Proof Theory: From the Foundations of Mathematics to Applications in Core Mathematics

Ulrich Kohlenbach
Department of Mathematics
Technische Universität Darmstadt

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Background: David Hilbert’s Program

Since the 19th century, noneffective (set-theoretic) principles became increasingly important. The issue of their legitimacy led Hilbert to the program:

Establish that uses of these higher noneffective/transfinite (,,ideal") principles in proofs of combinatorial/finitistic (,,real") propositions can be eliminated, at least in principle.

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Show the consistency of I by finitistic means.
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**In particular:** Show the consistency of $\mathcal{I}$ by finitistic means.
Impossibility of the program (in the narrow sense)

**Theorem** [K. Gödel 1931]

For no nontrivial consistent theory $\mathcal{T}$ is it possible to prove the consistency of $\mathcal{T}$ in $\mathcal{T}$ itself.
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Reduce the consistency of a theory $\mathcal{T}_1$ to that of a prima facie more constructive theory $\mathcal{T}_2$. 
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In ordinary mathematics: the “Gödel Phenomenon” is extremely rare. Usually, “ideal” principles can be replaced by suitable more elementary ones. However: this can be very difficult to accomplish.
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‘What more do we know if we have proved a theorem by restricted means than if we merely know that it is true? (G. Kreisel, 50’s)
Extractive Proof Theory (G. Kreisel):
New results by logical analysis of proofs

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Goal: Additional information on $C$:
- quantitative information: explicit effective bounds,
- qualitative aspects:
  - existence of bound that is independent from certain parameters,
  - generalizations of proofs: weakening of premises.
Let $T_1$ and $T_2$ be theories with languages $L(T_1)$ and $L(T_2)$.
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**Interpret** propositions $A$ from $\mathcal{L}(\mathcal{T}_1)$ (inductively over the logical structure of $A$) by propositions $A^I$ from $\mathcal{L}(\mathcal{T}_2)$.
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Proof Interpretations

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**Central Method:** Modern extensions of Gödel’s 1958 (developed as part of modified Hilbert program) **Functional (‘Dialectica’) Interpretation**! Uses embedding into systems based in **intuitionistic logic** (Brouwer).
“Proof Mining” in core mathematics

During the last 20 years this proof-theoretic approach has resulted in numerous new quantitative results as well as qualitative uniformity results in particular in: nonlinear analysis, fixed point theory, ergodic theory, topological dynamics, approximation theory etc.

General logical metatheorems explain these applications as instances of logical phenomena (K. 2005, Gerhardy/K. 2008, TAMS).

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If $A$ is existential, then general logical metatheorems (K. 2005) guarantee the extractability of effective bounds on ‘$\exists$’ that are independent from parameters $x$ from compact metric spaces $K$ (if separability is used) and metrically bounded subsets of abstract spaces $X$ that are not assumed to be separable (provided $X$ belongs to a sufficiently uniformly axiomatizable class of spaces).

Examples of such spaces $X$: metric, normed, Hilbert, uniformly convex, uniformly smooth, hyperbolic, CAT(0) spaces (but e.g. not separable, strictly convex or uniformly Gateaux differentiable spaces).
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Often a third approach to the foundations of mathematics

- **Formalism** (Hilbert)

is discussed as yet another option.
Mental constructions ‘by abandoning Kant’s a-priority of space but adhering more resolutely to the a-priority of time’ (Brouwer 1912). ‘Intuitionistic mathematics is an essentially language-less mental structure which comes into being by the self-unfolding of the two-ity as the Primordial Intuition’ (Brouwer 1947)
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- Ontological solipsism: ‘the only thing that is real to me is my own self at this moment, surrounded by a wealth of images in which the self believes and which make the self live. ... A second reality, independent of my self and corresponding to these images, is out of the question.’ (Brouwer 1898).
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- Logic is secondary to mathematics.

- Law-of-excluded middle on infinite domains in general meaningless.
Er is positief bewijs!

100 jaar na dato wordt de wiskundige L.E.J. Brouwer (1881-1966) geëerd met een eigen postzegel.

in 2007 is het 100 jaar geleden dat L.E.J. Brouwer (1881 – 1966) de stelling van Aristoteles verwierp. Brouwer vond dat een wiskundige stelling pas waar is als er "positief bewijs" is. Brouwer is de grondlegger van de intuitionistische wiskunde. Naar hem is o.a. de deeltjesstelling vernoemd. Iedere drie jaar reikt het Koninklijk Wiskundig Genootschap de Brouwer medaille uit aan een belangrijk wiskundige. Voor meer informatie: www.wiswa.nl
Both platonism and intuitionism have a **metaphysical** and a **formalistic** reading.
Interludium continued

- Both platonism and intuitionism have a **metaphysical** and a **formalist** reading.

