

# Structuralism, Invariance, and Univalence

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- ▶ Voevodsky has introduced a new axiom with both geometric and logical significance: the Univalence Axiom.
- ▶ UA captures a familiar aspect of informal mathematical practice: one can identify isomorphic objects.
- ▶ It is incompatible with conventional foundations, but is a powerful addition to homotopy type theory.

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- ▶ The Cauchy reals are isomorphic to the Dedekind reals, so these are *the same* number field,  $\mathbb{R}$ .
- ▶ the unit interval  $[0, 1]$  is homeomorphic to the closed interval  $[0, 2]$ , so these are *the same* topological space, say  $I$ .

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The group of automorphisms of  $\mathbb{Z}$  consists of functions, not numbers. The Cauchy reals are sequences of rationals, i.e. functions  $\mathbb{N} \rightarrow \mathbb{Q}$ , and the Dedekind reals are subsets of rationals. Mathematical objects are often constructed out of other ones, and so also have some residual structure: i.e. the real number field actually consisting of functions or of subsets. So there is a clear sense in which (PS) is simply false.



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We may want to restrict the properties that can occur on the right, in order for this to really be a weaker the condition.

So we have the reformulation:

$$A \cong B \Leftrightarrow \text{for all relevant properties } P, P(A) \Rightarrow P(B) \quad (\text{PS}')$$

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Let us consider the terms of (PS) in turn.



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The structure of  $A$  is the same as the structure of  $B$  just in case  $A$  and  $B$  are isomorphic. So (1) is true, but it's the definition of "structure" in terms of isomorphism, not the other way around.

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such that

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This makes reference to “structure-preserving maps”, “composition”  $\circ$ , and “identity” maps  $1_A$ , but these are primitive concepts of category theory, and need not be further defined. Note that “isomorphism” is always relative to a given category, which determines a kind of structure.

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But there are lots of other examples where the notion of morphism is given directly, and that then determines the corresponding notion of “structure” – e.g. “differentiable structure” or “smooth structure”.

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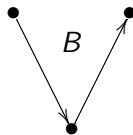
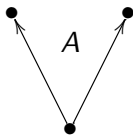
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Any functor preserves isomorphisms, so we can distinguish non-isomorphic structures by taking them to non-isomorphic objects by some functor.

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Since  $\text{Hom}(I, -)$  is a functor, it follows immediately that  $A$  and  $B$  cannot be isomorphic.

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That is, “isomorphic objects have all the same structural properties” (and conversely).

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One way to do this was functoriality. Let us consider another common way.

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The correspondence is called “Curry-Howard” or “Propositions-as-Types”. It is given by:

$$\text{term} : \text{Type} \quad \approx \quad \text{proof} : \text{Proposition}$$

The terms of a type can be regarded as the proofs of the corresponding proposition.

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The proof is a straightforward induction on the form of  $P(X)$ . We record this as the following *Principle of Invariance* for type theory:

All definable properties are isomorphism invariant. (PI)

## IV. Identity

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In *impredicative* type theory, one can define identity by Leibniz’s Law as:

$$A = B := \forall P. P(A) \rightarrow P(B),$$

But this is identity between terms  $A, B : X$  of some type  $X$ .

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So, more explicitly:

$$A =_X B := \forall P : \mathcal{P}(X). P(A) \rightarrow P(B),$$

which is not the relation we are looking for, i.e. the one between *types*  $X$  and  $Y$ .

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on each type  $X$ , extending the Curry-Howard correspondence. Its rules allow the inference,

$$\frac{a =_X b \quad P(a)}{P(b)}$$

for any predicate  $P(x)$  on  $X$ .

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If we add a universe  $U$  of all (small) types, and then it too has an identity relation satisfying:

$$\frac{A =_U B \quad P(A)}{P(B)}$$

for any definable property  $P(X)$  of types.

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We therefore have the converse of (PS), namely the expected statement:

Identical objects are isomorphic.



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Namely, taking

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we infer as before:

$$A \cong B \Rightarrow A =_U B,$$

i.e. the Principle of Structuralism.

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Is that really possible?

## V. What does “are” mean?

In type theory, every proposition is a type. For instance, corresponding to the proposition  $A \cong B$  we have the type of isomorphisms between  $A$  and  $B$ :

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Similarly, corresponding to the proposition  $A =_U B$  there is the *identity type*,

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## V. What does “are” mean?

