Fully abstract semantics of $\lambda\mu$ in the π-calculus

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Summary

- We focus on an output-based interpretation of $\lambda\mu\mathbf{x}$, the $\lambda\mu$-calculus with explicit substitution (and thereby of λ, $\lambda\mathbf{x}$, and $\lambda\mu$), into the synchronous π-calculus – enriched with pairing – that have their origin in mathematical logic.

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- Our interpretation respects one-step reduction.
- Using a notion of type assignment for the π-calculus that uses the type constructor $\rightarrow$, we can show that all Curry-assignable types to λ and $\lambda\mu$-terms are preserved by the interpretations.

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- Our interpretation respects one-step reduction.
- Using a notion of type assignment for the π-calculus that uses the type constructor $\rightarrow$, we can show that all Curry-assignable types to λ and $\lambda\mu$-terms are preserved by the interpretations.
- (Non) Termination of reduction is preserved.
- We will define a number of (weak) equivalences for $\lambda\mu$ that all correspond, and are respected by our interpretation with respect to contextual equivalence on processes, in a fully abstract way.

Parigot's $\lambda\mu$

Terms: $M, N \;::=\; x \mid \lambda x.M \mid MN \mid \mu\alpha.[\beta]M$

Type assignment: $A, B \;::=\; \varphi \mid A{\rightarrow}B$

$$(Ax): \frac{}{\Gamma, x{:}A \vdash x : A \mid \Delta} \qquad (\mu): \frac{\Gamma \vdash M : B \mid \alpha{:}A, \Delta}{\Gamma \vdash \mu\alpha.[\beta]M : A \mid \beta{:}B{+}\Delta} \qquad \frac{\Gamma \vdash M : B \mid \alpha{:}A, \Delta}{\Gamma \vdash \mu\alpha.[\alpha]M : A \mid \Delta}$$

$$({\rightarrow}I): \frac{\Gamma, x{:}A \vdash M : B \mid \Delta}{\Gamma \vdash \lambda x.M : A{\rightarrow}B \mid \Delta} \qquad ({\rightarrow}E): \frac{\Gamma \vdash M : A{\rightarrow}B \mid \Delta \quad \Gamma \vdash N : A \mid \Delta}{\Gamma \vdash MN : B \mid \Delta}$$

Reduction: *logical* : $(\lambda x.M)N \;\rightarrow\; M[N/x]$

structural : $(\mu\alpha.[\beta]M)N \;\rightarrow\; \mu\gamma.([\beta]M[N{\cdot}\gamma/\alpha])$

where $M[N{\cdot}\gamma/\alpha]$ is the structural substitution, defined by

$$([\alpha]M)[N{\cdot}\gamma/\alpha] \;\triangleq\; [\gamma](M\,[N{\cdot}\gamma/\alpha])N$$

etc.

$\lambda\mu$x: $\lambda\mu$ with explicit substitution

We make both substitutions explicit:

$$M, N \;::=\; x \mid \lambda x.M \mid MN \mid M\langle x := N\rangle \mid \mu\alpha.[\beta]M \mid M\langle \alpha := N{\cdot}\gamma\rangle$$

Reduction $\rightarrow_{\mathbf{x}}$ rules change accordingly, as in, for example,

$$\begin{array}{rcl} x\langle x := L\rangle \rightarrow L & & M\langle x := L\rangle \rightarrow M \qquad x \notin \mathit{fv}(M) \\ (MN)\langle x := L\rangle & \rightarrow & M\langle x := L\rangle N\langle x := L\rangle \\ ([\alpha]M)\langle \alpha := N{\cdot}\gamma\rangle & \rightarrow & [\gamma](M\langle \alpha := N{\cdot}\gamma\rangle)\,N \\ (PQ)\langle \alpha := N{\cdot}\gamma\rangle & \rightarrow & P\langle \alpha := N{\cdot}\gamma\rangle\, Q\langle \alpha := N{\cdot}\gamma\rangle \end{array}$$

For type assignment, we add the rules

$$(\textit{T-cut}) : \frac{\Gamma, x{:}A \vdash M : B \mid \Delta \quad \Gamma \vdash N : A \mid \Delta}{\Gamma \vdash M\langle x := N\rangle : B \mid \Delta}$$

$$(\textit{C-cut}) : \frac{\Gamma \vdash M : C \mid \alpha{:}A{\rightarrow}B, \Delta \quad \Gamma \vdash N : A \mid \Delta}{\Gamma \vdash M\langle \alpha := N{\cdot}\gamma\rangle : C \mid \gamma{:}B{+}\Delta}$$

Explicit head reduction for $\lambda\mu\mathbf{x}$

We define *explicit head reduction* $\rightarrow_{\mathbf{x}\mathrm{H}}$ on $\lambda\mu\mathbf{x}$ as $\rightarrow_{\mathbf{x}}$, but for:

1. $$(PQ)\langle x := N\rangle \rightarrow (P\langle x := N\rangle\, Q)\langle x := N\rangle$$
   $$(PQ)\langle \alpha := N{\cdot}\gamma\rangle \rightarrow (P\langle \alpha := N{\cdot}\gamma\rangle\, Q)\langle \alpha := N{\cdot}\gamma\rangle$$