- Formalism is a way of viewing the **ideal elements** (Hilbert) of a metaphysical reading of platonism/intuitionism as mere **regulative ideas** (Kant) that are useful in proving facts about **real statements** (Hilbert) of mathematics rather than as real objects themselves.
The metaphysical readings of these philosophies have had important impact on modern mathematics and computer science:

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- Platonism $\rightarrow$ (naive) set theory as a universal ontology of mathematics.
- Intuitionism $\rightarrow$ Mathematics as mental constructions $\rightarrow$ (constructive) mathematics as a high level programming language: ‘Proof as Programs’ (Curry-Howard Correspondence), Computational Mathematics.
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- **Platonism** → (naive) set theory as a universal ontology of mathematics.
- **Intuitionism** → Mathematics as mental constructions → (constructive) mathematics as a high level programming language: ‘Proof as Programs’ (Curry-Howard Correspondence), Computational Mathematics.

We will discuss

- **Mathematical uses** of a formalistic reading of platonism/intuitionism.
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Finitism
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‘Proof Mining’: Applied Foundations

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- A kind of ‘finitistic’ analysis (relative to the ideal elements) of the latter by exhibiting explicit constructions hidden in the use of quantifiers.
- The finitistic use of the ‘ideal elements’ makes it possible to replace them by finitary versions (without the proof noticing the difference!).
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This in the end gives some form of *finitistic realization of \( \forall \exists \)-theorems* proven together with an elementary proofs.
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For semi-intuitionistic proofs the restriction to ∀∃-theorem can be overcome: arbitrary logical complexity allowed!
The running theme: convergence statements in analysis

Let $(x_n)$ be a Cauchy sequence in a metric space $(X, \rho)$, i.e.

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\forall k \in \mathbb{N} \ \exists n \in \mathbb{N} \ \forall i, j \geq n (\rho(x_i, x_j) \leq 2^{-k}) \in \forall \exists \forall
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is noneffectively equivalent to

\[
\forall k \in \mathbb{N} g \in \mathbb{N}^{\mathbb{N}} \exists n \in \mathbb{N} \forall i, j \in [n; n + g(n)] (\rho(x_i, x_j) < 2^{-k}) \in \forall \exists
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Herbrand normal form or metastability (Tao).
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A bound \(\Phi(k, g)\) on ‘\(\exists n\)’ in the latter formula is a rate of metastability (introduced by Kreisel in 1951 as no-counterexample interpretation).
Effective full rates of convergence?

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  \[ \rho(x_n, f(x_n)) \to 0, \]
  even when \((x_n)\) may not converge to a fixed point of \(f\).
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- Possible for \((x_n)\) if sequence converges to **unique** fixed point/solution.

Extraction of modulus of uniqueness \(\Phi:\mathbb{IR}^* \to \mathbb{IR}^*\):\]

\[ \forall \varepsilon > 0, \quad \forall x, y \in X, \quad \rho(x, f(x)), \rho(y, f(y)) < \Phi(\varepsilon) \rightarrow \rho(x, y) < \varepsilon \]

gives rate of convergence (or – in the noncompact case – existence at all)! Numerous applications in analysis!
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Rates of Asymptotic Regularity
Asymptotic regularity of pseudocontractions

Let $X$ be a Banach space, $C \subset X$ a bounded convex subset and $f : C \to C$ a (Lipschitzian) pseudocontraction (F.E. Browder), i.e.

\[ \forall u, v \in C \forall \lambda > 1 \left( \left\| (\lambda I - f)(u) - (\lambda I - f)(v) \right\| \leq (\lambda - 1) \left\| u - v \right\| \right). \]

This generalizes the class of nonexpansive mappings.