In type theory, every proposition is a type. For instance, corresponding to the proposition  $A \cong B$  we have the type of isomorphisms between  $A$  and  $B$ :

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There is also a neat geometric interpretation of this type ...

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We just showed that there is a map,

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In particular, there is then a map coming back,

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which may reasonably be read “isomorphic objects are identical”.

This is just the Principle of Structuralism. It implies the inference we just doubted was even possible:

$$A \cong B \Rightarrow A =_U B.$$



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The Univalence Axiom is not a philosophical claim, but rather a precise mathematical axiom in a new system for the foundations of mathematics.

It has a more general form, namely that *equivalence of objects is equivalent to identity*,

$$\text{Eq}(A, B) \simeq \text{Id}(A, B) \quad (\text{UA})$$

where the notion of “equivalence” is a broad generalization of isomorphism that subsumes also logical equivalence, categorical equivalence, and homotopy equivalence.

## V. What does “are” mean?

Since the rules of identity permit substitution of identicals, one consequence of (UA) is the invariance schema that we called the Principle of Invariance:

$$\frac{A \cong B \quad P(A)}{P(B)}$$

but now for *all* properties  $P(X)$ .

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Thus even in this extended system, it still holds that *all definable properties are structural*.

But note that this requires that we also add the Univalence Axiom when we add identity and a universe.

## V. What does “are” mean?

In sum, (UA) is a new axiom of logic that not only makes sense of, but actually implies, the Principle of Structuralism:

Isomorphic objects are identical. (PS)

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Note: Given a sequence of univalent universes, the equivalence between equivalence and identity itself becomes an identity — so a consequence of UA is that “equivalence is identity”:

$$(A \simeq B) = (A = B)$$

This seems quite radical from a conventional point of view. It is incompatible with classical foundations in set theory. It is consistent with a type-theoretic foundation, and that is part of this remarkable new insight.



## VI. Background

From a foundational perspective, the Univalence Axiom is certainly radical and unexpected, but it is not entirely without historical precedent.

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The first edition of *Principia Mathematica* used an intensional type theory, but the axiom of reducibility implied that every function has an extensionally equivalent predicative replacement. This spoiled the interpretation of functions as expressions and the substitutional interpretation of quantification, what may be called the “syntactic interpretation”, as favored by Russell on the days when he was a constructivist.

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In the second edition, Russell states a new principle that he attributes to Wittgenstein: “a function can only occur in a proposition through its values”. He says that this justifies an axiom of extensionality, doing at least some of the work of the axiom of reducibility.

## VI. Background

Wittgenstein has noticed that all functions that can be explicitly defined are in fact extensional, and so one can consistently add the axiom of extensionality without destroying the syntactic interpretation.

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This is similar to our move: the Univalence Axiom is also a very general extensionality principle; it implies the extensionality laws:

$$\begin{aligned} p \leftrightarrow q &\longrightarrow p = q \\ \forall x(fx \leftrightarrow gx) &\longrightarrow f = g \end{aligned}$$

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$$\begin{aligned} p \leftrightarrow q &\longrightarrow p = q \\ \forall x(fx \leftrightarrow gx) &\longrightarrow f = g \end{aligned}$$

Since nothing in syntax violates UA, we can add it to the system and still maintain the good properties of syntax, like invariance under isomorphism.

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Various people have pursued this idea further in recent times:

1. McGee:  $\infty$ -FOL is invariant under all iso.s,
2. Feferman: FOL w/o = is invariant under all surj. homo.s,
3. Bonnay: FOL is invariant under all partial iso.s,
4. Awodey & Forssell: FOL is invariant under all continuous iso.s.

# Conclusion

The results of homotopy type theory presented here show that the entire system of constructive type theory is invariant under the much more general notion of *homotopy equivalence* – “same shape” – a much larger class of maps than Tarski’s bijections.

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The Univalence Axiom takes this insight,

$$A \simeq B \ \& \ P(A) \Rightarrow P(B),$$

i.e. “all properties are invariant”, and turns it into a simple and beautiful, new logical axiom:

$$(A \simeq B) \simeq (A = B).$$

So that “logical identity” is equivalent to equivalence.

# Conclusion

Finally, as a consequence of (UA), together with the very definition of structure (DS), we then have that two mathematical objects are identical iff they have the same structure:

$$\text{str}(A) = \text{str}(B) \Leftrightarrow A = B.$$

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In other words, mathematical objects *are* structures.  
Could there be a stronger formulation of structuralism?