2. We add two substitution rules:

$$M\langle x := N\rangle\langle y := L\rangle \rightarrow M\langle y := L\rangle\langle x := N\rangle\langle y := L\rangle \qquad x = \mathit{hv}(M)$$
$$M\langle \alpha := N{\cdot}\gamma\rangle\langle \beta := L{\cdot}\delta\rangle \rightarrow M\langle \beta := L{\cdot}\delta\rangle\langle \alpha := N{\cdot}\gamma\rangle\langle \beta := L{\cdot}\delta\rangle \quad \alpha = \mathit{hn}(M)$$

3. We remove the contextual rules:

$$M \rightarrow N \;\Rightarrow\; \begin{cases} LM & \rightarrow\; LN \\ L\langle x := M\rangle & \rightarrow\; L\langle x := N\rangle \\ L\langle \alpha := M{\cdot}\gamma\rangle & \rightarrow\; L\langle \alpha := N{\cdot}\gamma\rangle \end{cases}$$

The last change makes $\rightarrow_{\mathbf{x}\mathrm{H}}$ a *head* reduction; it is not lazy, since we allow reduction under λ. Only the head variable of a term ever gets replaced; the substitution for the other occurrences is ‘pending’.

The π-calculus with pairing

Definition *Channel names* and *data* are defined by:

$$a, b, c, d, x, y, z \qquad \textit{names} \qquad\qquad p ::= a \mid \langle a,b \rangle \qquad \textit{data}$$

$$
\begin{array}{llllll}
P, Q ::= & 0 & \textit{Nil} & \mid & a(x).P & \textit{Input} \\
\mid & P \mid Q & \textit{Composition} & \mid & \overline{a}\langle p \rangle.P & \textit{Output} \\
\mid & !P & \textit{Replication} & \mid & \textit{let } \langle x,y \rangle = z \textit{ in } P & \textit{Let construct} \\
\mid & (\nu a)P & \textit{Restriction} & & &
\end{array}
$$

We write $a(y,z).P$ for $a(x).\textit{let } \langle y,z \rangle = x \textit{ in } P$, $\overline{a}\langle b,c \rangle.P$ for $\overline{a}\langle\langle b,c \rangle\rangle.P$, and $(\nu mn)P$ for $(\nu m)(\nu n)P$.

We extend congruence by: $\textit{let } \langle x,y \rangle = \langle a,b \rangle \textit{ in } R \equiv R[a/x, b/y]$.

We write $P \sim_{\mathrm{C}} Q$ (and call P and Q *contextually equivalent*): for all contexts $\mathrm{C}[\cdot]$, and for all n, $\mathrm{C}[P]\Downarrow n \iff \mathrm{C}[Q]\Downarrow n$, where $\mathrm{C}[P]\Downarrow n$ stands for $\mathrm{C}[P]$ will eventually *output* on n.

Logical translation of $\lambda\mu\mathbf{x}$ terms

$$
\begin{aligned}
\ulcorner x \lrcorner a &\triangleq x \rightarrow a \\
\ulcorner \lambda x.M \lrcorner a &\triangleq (\nu x b)\,(\ulcorner M \lrcorner b \mid \overline{a}\langle x,b\rangle) \\
\ulcorner MN \lrcorner a &\triangleq (\nu c)\,(\ulcorner M \lrcorner c \mid \; c(v,d).\,(\ulcorner v := N \lrcorner \mid \; d \rightarrow a)) \\
\ulcorner M \langle x := N\rangle \lrcorner a &\triangleq (\nu x)\,(\ulcorner M \lrcorner a \mid \ulcorner x := N \lrcorner\,) \\
\ulcorner x := N \lrcorner &\triangleq \,!\ulcorner N \lrcorner x
\end{aligned}
$$

This is the encoding of [vBV-CONCUR'09]; we name the implicit output of terms with a (where $x \rightarrow a$ stands for $x(w).\overline{a}\langle w\rangle$.)

Logical translation of $\lambda\mu\mathbf{x}$ terms

$$\begin{array}{rcl}
\ulcorner x \lrcorner a & \triangleq & x \rightarrow a \\
\ulcorner \lambda x.M \lrcorner a & \triangleq & (\nu x b)\,(\ulcorner M \lrcorner b \,|\, \overline{a}\langle x,b\rangle) \\
\ulcorner MN \lrcorner a & \triangleq & (\nu c)\,(\ulcorner M \lrcorner c \,|\; c(v,d).\,(\ulcorner v := N \lrcorner \,|\; d \rightarrow a)) \\
\ulcorner M\langle x := N\rangle \lrcorner a & \triangleq & (\nu x)\,(\ulcorner M \lrcorner a \,|\, \ulcorner x := N \lrcorner\,) \\
\ulcorner x := N \lrcorner & \triangleq & !\,\ulcorner N \lrcorner x \\
 & & \\
\ulcorner \mu\gamma.[\beta]M \lrcorner a & \triangleq & \\
\ulcorner M\langle \beta := N \cdot \gamma\rangle \lrcorner a & \triangleq &
\end{array}$$

The cases to add.