$A := I - f$ is accretive (and dissipative) which in Hilbert space is equivalent to being monotone.

\[ \forall u, v \in C \left( \langle Au - Av, u - v \rangle \geq 0 \right). \]

In 1974 Bruck considered the following iteration schema

\[ x_{n+1} := (1 - \lambda_n)x_n + \lambda_n f(x_n) - \lambda_n \theta_n (x_n - x_1), \]

for suitable $(\lambda_n)$, $(\theta_n)$ in $(0, 1]$, and showed asymptotic regularity and strong convergence (towards a fixed point) results in Hilbert space.
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Asymptotic regularity for Lipschitzian pseudocontractions in arbitrary Banach spaces

Theorem (Chidume, Zegeye 2004): \( \lim_{n \to \infty} \| x_n - f(x_n) \| = 0 \), where

\( \begin{align*}
\text{(i) } & \lim \theta_n = 0, \\
\text{(ii) } & \sum_{n=1}^{\infty} \lambda_n \theta_n = \infty, \\
\text{(iii) } & \lim \frac{\lambda_n}{\theta_n} = 0, \\
\text{(iv) } & \lim \frac{\theta_{n-1} - 1}{\theta_n} = 0, \\
\text{(v) } & \lambda_n (1 + \theta_n) \leq 1.
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(ii) \( \sum_{n=1}^{\infty} \lambda_n \theta_n = \infty \),
(iii) \( \lim \frac{\lambda_n}{\theta_n} = 0 \),
(iv) \( \lim \frac{\theta_n - 1}{\lambda_n \theta_n} = 0 \),
(v) \( \lambda_n (1 + \theta_n) \leq 1 \).

Let \( M \geq \text{diam}(C) \) and \( (\lambda_n), (\theta_n) \subset (0, 1] \) with rates of conv./div.

\( R_i : (0, \infty) \to \mathbb{N} \)

1. \( \forall \varepsilon > 0 \forall n \geq R_1(\varepsilon) (\theta_n \leq \varepsilon) \),
2. \( \forall x \in (0, \infty) \left( \sum_{n=1}^{R_2(x)} \lambda_n \theta_n \geq x \right) \),
3. \( \forall \varepsilon > 0 \forall n \geq R_3(\varepsilon) (\lambda_n \leq \theta_n \varepsilon) \),
4. \( \forall \varepsilon > 0 \forall n \geq R_4(\varepsilon) \left( \left| \frac{\theta_n - 1}{\lambda_n \theta_n} \right| \leq \varepsilon \right) \).
Rate of convergence extracted from Chidume/Zeggeye (2004):

\[ \forall \epsilon > 0 \, \forall n \geq \Psi(M, L, R_1, R_2, R_3, R_4, \epsilon) \left( \|x_n - f(x_n)\| < \epsilon \right) \]

where

\[ \Psi(M, L, R_1, R_2, R_3, R_4, \epsilon) = \max\{N_2(C) + 1, R_1(\epsilon^2 M) + 1\} \]

and

\[ N_1(\epsilon) := \max\{R_3(\epsilon^4 M^2 (2 + L)), R_4(\sqrt{\epsilon} M^2 + 1 - 1)\} \]

\[ N_2(x) := R_2(x^2) + 1, \]

\[ C := 8 (1 + L)^2 M^2 \epsilon^2 + 2 \left( N_1(\epsilon^2 8 (1 + L)^2) - 1 \right) R_2(\epsilon^4 M^2 (2 + L)) . \]
Rate of convergence extracted from Chidume/Zegye (2004):

**Theorem (D. Körnlein/K. Nonlinear Analysis 2011)**

\[ \forall \epsilon > 0 \forall n \geq \psi (M, L, R_1, R_2, R_3, R_4, \epsilon) \left( \| x_n - f x_n \| < \epsilon \right) \]
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\[
\Psi (M, L, R_1, R_2, R_3, R_4, \varepsilon) = \max \{ N_2 (C) + 1, R_1 \left( \frac{\varepsilon}{2M} \right) + 1 \}
\]

and

\[
N_1 (\varepsilon) := \max \left\{ R_3 \left( \frac{\varepsilon}{4M^2 (2 + L)} \right), R_4 \left( \sqrt{\frac{\varepsilon}{M^2}} + 1 - 1 \right) \right\},
\]

\[
N_2 (x) := R_2 \left( \frac{x}{2} \right) + 1,
\]

\[
C := \frac{8 (1 + L)^2 M^2}{\varepsilon^2} + 2 \left( N_1 \left( \frac{\varepsilon^2}{8 (1 + L)^2} \right) - 1 \right).
\]
Rates of Full Convergence in Case of Uniqueness
Example 1: Kirk’s theorem for asymptotic contractions

Definition (Kirk JMAA03)

$(X, d)$ metric space. $f : X \to X$ is an **asymptotic contraction** with moduli $\Phi, \Phi_n : [0, \infty) \to [0, \infty)$ if $\Phi, \Phi_n$ are continuous, $\Phi(s) < s$ for all $s > 0$ and

$$
\forall n \in \mathbb{N} \forall x, y \in X (d(f^n(x), f^n(y)) \leq \Phi_n(d(x, y)),
$$

and $\Phi_n \to \Phi$ uniformly on the range of $d$. 

