1 \quad \textbf{and} \quad \varphi_2(3) = 3,\ \varphi_2(2) = 1 = \varphi_2(1)$$

we have that $U(x_1, x_2) = q(\varphi_1(x_1), \varphi_2(x_2))$ represents $\precsim$.

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Merci de votre attention!

Obrigado pela vossa atenção!

Thank you for your attention!

Grazie mille per la vostra attenzione!

Going back...

Consider $f\colon \mathbf{X} \to X$.

Problem 1: Determine whether f is pseudo-polynomial function.

Problem 2: Find all possible factorizations $f = p\left(\varphi_1, \ldots, \varphi_n\right)$.

Birkhoff-Priestley representation of distributive lattices

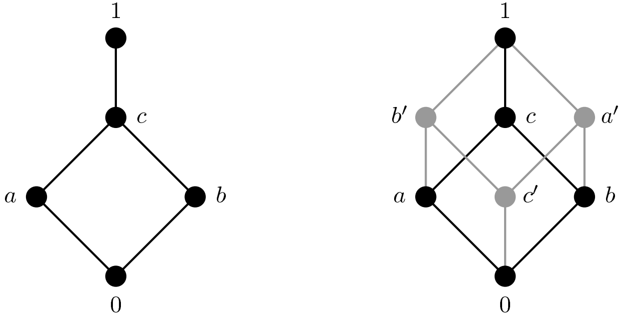
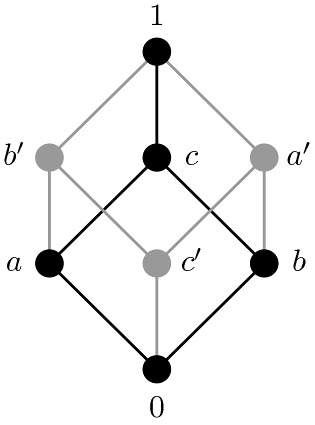


Figure : A distributive lattice X and its representation into the **Boolean lattice** $B(X)$

NB: Every Boolean lattice can be thought of as the **power-set** of some set

NB: Each $a \in B(L)$ has a **complement** a': $a \vee a' = 1$ and $a \wedge a' = 0$

Closure and interior operators from $B(X)$ to X



closure operator: $\mathrm{cl}\,(y) = \bigwedge_{\substack{x \in X \\ x \geq y}} x$

interior operator: $\mathrm{int}\,(y) = \bigvee_{\substack{x \in X \\ x \leqslant y}} x$

Examples:

$\mathrm{cl}\,(a') = \mathrm{cl}\,(b') = \mathrm{cl}\,(c') = 1$

$\mathrm{int}\,(a') = b, \ \mathrm{int}\,(b') = a, \ \mathrm{int}\,(c') = 0$

NB: cl and int are valued in X and both are the identity when restricted to X

Towards factorizations of pseudo-polynomials...

Given $f\colon \mathbf{X} \to X$ define the polynomial $p_f\colon X^n \to X$

$$p_f(\mathbf{x}) := \bigvee_{I \subseteq [n]} \Big(f(\widehat{\mathbf{1}}_I) \wedge \bigwedge_{i \in I} x_i \Big)$$

and functions $\Phi_i^-, \Phi_i^+\colon X_i \to X$, $i \in [n]$,

$$\Phi_i^-(a_i) := \bigvee_{\substack{\mathbf{x} \in \mathbf{X} \\ x_i = a_i}} \mathrm{cl}\big(f(\mathbf{x}) \wedge f(\mathbf{x}_i^0)'\big) \quad \text{and} \quad \Phi_i^+(a_i) := \bigwedge_{\substack{\mathbf{x} \in \mathbf{X} \\ x_i = a_i}} \mathrm{int}\big(f(\mathbf{x}) \vee f(\mathbf{x}_i^1)'\big)$$

A bit of intuition...

1. For $u \leqslant m \leqslant w, v \in X$: $m = \mathrm{median}(u, v, w)$ **iff** $m \wedge u' \leqslant v \leq m \vee w'$
2. If $f(\mathbf{x}) = \mathrm{median}\big(f(\mathbf{x}_i^{0_i}), \varphi_i(x_i), f(\mathbf{x}_i^{1_i})\big)$ (pseudo-median decomp.), **then**

$$f(\mathbf{x}) \wedge f(\mathbf{x}_i^{0_i})' \leqslant \varphi_i(x_i) \leqslant f(\mathbf{x}) \vee f(\mathbf{x}_i^{1_i})'$$

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If $f\colon \mathbf{X} \to X$ is pseudo-polynomial, then $f = p_f(\varphi_1, \ldots, \varphi_n)$ for $\varphi_i \in \{\Phi_i^-, \Phi_i^+\}$

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Characterization of pseudo-polynomials...