Logical translation of $\lambda\mu\mathbf{x}$ terms

$$
\begin{array}{rcl}
\ulcorner x \lrcorner\, a & \triangleq & x \rightarrow a \\
\ulcorner \lambda x.M \lrcorner\, a & \triangleq & (\nu xb)\,(\ulcorner M \lrcorner\, b \,|\, \overline{a}\langle x,b\rangle) \\
\ulcorner MN \lrcorner\, a & \triangleq & (\nu c)\,(\ulcorner M \lrcorner\, c \,|\; c(v,d).\,(\ulcorner v := N \lrcorner \;|\; d \rightarrow a)) \\
\ulcorner M\langle x := N\rangle \lrcorner\, a & \triangleq & (\nu x)\,(\ulcorner M \lrcorner\, a \,|\, \ulcorner x := N \lrcorner\,) \\
\ulcorner x := N \lrcorner & \triangleq & !\,\ulcorner N \lrcorner\, x \\
 & & \\
\ulcorner \mu\gamma.[\beta]M \lrcorner\, a & \triangleq & \ulcorner M[a/\gamma] \lrcorner\, \beta \\
\ulcorner M\langle \beta := N\cdot\gamma\rangle \lrcorner\, a & \triangleq &
\end{array}
$$

μ-abstraction and naming need no extra construction! : the π-calculus is ‘classical’ enough on its own.

Logical translation of $\lambda\mu\mathbf{x}$ terms

$$
\begin{array}{rcl}
\ulcorner x \lrcorner\, a & \triangleq & x \rightarrow a \\
\ulcorner \lambda x.M \lrcorner\, a & \triangleq & (\nu x b)\,(\ulcorner M \lrcorner\, b \,|\, \overline{a}\langle x,b\rangle) \\
\ulcorner MN \lrcorner\, a & \triangleq & (\nu c)\,(\ulcorner M \lrcorner\, c \,|\; c(v,d).\,(\ulcorner v := N \lrcorner \;|\; d \rightarrow a)) \\
\ulcorner M\langle x := N\rangle \lrcorner\, a & \triangleq & (\nu x)\,(\ulcorner M \lrcorner\, a \,|\, \ulcorner x := N \lrcorner\,) \\
\ulcorner x := N \lrcorner & \triangleq & !\,\ulcorner N \lrcorner\, x
\end{array}
$$

$$
\begin{array}{rcl}
\ulcorner \mu\gamma.[\beta]M \lrcorner\, a & \triangleq & \ulcorner M[a/\gamma] \lrcorner\, \beta \\
\ulcorner M\langle \beta := N\cdot\gamma\rangle \lrcorner\, a & \triangleq & (\nu \beta)\,(\ulcorner M \lrcorner\, a \,|\, \ulcorner \beta := N\cdot\gamma \lrcorner\,)
\end{array}
$$

Here $\ulcorner \beta := N\cdot\gamma \lrcorner$ needs to ‘feed’ the parts ‘called’ β in $\ulcorner M \lrcorner\, a$ with $\ulcorner N \cdot \gamma \lrcorner$, in the same way that $c(v,d).\,(\ulcorner x := N \lrcorner \,|\, d \rightarrow a)$ ‘feeds’ $\ulcorner M \lrcorner\, c$ with $\ulcorner N \cdot a \lrcorner$, so . . .

Logical translation of $\lambda\mu\mathbf{x}$ terms

$$
\begin{array}{rcl}
\ulcorner x \lrcorner\, a & \triangleq & x \to a \\
\ulcorner \lambda x.M \lrcorner\, a & \triangleq & (\nu x b)\,(\ulcorner M \lrcorner\, b \mid \overline{a}\langle x,b\rangle) \\
\ulcorner MN \lrcorner\, a & \triangleq & (\nu c)\,(\ulcorner M \lrcorner\, c \mid \ c(v,d).\,(\ulcorner v := N \lrcorner \mid \ d \to a)) \\
\ulcorner M\langle x := N\rangle \lrcorner\, a & \triangleq & (\nu x)\,(\ulcorner M \lrcorner\, a \mid \ulcorner x := N \lrcorner\,) \\
\ulcorner x := N \lrcorner & \triangleq & !\,\ulcorner N \lrcorner\, x
\end{array}
$$

$$
\begin{array}{rcl}
\ulcorner \mu\gamma.[\beta]M \lrcorner\, a & \triangleq & \ulcorner M[a/\gamma] \lrcorner\, \beta \\
\ulcorner M\langle \beta := N{\cdot}\gamma\rangle \lrcorner\, a & \triangleq & (\nu \beta)\,(\ulcorner M \lrcorner\, a \mid \ulcorner \beta := N{\cdot}\gamma \lrcorner\,) \\
\ulcorner \beta := N{\cdot}\gamma \lrcorner & \triangleq & !\,\beta(v,d).\,(\ulcorner v := N \lrcorner \mid !\, d \to \gamma)
\end{array}
$$

Notice that β might be used as name for more than one subterm of M, so we need to replicate the ‘explicit structural substitution’.

