Auctions

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General Problem : resource allocation

An engineer or computer science view about how to allocate a set of goods.

- universal mechanism
- anonymous mechanism
- Centralised mechanisms for allocating goods
  - auctions for single goods
  - multi-unit auctions
  - combinatorial auctions
    - winner determination
    - representation language
- Decentralised mechanisms

How to allocate scarce resources ?

Many agents desire to obtain the use of a scarce resource

- if it is not scarce, maybe all agents can use it
- otherwise, what is a reasonable way for allocating that resource ?
- ➩ allocate the resources to those that value them the most : **efficiency**

Terminology :

- The agents that want the use of a resource are the **bidders**
- An **auctioneer** is an agent that runs the auction and who is in charge of allocating a resource to one of the bidders
- Bidders want to obtain the resource for a *minimum* price
- An auctioneer wants a *maximum* price
- An auctioneer chooses the *type of auctions*
- Bidders choose their *strategy* for participating to the auction.

Dimensions of an auction protocol

- Are bids made by the agents known to each other ?
  - **open cry** : bids are common knowledge
  - **sealed-bid** : bids are only known by the auctioneer
- One shot vs multiple bids
  - **one shot** : each agent makes a single bid and auctioneer announces the winner and the price.
  - **ascending auctions** : multiple bids that are increasing
  - **descending auctions** : multiple bids that are decreasing
- Who is the winner ? (winner determination problem)
  - trivial question for the single-item auction
  - not so trivial for combinatorial auctions
- What price does the winner pays ?

English Auctions

- open cry
- ascending auction : agents can place a bid higher than the current highest bid. When no more bids are placed, the auction terminates.
- the winner is the highest bidder
- the winner pays the amount of her bid

The auctioneer can set a **reservation price** : if no bidder is willing to bid that price, the auctioneer keep the resource.

Dutch Auctions

- open cry
- descending auction : auctioneer starts announcing a very high value and then continuously lowers the offer price until an agent makes a bid. The auction then terminates.
- the winner is the highest bidder
- the winner pays the amount of her bid (i.e. the price announced by the auctioneer right before the bidder made the bid)

First-price sealed-bid auctions

- Sealed-bid.
- One shot.
- The winner is the highest bidder.
- The winner pays the amount of her bid.

Second-price sealed-bid auctions – Vickrey Auctions

- William Vickrey (Nobel Prize in Economics 1996).
- Sealed-bid.
- One shot.
- The winner is the highest bidder.
- The winner pays the amount of the second highest bid.

Variants

- mix of English and Dutch auctions : first like a Dutch auction, but when the first bidder bids, other may outbit her !
- with a deadline
- candle auction : random stopping time.
- sharing the surplus differently (ex Cavallo AAMAS 2006)

Economics view

- the seller is unsure about the values that bidders attach to the object otherwise, the problem would be easy !
- **private values model**
  each bidder *knows* the value of the object to himself.
  knowledge of the values of other bidders would have no effect on this value !
- This may not be reasonable for a good that could be sold again.
- **interdependent values**
  the bidders may be *unsure* about the value of the object
  has an estimate or some privately known signal correlated to the value.
  ➪ values are unknown and may be affected by information available to other bidders
- special case of **pure common values**
  value is the same for each bidder
  (ex : land with an unknown amount of oil)

Choosing the auction type

- big difference in implementation and organisation
- not so different for *rational* decision makers !

- Dutch and first-price sealed bid are **strategically equivalent**.

  There is no information shared : bidding an amount in FPSB is equivalent to offering to buy at that amount in a Dutch auction.

- when values are **private** an English auction is *weakly* equivalent to a Vickrey auction.
  - interdependent values : more information is made available after each bid ⇒ the item value is updated
  - private values : bid for the value

  the two auctions are *not* strategically equivalent.

Performance

- seller's perspective : what is the revenue ?
- society's perspective : efficiency – does the item falls in the hand of the bidder with the highest valuation (*ex post*).
  n.b. there is a litterature about values with resale
- susceptibility to lies and collusion

Private Value Auction – the symmetric model

- set N of n buyers
- Bidder i assigns a value of X_i to the object
- Each X_i is independently and identically distributed on some interval $[0,\omega]$ according to an *increasing* distribution function F
- F admits a continuous density f (i.e. $F' = f$)
- $\mathbb{E}[X_i] < \infty$
- F and n are *common knowledge*
- i knowns only the realisation x_i of X_i
- Buyers are *risk neutral* : they maximise expected utility.
- Buyers do not have budget constraints (they can pay x_i)
- a strategy is a function that maps the value of the object to a bid

$$\beta_i : [0,\omega] \rightarrow \mathbb{R}$$

- interested in symmetrical equilibrium

Second Price auction

It is equivalent to English auctions.

Payoffs

$$\pi_i = \begin{cases} x_i - \max_{j \neq i} b_j & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

If there is a **tie**, i.e. $b_i = \max_{j \neq i} b_j$
the object goes to one of the highest bidder with equal probability.

Theorem

In a second price sealed bid auction with *private values*, it is a *wealkly* dominant strategy to bid according to $b_i^{II}(x) = x$

Proof

bidding less than x_i can never increase profit and may decrease it.
Idem for bidding more than x_i □

Some notation

- Let Y_1 be the highest of $n-1$ independently drawn values
- Let us call G the distribution function of Y_1

Expected payment

$$
\begin{aligned}
m^{II}(x) &= \mathbb{P}[win] \times \mathbb{E}[\text{2nd highest bid} \mid x \text{ is the highest bid}] \\
&= \mathbb{P}[win] \times \mathbb{E}[\text{2nd highest value} \mid x \text{ is the highest bid}] \\
&= G(x) \times \mathbb{E}[Y_1 \mid Y_1 < x]
\end{aligned}
$$

First price auctions

Payoffs

$$\pi_i = \begin{cases} x_i - b_i & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

Strategy

- Bidding x_i guarantee a payoff of 0...
- Bidding $\beta_i(x_i) > x_i$ may lead to negative utility.