Theorem (Kirk JMAA03)

$(X, d)$ complete metric space, $f : X \to X$ continuous asymptotic contraction with some orbit bounded. Then $f$ has a unique fixed point $p \in X$ and $(f^n(x_0))$ converges to $p$ for each $x_0 \in X$.

(Proof uses ultrapower structures and hence the axiom of choice!)
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• Using Gerhardy’s result, E.M.Briseid (JMAA 2007) constructed an effective **full rate of convergence**.

\[(f_n(x_0))\] is redundant to assume: rate of convergence using only \[b \geq d(x, f(x))\] (Fixed Point Theory 2007, Int. J. Math. Stat. 2010).

E.M.Briseid showed that for bounded metric spaces the existence of \(x_0\)-uniform rate of convergence implies that \(f\) is asymptotically contractive (JMAA 2007). Also: new uniformity results generalizing Reich et al (2007).
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Example 2: Generalized $p$-contractions

**Definition:** [Rhoades 1977] $(X, d)$ metric space and $p \in \mathbb{N}$.

$f : X \to X$ is called **generalized $p$-contractive** if

$$\forall x, y \in X (x \neq y \Rightarrow d(f^p(x), f^p(y)) < \text{diam} \{x, y, f^p(x), f^p(y)\}).$$

Theorem: [Kincses/Totik 1990] $(K, d)$ compact metric space and $f : K \to K$ continuous and generalized $p$-contractive for some $p \in \mathbb{N}$.

Then $f$ has a unique fixed point $\xi$ and for every $x \in K \lim_{n \to \infty} f^n(x) = \xi.$
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**Theorem:** [Kincses/Totik 1990]

$(K, d)$ **compact** metric space and $f : K \to K$ continuous and generalized $p$-contractive for some $p \in \mathbb{N}$. Then $f$ has a unique fixed point $\xi$ and for every $x \in K$ \(\lim_{n \to \infty} f^n(x) = \xi\).
**Definition:** [Briseid, J. Nonlinear Convex Anal. 2008]

A $(X, d)$ metric space, $p \in \mathbb{N}$. $f : X \to X$ is called **uniformly generalized $p$-contractive** with modulus $\eta : \mathbb{Q}_+^* \to \mathbb{Q}_+^*$ if for all $x, y \in X$, $\varepsilon \in \mathbb{Q}_+^*$

$$d(x, y) > \varepsilon \rightarrow d(f^p(x), f^p(y)) + \eta(\varepsilon) < \text{diam} \{x, y, f^p(x), f^p(y)\}.$$
New existence without compactness

Theorem (Briseid, J. Nonlinear Convex Anal. 2008)

\((X, d)\) \textbf{complete} metric space and \(p \in \mathbb{N}\). \(f : X \to X\) be a \textbf{uniformly} continuous and \textbf{uniformly} generalized \(p\)-contractive with moduli \(\omega, \eta\). Let \((f^n(x_0))\) be bounded by \(b \in \mathbb{Q}^*_+\). Then \(f\) has a unique fixed point \(\xi\) and \((f^n(x_0))\) converges to \(\xi\) with rate of convergence \(\Phi : \mathbb{Q}^*_+ \to \mathbb{N}\),

\[
\Phi(\varepsilon) := \begin{cases} 
 p\left[\frac{(b - \varepsilon)}{\rho(\varepsilon)}\right] & \text{if } b > \varepsilon, \\
 0, & \text{otherwise}
\end{cases}
\]

with

\[
\rho(\varepsilon) := \min \left\{ \eta(\varepsilon), \frac{\varepsilon}{2}, \eta\left(\frac{1}{2} \omega^p(\frac{\varepsilon}{2})\right) \right\}.
\]
New existence without compactness