Logical translation of $\lambda\mu\mathbf{x}$ terms

$$
\begin{array}{rcl}
\ulcorner x \lrcorner\, a & \triangleq & x \rightarrow a \\
\ulcorner \lambda x.M \lrcorner\, a & \triangleq & (\nu x b)\,(\ulcorner M \lrcorner\, b \mid \overline{a}\langle x,b\rangle) \\
\ulcorner MN \lrcorner\, a & \triangleq & (\nu c)\,(\ulcorner M \lrcorner\, c \mid !c(v,d).\,(\ulcorner v := N \lrcorner \mid !d \rightarrow a)) \\
\ulcorner M\langle x := N\rangle \lrcorner\, a & \triangleq & (\nu x)\,(\ulcorner M \lrcorner\, a \mid \ulcorner x := N \lrcorner\,) \\
\ulcorner x := N \lrcorner & \triangleq & !\ulcorner N \lrcorner\, x \\
 & & \\
\ulcorner \mu\gamma.[\beta]M \lrcorner\, a & \triangleq & \ulcorner M[a/\gamma] \lrcorner\, \beta \\
\ulcorner M\langle \beta := N\cdot\gamma\rangle \lrcorner\, a & \triangleq & (\nu \beta)\,(\ulcorner M \lrcorner\, a \mid \ulcorner \beta := N\cdot\gamma \lrcorner\,) \\
\ulcorner \beta := N\cdot\gamma \lrcorner & \triangleq & !\beta(v,d).\,(\ulcorner v := N \lrcorner \mid !d \rightarrow \gamma)
\end{array}
$$

Since for $M \in \lambda\mu$, c might be used inside $\ulcorner M \lrcorner\, c$ as well, i.e. is not just the name of the term itself, we need to replicate also the $c(v,d).\,(\ulcorner v := N \lrcorner \mid d \rightarrow a)$, as well as the forwarder $d \rightarrow a$.

Logical translation of $\lambda\mu\mathbf{x}$ terms

$$
\begin{array}{rcl}
\ulcorner x \lrcorner\, a & \triangleq & x(u).\,!u \rightarrow a \\
\ulcorner \lambda x.M \lrcorner\, a & \triangleq & (\nu xb)\,(\ulcorner M \lrcorner\, b \mid \overline{a}\langle x,b\rangle) \\
\ulcorner MN \lrcorner\, a & \triangleq & (\nu c)\,(\ulcorner M \lrcorner\, c \mid !c(v,d).\,(\ulcorner v := N \lrcorner \mid !d \rightarrow a)) \\
\ulcorner M\langle x := N\rangle \lrcorner\, a & \triangleq & (\nu x)\,(\ulcorner M \lrcorner\, a \mid \ulcorner x := N \lrcorner) \\
\ulcorner x := N \lrcorner & \triangleq & !\,(\nu w)\ \overline{x}\langle w\rangle.\,\ulcorner N \lrcorner\, w
\end{array}
$$

$$
\begin{array}{rcl}
\ulcorner \mu\gamma.[\beta]M \lrcorner\, a & \triangleq & \ulcorner M[a/\gamma] \lrcorner\, \beta \\
\ulcorner M\langle \beta := N\cdot\gamma\rangle \lrcorner\, a & \triangleq & (\nu\beta)\,(\ulcorner M \lrcorner\, a \mid \ulcorner \beta := N\cdot\gamma \lrcorner) \\
\ulcorner \beta := N\cdot\gamma \lrcorner & \triangleq & !\beta(v,d).\,(\ulcorner v := N \lrcorner \mid !d \rightarrow \gamma)
\end{array}
$$

As is the case for Milner's translation and in contrast to the interpretation for the λ-calculus, a guard is placed on the replicated terms. This is to make sure that $(\nu x)\,(\ulcorner x := N \lrcorner) \sim_{\mathrm{c}} \mathit{0}$.

Logical translation of $\lambda\mu\mathbf{x}$ terms

$$
\begin{aligned}
\llbracket x \rrbracket a &\triangleq x(u).!u \rightarrow a \\
\llbracket \lambda x.M \rrbracket a &\triangleq (\nu xb)(\llbracket M \rrbracket b \mid \overline{a}\langle x,b\rangle) \\
\llbracket MN \rrbracket a &\triangleq (\nu c)(\llbracket M \rrbracket c \mid !c(v,d).(\llbracket v := N \rrbracket \mid !d \rightarrow a)) \\
\llbracket M\langle x := N\rangle \rrbracket a &\triangleq (\nu x)(\llbracket M \rrbracket a \mid \llbracket x := N \rrbracket) \\
\llbracket x := N \rrbracket &\triangleq !(\nu w)\, \overline{x}\langle w\rangle.\llbracket N \rrbracket w
\end{aligned}
$$

$$
\begin{aligned}
\llbracket \mu\gamma.[\beta]M \rrbracket a &\triangleq \llbracket M[a/\gamma] \rrbracket \beta \\
\llbracket M\langle \beta := N\cdot\gamma\rangle \rrbracket a &\triangleq (\nu \beta)(\llbracket M \rrbracket a \mid \llbracket \beta := N\cdot\gamma \rrbracket) \\
\llbracket \beta := N\cdot\gamma \rrbracket &\triangleq !\beta(v,d).(\llbracket v := N \rrbracket \mid !d \rightarrow \gamma)
\end{aligned}
$$

Notice the similarity between

$$
\begin{aligned}
\llbracket MN \rrbracket a &= (\nu c)(\llbracket M \rrbracket c \mid !c(v,d).(\llbracket v := N \rrbracket \mid !d \rightarrow a)) \\
\llbracket M\langle c := N\cdot\gamma\rangle \rrbracket a &= (\nu c)(\llbracket M \rrbracket a \mid !c(v,d).(\llbracket v := N \rrbracket \mid !d \rightarrow \gamma))
\end{aligned}
$$

Structural substitution is a distributed application! Therefore . . .