Fact

If f is a pseudo-polynomial, **then** it satisfies $f(\mathbf{x}_i^{0_i}) \leqslant f(\mathbf{x}) \leqslant f(\mathbf{x}_i^{1_i})$ (BCi)

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$f : \mathbf{X} \to X$ is a pseudo-polynomial **iff** for every $i \in [n]$: (BCi) and

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Finding all factorizations...

Let $f\colon \mathbf{X} \to X$ be a pseudo-polynomial function.

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$\varphi_i\colon X_i \to X$ satisfying (BC) appears in a factorization of f **iff** $\Phi_i^- \leq \varphi_i \leq \Phi_i^+$.

In particular: $f = p_f\left(\varphi_1, \ldots, \varphi_n\right)$ **where**

$$p_f(\mathbf{x}) = \bigvee_{I \subseteq [n]} \left(f(\widehat{\mathbf{1}}_I) \wedge \bigwedge_{i \in I} x_i\right).$$

Proposition (C. & Waldhauser): Fix $\varphi_i\colon X_i \to X$ as above...

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Basically, the solutions of an interpolation problem...

Find all $p \colon X^n \to X$ **s.t.** $p\left(\mathbf{1}_I\right) = f\left(\widehat{\mathbf{1}}_I\right)$ for all $I \subseteq [n]$

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$$p^-(\mathbf{x}) := \bigvee_{I \subseteq [n]} \Big(c_I^- \wedge \bigwedge_{i \in I} x_i\Big) \quad \text{where} \quad c_I^- := \mathrm{cl}\Big(f(\widehat{\mathbf{1}}_I) \wedge \bigwedge_{i \notin I} \varphi_i(0)'\Big),$$

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Finding all polynomial functions

Proposition (C. & Waldhauser): Fix $\varphi_i \colon X_i \to X$ as above...

Then we can effectively compute two polynomials $p^-, p^+ \colon X^n \to X$ **s.t.**

$$f = p(\varphi_1, \dots, \varphi_n) \quad \textbf{iff} \quad p^- \leqslant p \leqslant p^+.$$

Basically, the solutions of an interpolation problem...

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Example: Airlines

Question: Are preferences about travelling with four airlines $\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3, \mathrm{A}_4$ in economy class (E) and first class (F) modelled by pseudo-polynomial functions?

x_1	x_2	$f(x_1, x_2)$
A_1	E	B
A_1	F	B
A_2	E	B
A_2	F	D
A_3	E	N
A_3	F	G
A_4	E	N
A_4	F	V

Table: shows a customer's evaluation of these eight options, where $\mathrm{B}, \mathrm{N}, \mathrm{G}, \mathrm{V}, \mathrm{D}$ stand for bad, neutral, good, very good, and don't know

Defines: overall utility $f : X_1 \times X_2 \to X$ where

$$\begin{aligned} X_1 &:= \{\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3, \mathrm{A}_4\}, \quad X_2 := \{\mathrm{E}, \mathrm{F}\} \\ X &:= \{\mathrm{B}, \mathrm{N}, \mathrm{G}, \mathrm{V}, \mathrm{D}\} \end{aligned}$$

Assume: D is better than B, worse than G, and incomparable with N, hence ...

NB: $0_{X_2} = \mathrm{E}$ and $1_{X_2} = \mathrm{F}$, but 0_{X_1} and 1_{X_1} are not clear.

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NB1: $\Phi_k^- \leq \Phi_k^+$ for $k = 1, 2$. **Hence:** $f(x_1, x_2) = p(\varphi_1(x_1), \varphi_2(x_2))$

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Case $(\varphi_1, \varphi_2) = \left(\Phi_1^+, \Phi_2^+\right)$**:** 2 possibilities $p^- = y_1 \wedge y_2$ and $p^+ = p_f$

Thus: f has altogether three factorizations, namely,

$$\begin{aligned} f(x_1, x_2) &= \Phi_1^-(x_1) \wedge \left(\Phi_2^-(x_2) \vee \mathrm{N}\right) \quad \text{or} \\ &= \Phi_1^+(x_1) \wedge \left(\Phi_2^+(x_2) \vee \mathrm{N}\right) \quad \text{or} \\ &= \Phi_1^+(x_1) \wedge \Phi_2^+(x_2) \end{aligned}$$

However: they are essentially the same, since $\Phi_2^- \vee \mathrm{N} = \Phi_2^+ \vee \mathrm{N} = \Phi_2^+$

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