**Theorem (Briseid, J. Nonlinear Convex Anal. 2008)**

$(X, d)$ complete metric space and $p \in \mathbb{N}$. $f : X \to X$ be a **uniformly** continuous and **uniformly** generalized $p$-contractive with moduli $\omega, \eta$. Let $(f^n(x_0))$ be bounded by $b \in \mathbb{Q}^*_+$. Then $f$ has a unique fixed point $\xi$ and $(f^n(x_0))$ converges to $\xi$ with rate of convergence $\Phi : \mathbb{Q}^*_+ \to \mathbb{N}$,

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**Existence** of fixed point by **uniform uniqueness** via **uniform continuity** and **uniform generalized $p$-contractivity!**
Fluctuation Bounds
An Example from Ergodic Theory

$X$ Hilbert space, $f : X \to X$ linear and $\|f(x)\| \leq \|x\|$ for all $x \in X$.

$$A_n(x) := \frac{1}{n+1} S_n(x), \text{ where } S_n(x) := \sum_{i=0}^{n} f^i(x) \quad (n \geq 0)$$
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Theorem (von Neumann Mean Ergodic Theorem)

For every $x \in X$, the sequence $(A_n(x))_n$ converges.
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Avigad/Gerhardy/Towsner (TAMS 2010):

in general no computable rate of convergence.

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**Theorem (Garrett Birkhoff 1939)**

Mean Ergodic Theorem holds for uniformly convex Banach spaces.

**Theorem (K./Leuștean, Ergodic Theor. Dynam. Syst. 2009)**

Let $X$ be a uniformly convex Banach space, $\eta$ a modulus of uniform convexity, and $f : X \to X$ as above, $b > 0$. Then for all $x \in X$ with $\|x\| \leq b$, all $\varepsilon > 0$, all $g : \mathbb{N} \to \mathbb{N}$:

$$\exists n \leq \Phi(\varepsilon, g, b, \eta) \forall i, j \in [n; n + g(n)] (\|A_i(x) - A_j(x)\| < \varepsilon),$$

where

$$\Phi(\varepsilon, g, b, \eta) := M \cdot \tilde{h}(K)(0),$$

with

$$M := \lceil 16b\varepsilon \rceil, \quad \gamma := \varepsilon \frac{1}{16} \eta\left(\frac{b}{8}\varepsilon\right),$$

$$K := \lceil b\gamma \rceil,$$

$$h(n) := 2(Mn + g(Mn)),$$

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$$M := \left\lceil \frac{16b}{\varepsilon} \right\rceil, \gamma := \frac{\varepsilon}{16} \eta \left(\frac{\varepsilon}{8b}\right), \quad K := \left\lceil \frac{b}{\gamma} \right\rceil,$$

$$h, \tilde{h} : \mathbb{N} \to \mathbb{N}, \quad h(n) := 2(Mn + g(Mn)), \quad \tilde{h}(n) := \max_{i \leq n} h(i).$$

**Computable rate of convergence** iff the **norm of limit is computable**!
Bounding the number of fluctuations

We say that \((x_n)\) admits \(k\) \(\varepsilon\)-fluctuations if there are \(i_1 \leq j_1 \leq \ldots i_k \leq j_k\) s.t. \(\|x_{j_n} - x_{i_n}\| \geq \varepsilon\) for \(n = 1, \ldots, k\).
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As a corollary to our analysis of Birkhoff’s proof, Avigad and Rute showed

**Theorem (Avigad, Rute (2012))**

\((A_n(x))\) admits at most

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2 \log(M) \cdot \frac{b}{\varepsilon} + \frac{b}{\gamma} \cdot 2 \log(2M) \cdot \frac{b}{\varepsilon} + \frac{b}{\gamma}
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Partly possible because Birkhoff’s proof only uses boundedly many (in the data) instances of the law-of-excluded-middle for \(\exists\)-statements!
Bounds on Metastability
Strong nonlin. ergodic theorem (Wittmann)

Halpern iterations:

\[ f : C \to C \text{ nonexpansive}, \quad x_0 \in C, \quad \alpha_n \in [0, 1], \quad x_{n+1} = \alpha_{n+1} x_0 + (1 - \alpha_{n+1}) f(x_n). \]

Using weak compactness, Wittmann proved in 1992:

Theorem (R. Wittmann 1992):

\[ C \subseteq X \text{ closed and convex}, \quad x_0 \in C \text{ and } \text{Fix}(f) \neq \emptyset. \]

Under suitable conditions on \((\alpha_n)\) (e.g. \(\alpha_n = \frac{1}{n+1}\)) \((x_n)\) converges strongly towards the fixed point of \(f\) that is closest to \(x_0\).