Logical translation of $\lambda\mu\mathbf{x}$ terms

$$
\begin{array}{rcl}
\llbracket x \rrbracket a & \triangleq & x(u).!u \rightarrow a \\
\llbracket \lambda x.M \rrbracket a & \triangleq & (\nu xb)(\llbracket M \rrbracket b \mid \overline{a}\langle x,b\rangle) \\
\llbracket MN \rrbracket a & \triangleq & (\nu c)(\llbracket M \rrbracket c \mid \llbracket c := N{\cdot}a \rrbracket) \\
\llbracket M\langle x := N\rangle \rrbracket a & \triangleq & (\nu x)(\llbracket M \rrbracket a \mid \llbracket x := N \rrbracket) \\
\llbracket x := N \rrbracket & \triangleq & !(\nu w)\,\overline{x}\langle w\rangle.\llbracket N \rrbracket w \\
\\
\llbracket \mu\gamma.[\beta]M \rrbracket a & \triangleq & \llbracket M[a/\gamma] \rrbracket \beta \\
\llbracket M\langle \beta := N{\cdot}\gamma\rangle \rrbracket a & \triangleq & (\nu\beta)(\llbracket M \rrbracket a \mid \llbracket \beta := N{\cdot}\gamma \rrbracket) \\
\llbracket \beta := N{\cdot}\gamma \rrbracket & \triangleq & !\beta(v,d).(\llbracket v := N \rrbracket \mid !d \rightarrow \gamma)
\end{array}
$$

This is the encoding of $\lambda\mu\mathbf{x}$.

(Classical) Type Assignment for π

$$(0): \frac{}{0 : \vdash} \qquad (!): \frac{P : \Gamma \vdash \Delta}{!P : \Gamma \vdash \Delta} \qquad (out): \frac{}{\overline{a}\langle b\rangle : b{:}A \vdash a{:}A, b{:}A} \ (a \neq b)$$

$$(\nu): \frac{P : \Gamma, a{:}A \vdash a{:}A, \Delta}{(\nu a)\, P : \Gamma \vdash \Delta} \qquad (pair\text{-}out): \frac{}{\overline{a}\langle b,c\rangle : b{:}A \vdash a{:}A{\rightarrow}B, c{:}B}$$

$$(|): \frac{P_1 : \Gamma \vdash \Delta \quad \cdots \quad P_n : \Gamma \vdash \Delta}{P_1 \mid \cdots \mid P_n : \Gamma \vdash \Delta} \qquad (let): \frac{P : \Gamma, y{:}B \vdash x{:}A, \Delta}{let\ \langle x,y\rangle = z\ in\ P : \Gamma + z{:}A{\rightarrow}B \vdash \Delta} \ (1)$$

$$(W): \frac{P : \Gamma \vdash \Delta}{P : \Gamma' \vdash \Delta'} \ (2) \qquad (in): \frac{P : \Gamma, x{:}A \vdash x{:}A, \Delta}{a(x).P : \Gamma + a{:}A \vdash \Delta}$$

(1) $y, z \notin \Delta; x \notin \Gamma$; (2) $\Gamma' \supseteq \Gamma, \Delta' \supseteq \Delta$.

This system is not trivial:

$$(\nu c)\,(x(u).\,!u{\rightarrow}c \mid\, !c(v,d).\,(!\,(\nu w)\ \overline{v}\langle w\rangle.\,x(u).\,!u{\rightarrow}w \mid\, !d{\rightarrow}a))$$

is not typeable.

Results in [vBV-IFIPTCS'12]

(*Type preservation*) : If $\Gamma \vdash_{\mu\mathbf{x}} M : A \mid \Delta$, then $[\![M]\!]\, a : \Gamma \vdash_{\pi} a{:}A, \Delta$.

Results in [vBV-IFIPTCS'12]

(*Type preservation*) : If $\Gamma \vdash_{\mu\mathbf{x}} M : A \mid \Delta$, then $[\![M]\!]\, a : \Gamma \vdash_{\pi} a{:}A, \Delta$.

(*Soundness*) : $M \rightarrow_{\mathbf{x}\mathsf{H}} N \Rightarrow [\![M]\!]\, a \rightarrow_{\pi}^{*}, \sim_{\mathsf{G}}, \sim_{\mathsf{R}} [\![N]\!]\, a$.

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(*Soundness*) : $M \rightarrow_{\mathbf{x}\mathsf{H}} N \Rightarrow \llbracket M \rrbracket a \rightarrow^{*}_{\pi}, \sim_{\mathsf{G}}, \sim_{\mathsf{R}} \llbracket N \rrbracket a$.

(*Operational Soundness*) : 1. $M \rightarrow^{*}_{\mathsf{xH}} N \Rightarrow \llbracket M \rrbracket a \sim_{\mathsf{C}} \llbracket N \rrbracket a$.

2. $M \uparrow_{\mathbf{x}\mathsf{H}} \Rightarrow \llbracket M \rrbracket a \uparrow_{\pi}$.

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(*Type preservation*) : If $\Gamma \vdash_{\mu\mathbf{x}} M : A \mid \Delta$, then $\llbracket M \rrbracket a : \Gamma \vdash_{\pi} a{:}A, \Delta$.