⇨ must bid $\beta_i(x_i) < x_i$:

- bidding closer to x_i increases the probability to win, *but* decreases the utility.

Theorem

Symmetric equilibrium strategies in a first price auction are given by

$$\beta_1(x) = E[Y_1 \mid Y_1 < x]$$

where Y_1 denotes the highest of $n-1$ independently drawn values

First price auctions : examples

- values are uniformly distributed on $[0,1]$
  $F(x) = x$, then $G(x) = x^{n-1}$

$$\beta^I(x) = \frac{n-1}{n}x$$

- values are exponentially distributed on $[0,+\infty[$ and there are only two bidders
  $F(x) = 1 - e^{-\lambda x}$ for some $\lambda > 0$ and $n = 2$

$$\begin{aligned} \beta^I(x) &= x - \int_0^x \frac{F(y)}{F(x)} dy \\ &= \frac{1}{\lambda} - \frac{xe^{-\lambda x}}{1 - e^{-\lambda x}} \end{aligned}$$

So no bidder bids more than $\frac{1}{2}$!

Expected payment
The buyer pays what she bids, so we have again

$$m^I(x) = G(x) \times \mathbb{E}[Y_1 \mid Y_1 < x]$$

The revenue of the seller is the sum of the expected payment of the bidders, so the two auctions yield the *same* **revenue** for the seller.

One can actually derive that the expected revenue of the seller is

$$\mathbb{E}[Rev] = \mathbb{E}[Y_2],$$

where Y_2 denote the second highest of n independent draws from F.

N.B. *On average*, the revenues are the same, not on specific instances (in some cases, the revenue of I is always better than II, and vice versa for other cases).

Revenue equivalence principle

We say an auction is **standard** is the highest bidder is awarded the item.

Theorem

If the values are independent and identically distributed and all bidders are risk neutral, then **any** symmetric and increasing equilibrium of **any** standard auction such that the expected payment of a bidder with 0 valuation is 0 yields the **same expected revenue** to the seller.

example :
if the values are uniformly distributed in $[0,1]$,
the expected revenue is $\frac{n-1}{n+1}$.

➩ can be used to derive equilibrium strategy
(e.g. strategy for an all pay auction or third price auction)

Choosing the auction type

It depends on the risk attitude of the bidders/auctioneer

- risk neutral bidders : Revenue-equivalence Theorem (Vickrey 1961)
- risk averse bidders : Dutch and First-price sealed bid yield higher revenue
- risk averse auctioneer : Vickrey or English auctions are better

some variants :

- with entry cost
- with uncertain number of bidders

Lies and Collusion

- all four auctions are susceptible to collusion :
  groups of agent can form and decide on a low bid and share revenue later.
  ➯ Optimal when the grand coalition is formed
  ➯ Auctioneer can try to avoid that bidders identify other bidders
- auctioneer may cheat for Vickrey auction.
- auctioneer may place “fake” bids (known as shills)

Multi unit auctions

Multi unit auctions

Instead of selling a single resource, an action may sell n copies of that resource.

ex1 : 10 resources

- bidder A : 5 copies, 20 per copy or nothing
- bidder B : 3 copies or less for 15 per copy
- bidder C : 5 copies for 15 per copy or nothing
- bidder D : 1 copy for 15

How to allocate the 10 copies ?

ex2 : n resources, bidders want only one item.

Bidder	A	B	C	D
Bid	25	20	15	8

- what price for each item ? (different or same ?)
- how many should the auctioneer actually sells ? (maybe an auctioneer is better off by selling less resources)

Examples

for scenario 2 with n resources for sale :

- best n bidders paying the price of the first loser
- best n bidders paying the price of the last winner
- run a sequence of single-resource auctions

For scenario 2, there is also a revenue equivalence theorem.

For scenario 1 :
we can determine the winners by choosing the social-welfare-maximising allocation if we know the valuation function of each agent i (for each number of copies, the function returns a value) $v_i : \{1...n\} \rightarrow \mathbb{R}$
⇨ it works by asking a lot of information to the bidders
⇨ one needs to specify a *language* for describing the valuation function

Winner determination in scenario 1 can be expressed as an *integer program*
⇨ computationally hard

Variants

- unlimited supplies (random sampling optimal price auction)
- sponsored search auctions
  not exactly multi units :
  - there are k spots to fill for advertisement
  - **but** the rank used for display play a role !

  Generalised first-price : the bidder with the highest j^{th} bid gets the j^{th} slot. If bidder i's ad receives a click, she pays the auctioneer b_i.
  ➾ not necessarily a pure equilibrium.

  Generalised second-price : the bidder with the highest j^{th} bid gets the j^{th} slot. If bidder i's ad is rank in slot j and receives a click, she pays the auctioneer b_{j+1}

Some sources

Yohav Shoham and Kevin Leyton Brown.
Multiagent Systems.
Cambridge University Press, 2009.

Chapter 11.1-11.2 auctions with single good, 11.3 Combinatorial auctions.

Vijay Krishna.
Auction theory.
Academic Press, 2010.

Books for economists, I used chapters 1, 2 and 3 (out of 18).

- Chapter 14 of *Multiagent Systems* by Michael Wooldridge (2009) is also about auctions, I find it a bit less advanced and less complete than the chapter by Shoham and Leyton-Brown.
- Shoham and Leyton-Brown also wrote chapter 7 "Mechanism Design and Auctions in the book *Multiagent Systems* edited by Gerhard Weiss (2013), but contains less information than the chapter in their own book