Remark: Wittmann's theorem is a nonlinear generalization of the Mean ergodic theorem: for \(\alpha_n = \frac{1}{n+1}\), \(C = X\) and linear \(f\), the Halpern iteration coincides with the Cesàro means.
**Halpern iterations:** $f : C \rightarrow C$ nonexpansive, $x_0 \in C$, $\alpha_n \in [0, 1]$

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**Strong nonlin. ergodic theorem (Wittmann)**

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General features of the logical analysis of the proof due Wittmann

In Wittmann's proof the use of weak compactness gets in the end eliminated via a quantitative projection argument but could be fully effectivized (K., Adv. Math. 2011).

Wittmann's result has been generalized to uniformly smooth Banach spaces (Shioji-Takahashi 1997) and CAT(0)-spaces (Saejung 2010) using Banach limits (AC).
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General features of the logical analysis of the proof due Wittmann

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- Wittmann’s result has been generalized to uniformly smooth Banach spaces (Shioji-Takahashi 1997) and CAT(0)-spaces (Saejung 2010) using Banach limits (AC).

\( X \subset CAT(0), C \subset X \) convex, \( diam(C) \leq M \), \((x_n)\) as above, \( \varepsilon \in (0, 1) \):

\[
\forall g : \mathbb{N}^\mathbb{N} \exists k \leq \Sigma(\varepsilon, g, M) \forall i, j \in [k; k + g(k)] \ (\rho(x_i, x_j) \leq \varepsilon),
\]
Rate of Metastability in Saejung’s Theorem


Let $X \subseteq \text{CAT}(0)$, $C \subseteq X$ be convex, $\text{diam}(C) \leq M$, $(x_n)$ as above, $\varepsilon \in (0, 1)$:

$$\forall g : \mathbb{N}^\mathbb{N} \exists k \leq \Sigma(\varepsilon, g, M) \forall i, j \in [k; k + g(k)] \ (\rho(x_i, x_j) \leq \varepsilon),$$

$$\Sigma(\varepsilon, g, M) := \left\lceil \frac{12M^2 \chi^*_L(\varepsilon^2/12)}{\varepsilon^2} \right\rceil, \text{ with } L := h^*_\left(\left\lceil \frac{M^2}{\varepsilon^2_0} \right\rceil \right)(0) + \left\lceil \frac{1}{\varepsilon_0} \right\rceil,$$

\[ X \in \text{CAT}(0), C \subseteq X \text{ convex}, \text{diam}(C) \leq M, (x_n) \text{ as above}, \varepsilon \in (0, 1) : \]

\[ \forall g : \mathbb{IN}^\mathbb{IN} \exists k \leq \Sigma(\varepsilon, g, M) \forall i, j \in [k; k + g(k)] (\rho(x_i, x_j) \leq \varepsilon), \]

\[ \Sigma(\varepsilon, g, M) := \left\lceil \frac{12M^2 \chi^*_k(\varepsilon^2/12)}{\varepsilon^2} \right\rceil, \text{ with } L := \tilde{h}^*(\lceil M^2/\varepsilon^2 \rceil)(0) + \left\lceil \frac{1}{\varepsilon_0} \right\rceil, \]

\[ \chi^*_k(\varepsilon) := \left\lfloor \frac{12M^2(k+1)}{\varepsilon} + \frac{144M^4(k+1)^2}{\varepsilon^2} \right\rfloor - 1, \varepsilon_0 := \varepsilon^2/24(d + 1)^2, \]

\[ \Theta_k(\varepsilon) := \left\lfloor \frac{3M^2 \chi^*_k(\varepsilon/3)}{\varepsilon} \right\rfloor, \Delta^*_k(\varepsilon, g) := \frac{\varepsilon}{3g_{\varepsilon, k}(\Theta_k(\varepsilon) - \chi^*_k(\varepsilon/3))}, \]

\[ g_{\varepsilon, k}(n) := n + g(n + \chi^*_k(\varepsilon/3)), \quad h(k) := \max \left\{ \left\lceil \frac{M^2}{\Delta^*_k(\varepsilon^2/4, g)} \right\rceil, k \right\} - k, \]