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(*Operational Soundness*) : 1. $M \rightarrow^{*}_{\mathrm{xH}} N \Rightarrow \llbracket M \rrbracket a \sim_{\mathrm{C}} \llbracket N \rrbracket a$.

2. $M \uparrow_{\mathbf{x}\mathrm{H}} \Rightarrow \llbracket M \rrbracket a \uparrow_{\pi}$.

(*Adequacy*) : $M =_{\mathrm{xH}} N \Rightarrow \llbracket M \rrbracket a \sim_{\mathrm{C}} \llbracket N \rrbracket a$.

Results in [vBV-IFIPTCS'12]

(*Type preservation*) : If $\Gamma \vdash_{\mu\mathbf{x}} M : A \mid \Delta$, then $[\![M]\!]\, a : \Gamma \vdash_{\pi} a{:}A, \Delta$.

(*Soundness*) : $M \rightarrow_{\mathbf{x}\mathsf{H}} N \Rightarrow [\![M]\!]\, a \rightarrow^{*}_{\pi}, \sim_{\mathsf{G}}, \sim_{\mathsf{R}} [\![N]\!]\, a$.

(*Operational Soundness*) : 1. $M \rightarrow^{*}_{\mathsf{xH}} N \Rightarrow [\![M]\!]\, a \sim_{\mathsf{C}} [\![N]\!]\, a$.

2. $M \Uparrow_{\mathbf{x}\mathsf{H}} \Rightarrow [\![M]\!]\, a \Uparrow_{\pi}$.

(*Adequacy*) : $M =_{\mathsf{xH}} N \Rightarrow [\![M]\!]\, a \sim_{\mathsf{C}} [\![N]\!]\, a$.

(*Operational completeness for* $\rightarrow_{\mathbf{x}\mathsf{H}}$) : If $[\![M]\!]\, a \rightarrow^{*}_{\pi} P$ then there exists N such that $P \rightarrow^{+}_{\pi}, \sim_{\mathsf{R}}, \sim_{\mathsf{G}} [\![N]\!]\, a$, and $M \rightarrow^{*}_{\mathsf{xH}} N$.

Results in [vBV-IFIPTCS'12]

(*Type preservation*) : If $\Gamma \vdash_{\mu\mathbf{x}} M : A \mid \Delta$, then $[\![M]\!]\, a : \Gamma \vdash_{\pi} a{:}A, \Delta$.

(*Soundness*) : $M \rightarrow_{\mathbf{x}\mathrm{H}} N \Rightarrow [\![M]\!]\, a \rightarrow_{\pi}^{*}, \sim_{\mathrm{G}}, \sim_{\mathrm{R}} [\![N]\!]\, a$.

(*Operational Soundness*) : 1. $M \rightarrow_{\mathrm{XH}}^{*} N \Rightarrow [\![M]\!]\, a \sim_{\mathrm{C}} [\![N]\!]\, a$.

2. $M \uparrow_{\mathbf{x}\mathrm{H}} \Rightarrow [\![M]\!]\, a \uparrow_{\pi}$.

(*Adequacy*) : $M =_{\mathrm{XH}} N \Rightarrow [\![M]\!]\, a \sim_{\mathrm{C}} [\![N]\!]\, a$.

(*Operational completeness for* $\rightarrow_{\mathbf{x}\mathrm{H}}$) : If $[\![M]\!]\, a \rightarrow_{\pi}^{*} P$ then there exists N such that $P \rightarrow_{\pi}^{+}, \sim_{\mathrm{R}}, \sim_{\mathrm{G}} [\![N]\!]\, a$, and $M \rightarrow_{\mathrm{XH}}^{*} N$.

(*Termination*) : 1. If $M \rightarrow_{\mathrm{XH}}^{nf} N$, then $[\![M]\!]\, a \downarrow_{\pi}$.

2. If $M \rightarrow_{\beta\mu}^{nf} N$, then $[\![M]\!]\, a \downarrow_{\pi}$.

3. Let $M \in \Lambda\mu$. If $[\![M]\!]\, a \downarrow_{\pi}$ then there exists $N \in \lambda\mu\mathbf{x}$ and L in $\rightarrow_{\beta\mu}^{*}$-head normal form such that $[\![M]\!]\, a \sim_{\mathrm{C}} [\![N]\!]\, a$, and $M \rightarrow_{\mathrm{XH}}^{nf} N$ and $N \rightarrow_{:=}^{nf} L$.

How about full abstraction?

The results of [vBV-IFIPTCS'12] essentially show: $M \rightarrow_{\mathsf{XH}}^{nf} N$ if and only if $[\![M]\!]\, a \rightarrow_{\mathsf{RG}}^{nf} [\![N]\!]\, a$.

Can we improve on this? Can we relate our encoding also to more traditional notions of equality, like $=_{\beta\mu}$?

First of all, take $\Delta = \lambda x.xx$ and $\Omega = \lambda y.yyy$, then we can show that $[\![\Delta\Delta]\!]\, a \sim_{\mathsf{C}} [\![\Omega\Omega]\!]\, a$ so both can never *input* nor *output*. But then:

$$
\begin{array}{ll}
[\![\lambda z.\Delta\Delta]\!]\, a & \stackrel{\Delta}{=} \\
(\nu z b)\, ([\![\Delta\Delta]\!]\, b \,|\, \overline{a}\langle z,b \rangle) & \sim_{\mathsf{C}} \\
(\nu z b)\, ([\![\Omega\Omega]\!]\, b \,|\, \overline{a}\langle z,b \rangle) & \stackrel{\Delta}{=} \\
[\![\lambda z.\Omega\Omega]\!]\, a &
\end{array}
$$

So we would need $\lambda z.\Delta\Delta$ and $\lambda z.\Omega\Omega$ to be equivalent in $\lambda\mu$: we need a notion of *weak* equivalence between terms, based on a notion of *weak* (lazy) head-normal forms: we need $\lambda\mu$-Lévy-Longo trees.

Weak equivalences for $\lambda\mu$

The $\lambda\mu$ *weak head-normal forms* are defined through the grammar:

$$\begin{array}{rcl} \boldsymbol{H}_{\mathsf{W}} & ::= & xM_1\cdots M_n \ (n \geq 0) \mid \lambda x.M \\ & \mid & \mu\alpha.[\beta]\boldsymbol{H}_{\mathsf{W}} \ \ (\alpha \neq \beta \text{ or } \alpha \notin \boldsymbol{H}_{\mathsf{W}}, \text{ and } \boldsymbol{H}_{\mathsf{W}} \neq \mu\gamma.[\delta]\boldsymbol{H}'_{\mathsf{W}}) \end{array}$$

Notice that we allow any term inside an abstraction. We use $\rightarrow_{\mathsf{WH}}$ for (implicit) weak head reduction, and $\rightarrow_{\mathsf{WxH}}$ for its explicit variant.

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Notice that we allow any term inside an abstraction. We use $\rightarrow_{\mathsf{wH}}$ for (implicit) weak head reduction, and $\rightarrow_{\mathsf{wxH}}$ for its explicit variant.

We define $\sim^{\mathsf{w}}_{\beta\mu}$ as the smallest congruence that contains:

$$\begin{array}{rcll} (\lambda x.M)N & \sim^{\mathsf{w}}_{\beta\mu} & M[N/x] & \\ (\mu\alpha.M)N & \sim^{\mathsf{w}}_{\beta\mu} & \mu\gamma.M[N{\cdot}\gamma/\alpha] & (\gamma \textit{ fresh}) \\ \mu\alpha.[\beta]\mu\gamma.[\delta]M & \sim^{\mathsf{w}}_{\beta\mu} & \mu\alpha.[\delta]M[\beta/\gamma] & \\ \mu\alpha.[\alpha]M & \sim^{\mathsf{w}}_{\beta\mu} & M & \textit{if } \alpha \notin M \\ M, N \textit{ have no weak hnf} & \Rightarrow & M \sim^{\mathsf{w}}_{\beta\mu} N & \end{array}$$

Equivalences $\sim^{w}_{\mathsf{H}}$ and $\sim^{w}_{\mathbf{x}\mathsf{H}}$ on terms

We have also two equivalences induced by $\rightarrow_{w\mathsf{H}}$ and $\rightarrow_{w\mathbf{x}\mathsf{H}}$.

$\sim^{w}_{\mathsf{H}}$ (*weak head-equivalence*) is co-inductively defined by: $M \sim^{w}_{\mathsf{H}} N$ if and only if either:

- if $M \rightarrow^{nf}_{w\mathsf{H}} \; xM_1 \cdots M_n \; (n \geq 0)$, then $N \rightarrow^{nf}_{w\mathsf{H}} \; xN_1 \cdots N_n$ and $M_i \sim^{w}_{\mathsf{H}} N_i$; or
- if $M \rightarrow^{nf}_{w\mathsf{H}} \; \lambda x.M'$, then $N \rightarrow^{nf}_{w\mathsf{H}} \; \lambda x.N'$ and $M' \sim^{w}_{\mathsf{H}} N'$; or
- if $M \rightarrow^{nf}_{w\mathsf{H}} \; \mu\alpha.[\beta]M'$, then $N \rightarrow^{nf}_{w\mathsf{H}} \; \mu\alpha.[\beta]N'$ (so $\alpha \neq \beta$ or $\alpha \notin fn(M')$, and $M' \neq \mu\gamma.[\delta]L$, and similarly for N') and $M' \sim^{w}_{\mathsf{H}} N'$.

Equivalences $\sim^{w}_{\mathsf{H}}$ and $\sim^{w}_{\mathbf{x}\mathsf{H}}$ on terms

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- if $M \rightarrow^{nf}_{w\mathsf{H}} \; xM_1 \cdots M_n \; (n \geq 0)$, then $N \rightarrow^{nf}_{w\mathsf{H}} \; xN_1 \cdots N_n$ and $\overrightarrow{M_i \sim^{w}_{\mathsf{H}} N_i}$; or
- if $M \rightarrow^{nf}_{w\mathsf{H}} \; \lambda x.M'$, then $N \rightarrow^{nf}_{w\mathsf{H}} \; \lambda x.N'$ and $M' \sim^{w}_{\mathsf{H}} N'$; or
- if $M \rightarrow^{nf}_{w\mathsf{H}} \; \mu\alpha.[\beta]M'$, then $N \rightarrow^{nf}_{w\mathsf{H}} \; \mu\alpha.[\beta]N'$ (so $\alpha \neq \beta$ or $\alpha \notin fn(M')$, and $M' \neq \mu\gamma.[\delta]L$, and similarly for N') and $M' \sim^{w}_{\mathsf{H}} N'$.