\[ h^*(k) := h\left(k + \left\lceil \frac{1}{\varepsilon_0} \right\rceil\right) + \left\lceil \frac{1}{\varepsilon_0} \right\rceil, \tilde{h}^*(k) := k + h^*(k). \]
Further consequences of the analysis
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- A **quadratic** full rate of convergence for the asymptotic regularity $\rho(x_n, f(x_n)) \to 0$:

$$\forall \varepsilon > 0 \forall n \geq \frac{4M}{\varepsilon} + \frac{16M^2}{\varepsilon^2} \ (\rho(x_n, f(x_n)) < \varepsilon).$$
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- Let $z_k$ be the unique fixed point of the contraction $f_k(x) := \frac{1}{k} x \ominus (1 - \frac{1}{k})f(x)$.

Then the analysis yields (essentially) polynomially in a given rate $\alpha$ of convergence of the **resolvent** $(z_k)$ a rate of convergence of $(x_n)$. 
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- Let \( z_k \) be the unique fixed point of the **contraction**
  \[ f_k(x) := \frac{1}{k} x \oplus (1 - \frac{1}{k})f(x). \]

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  of convergence of the **resolvent** \( (z_k) \) a rate of convergence of \( (x_n) \).

For effective \( X, f, x \) and \( z := \lim z_k \):

**Computable convergence for \( (x_n) \) iff \( \|z - x\| \) is computable.**
Further consequences of the analysis

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For effective $X, f, x$ and $z := \lim z_k$:

**Computable convergence for** $(x_n)$ **iff** $\|z - x\|$ **is computable.**

Also unbounded $C$ and modified Halpern iterations: $M \geq 4\|p - x_0\|$

(Schade/K., Fixed Point Theory Appl. 2012)
Other developments

Similar results for uniformly smooth Banach spaces (then bound depends on modulus of smoothness): Leustean/K., Phil. Trans. Royal Soc. A, 2012. This involves new logical metatheorem for uniformly smooth spaces and with uniformly continuous normalized duality map. Another strong nonlinear ergodic theorem due to Wittmann for mappings $\|f(x) + f(y)\| \leq \|x + y\|$ has been analyzed by P. Safarik, J. Math. Anal. Appl. 2012.

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• Rates of **asymptotic regularity** for **asymptotically nonexpansive mappings** in uniformly convex normed and $W$-hyperbolic spaces (Lambov/K., Proc. FPTA 2004, Leuştean/K., JEMS 2010).


Approximate fixed point property in **product spaces** (Leuştean/K., Nonlinear Anal.2007).


Rate of asymptotic regularity for Bruck’s iteration of Lipschitzian pseudocontractions in Hilbert space (Körnlein/K., 2013 submitted).
Philosophical conclusion from ‘proof mining’ enterprise

Rather than having ‘collapsed’ (S. Kripke), Hilbert’s program (in the broad sense) on the possibility of elimination of ideal elements from proofs of real statements has largely been vindicated for ordinary mathematical practice. To a large extent this is even true for the narrow interpretation of the program, i.e. the reduction to strict finitism: primitive recursive arithmetic: All bounds in this talk were primitive recursive. Striking exceptions exist (H. Friedman) but are very rare.
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Two advertisements

Tokyo-Darmstadt International PhD School on Mathematical Fluid Dynamics.
Likely to get a 2nd 4.5 year funding period including Proof Mining/Reverse Math. Projects.

Modulo final approval in May: Announcement of PhD positions for proof-theorists in this project this summer!

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Applied Proof Theory: Proof Interpretations and their Use in Mathematics

Ulrich Kohlenbach presents an applied form of proof theory that has led in recent years to new results in number theory, approximation theory, nonlinear analysis, geodesic geometry and ergodic theory (among others). This applied approach is based on logical transformations (so-called proof interpretations) and concerns the extraction of effective data (such as bounds) from *prima facie* ineffective proofs as well as new qualitative results such as independence of solutions from certain parameters, generalizations of proofs by elimination of premises.

The book first develops the necessary logical machinery emphasizing novel forms of Gödel's famous functional („Dialectica“) interpretation. It then establishes general logical metatheorems that connect these techniques with concrete mathematics. Finally, two extended case studies (one in approximation theory and one in fixed point theory) show in detail how this machinery can be applied to concrete proofs in different areas of mathematics.