$\sim^{w}_{\mathbf{x}\mathsf{H}}$ (*explicit weak head-equality*) is co-inductively by: $M \sim^{w}_{\mathbf{x}\mathsf{H}} N$ if and only if both $M \rightarrow^{nf}_{w\mathbf{x}\mathsf{H}} \; M'\,\mathbf{S}$ and $N \rightarrow^{nf}_{w\mathbf{x}\mathsf{H}} \; N'\,\mathbf{S}'$, and either:

- if $M' = xM_1 \cdots M_n \; (n \geq 0)$, then $N = xN_1 \cdots N_n$ (so $x \notin \mathbf{S}$, $x \notin \mathbf{S}'$) and $\overrightarrow{M_i\,\mathbf{S} \sim^{w}_{\mathbf{x}\mathsf{H}} N_i\,\mathbf{S}'}$; ...

The other cases are similar.

Weak Approximation Semantics for $\lambda\mu$

1. The set of $\lambda\mu$'s *weak approximants* $\mathcal{A}_w$ wrt $\rightarrow_{\beta\mu}$ is defined by:

$$A ::= \bot \mid \lambda x.A \mid xA_1 \cdots A_n \quad (n \geq 0)$$
$$\mid \mu\alpha.[\beta]A \quad (\alpha \neq \beta \text{ or } \alpha \notin A,\ A \neq \mu\gamma.[\delta]A',\ A \neq \bot)$$

   Notice that, in $\lambda x.A$, the body A can be $\bot$: this characterises *weak* hnf.

2. The relation $\sqsubseteq \subseteq \mathcal{A}_w \times \lambda\mu$ is defined as the smallest preorder that is the compatible extension of $\bot \sqsubseteq M$.
3. The set of *weak approximants* of M, $\mathcal{A}_w(M)$, is defined as:

$$\{A \in \mathcal{A}_w \mid \exists N \in \lambda\mu\ [M \rightarrow^*_{\beta\mu} N \ \&\ A \sqsubseteq N]\}$$

4. $M \sim^w_{\mathcal{A}} N \triangleq \mathcal{A}_w(M) = \mathcal{A}_w(N)$.

Notice that $\mathcal{A}_w(\lambda z.\Delta\Delta) = \{\bot, \lambda z.\bot\} = \mathcal{A}_w(\lambda z.\Omega\Omega)$. We can define the *$\lambda\mu$-Lévy-Longo tree* for M by $\sqcup\mathcal{A}_w(M)$.

Full abstraction

We show: let $M, N \in \lambda\mu$, then

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$$[\![M]\!]\, a \sim_{\mathsf{C}} [\![N]\!]\, a$$

$$\Longleftrightarrow$$

$$M \sim^{W}_{\mathbf{x}\mathsf{H}} N$$

Full abstraction

We show: let $M, N \in \lambda\mu$, then

$$[\![M]\!]\, a \sim_{\mathsf{C}} [\![N]\!]\, a$$

$$\Longleftrightarrow$$

$$M \sim^{\mathsf{W}}_{\mathbf{x}\mathsf{H}} N$$

$$\Longleftrightarrow$$

$$M \sim^{\mathsf{W}}_{\mathsf{H}} N$$

Full abstraction

We show: let $M, N \in \lambda\mu$, then

$$[\![M]\!]\,a \sim_{\mathsf{c}} [\![N]\!]\,a$$

$$\Longleftrightarrow$$

$$M \sim^{\mathsf{w}}_{\mathbf{x}\mathsf{H}} N$$

$$\Longleftrightarrow$$

$$M \sim^{\mathsf{w}}_{\mathsf{H}} N$$

$$\Longleftrightarrow$$

$$M \sim^{\mathsf{w}}_{\mathcal{A}} N$$

Full abstraction

We show: let $M, N \in \lambda\mu$, then

$$[\![M]\!]\, a \sim_{\mathsf{C}} [\![N]\!]\, a$$

$$\Longleftrightarrow$$

$$M \sim^{w}_{\mathbf{x}\mathsf{H}} N$$

$$\Longleftrightarrow$$

$$M \sim^{w}_{\mathsf{H}} N$$

$$\Longleftrightarrow$$

$$M \sim^{w}_{\mathcal{A}} N$$

$$\Longleftrightarrow$$

$$M \sim^{w}_{\beta\mu} N$$

Full abstraction

We show: let $M, N \in \lambda\mu$, then

$$\llbracket M \rrbracket\, a \sim_{\mathsf{C}} \llbracket N \rrbracket\, a$$

$$\Longleftrightarrow$$

$$M \sim^{\mathsf{w}}_{\mathbf{x}\mathsf{H}} N$$

$$\Longleftrightarrow$$

$$M \sim^{\mathsf{w}}_{\mathsf{H}} N$$

$$\Longleftrightarrow$$

$$M \sim^{\mathsf{w}}_{\mathcal{A}} N$$

$$\Longleftrightarrow$$

$$M \sim^{\mathsf{w}}_{\beta\mu} N$$

So, although the π *implements only* the explicit head reduction for $\lambda\mu$, our interpretation $\llbracket \cdot \rrbracket \cdot$ gives a fully-abstract semantics.

Questions

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