Copatterns: Programming Infinite Structures by Observations

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From Codata to Coalgebras

Algebras and Coalgebras

Patterns and Copatterns

Defining Fibonacci Numbers by Copattern Matching

Simulating Codata Types in Coalgebras

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Conclusion
Coalgebras in Functional Programming

- Originally functional programming based on
  - function types,
  - inductive data types.
- In computer science, many computations are interactive.
- Since interactions might go on forever (if not terminated by the user), they correspond to non-wellfounded data types
  - Streams, which are infinite lists,
  - non-wellfounded trees (IO-trees).
Codata Type

- **Idea of Codata Types:**
  
  ```
  codata Stream : Set where
  cons : N → Stream → Stream
  ```

- **Same definition as inductive data type but we are allowed to have infinite chains of constructors**
  
  ```
  cons n₀ (cons n₁ (cons n₂ ⋅ ⋅ ⋅))
  ```

- **Problem 1:** Non-normalisation.

- **Problem 2:** Equality between streams is equality between all elements, and therefore undecidable.

- **Problem 3:** Underlying assumption is
  
  $$\forall s : \text{Stream}. \exists n, s'. s = \text{cons } n \ s'$$

  which results in undecidable equality.
Subject Reduction Problem

- In order to repair problem of normalisation restrictions on reductions were introduced.
- Resulted in Coq in a long known problem of **subject reduction**.
- In order to avoid this, in Agda dependent elimination for coalgebras disallowed.
  - Makes it difficult to use.
Problem of Subject reduction:

```haskell
data _==_ {A : Set} (a : A) : A → Set where
  refl : a == a

codata Stream : Set where
  cons : ℕ → Stream → Stream

zeros : Stream
zeros = cons 0 zeros

force : Stream → Stream
force s = case s of (cons x y) → cons x y

lem1 : (s : Stream) → s == force(s))
lem1 s = case s of (cons x y) → refl

lem2 : zeros == cons 0 zeros
lem2 = lem1 zeros
lem2 → refl  but ¬(refl : zeros == cons 0 zeros)
```
Solution is to follow the long established categorical formulation of coalgebras.
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Initial F-Algebras

- Inductive data types correspond to initial F-Algebras.
- E.g. the natural numbers can be formulated as

  \[ F(X) = 1 + X \]
  \[ \text{intro} : F(\mathbb{N}) \to \mathbb{N} \]
  \[ \text{intro (inl } \ast \text{)} = 0 \]
  \[ \text{intro (inl } n \text{)} = S \ n \]

and we get the diagram

\[
\begin{align*}
1 + \mathbb{N} & = F(\mathbb{N}) \quad \text{intro} \quad \mathbb{N} \\
1 + g & = F(g) \\
1 + A & = F(A) \quad f \quad A
\end{align*}
\]
Existence of unique $g$ corresponds to unique iteration (example $\mathbb{N}$):

\[
\begin{align*}
1 + \mathbb{N} & \xrightarrow{\text{intro}} \mathbb{N} \\
1 + g & \xrightarrow{\exists! g} \\
1 + A & \xrightarrow{f} A
\end{align*}
\]

\[
\begin{align*}
g 0 &= g(\text{intro inl}) = f \text{ inl} \\
g (S \, n) &= g(\text{intro (inr n)}) = f (\text{inr (g n)})
\end{align*}
\]

By choosing arbitrary $f$ we can define $g$ by pattern matching on its argument $n$:

\[
\begin{align*}
g 0 &= a_0 \\
g (S \, n) &= f (g \, n) \text{ for some } f : \mathbb{N} \to \mathbb{N}
\end{align*}
\]
Recursion and Induction

From the principle of unique iteration one can derive the principle of recursion:
Assume

\[ a_0 : A \]
\[ f_0 : \mathbb{N} \to A \to A \]

We can then define \( g : \mathbb{N} \to A \) s.t.

\[ g \ 0 \ = \ a_0 \]
\[ g \ (S \ n) \ = \ f_0 \ n \ (g \ n) \]

Induction is as recursion but now

\[ g : (n : \mathbb{N}) \to A \ n \]
Final coalgebras $F^\infty$ are obtained by reversing the arrows in the diagram for $F$-algebras:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & F(A) \\
\vert & & \vert \\
\exists! g & \downarrow & F(g) \\
F^\infty & \xrightarrow{\text{case}} & F(F^\infty)
\end{array}
\]
Consider Streams = $F^\infty$ where $F(X) = \mathbb{N} \times X$:

$$
\begin{array}{ccc}
A & \xrightarrow{f} & \mathbb{N} \times A \\
\downarrow & & \downarrow \\
\text{Stream} & \xrightarrow{\text{case}} & \mathbb{N} \times \text{Stream} \\
\exists! g & & \text{id} \times g
\end{array}
$$

Let

$$\text{case } s = \langle \text{head } s, \text{tail } s \rangle$$

and

$$f \ a = \langle f_0 \ a, f_1 \ a \rangle$$
Guarded Recursion

Resulting equations:

\[
\begin{align*}
\text{head} (g \ a) &= f_0 \ a \\
\text{tail} (g \ a) &= g (f_1 \ a)
\end{align*}
\]
Example of Guarded Recursion

\[
\begin{align*}
\text{head} \ (g \ a) & = \ f_0 \ a \\
\text{tail} \ (g \ a) & = \ g \ (f_1 \ a)
\end{align*}
\]

describes a schema of guarded recursion (or better coiteration)
As an example, with \( A = \mathbb{N}, \ f_0 \ n = n, \ f_1 \ n = n + 1 \) we obtain:

\[
\begin{align*}
\text{inc} : \mathbb{N} & \rightarrow \text{Stream} \\
\text{head} \ (\text{inc} \ n) & = \ n \\
\text{tail} \ (\text{inc} \ n) & = \ \text{inc} \ (n + 1)
\end{align*}
\]
Corecursion

In coiteration we need to make in `tail` always a recursive call:

\[ \text{tail} \ (g \ a) = g \ (f_1 \ a) \]

Corecursion allows for `tail` to escape into a previously defined stream. Assume

\[
\begin{align*}
A &: \text{Set} \\
f_0 &: A \to \mathbb{N} \\
f_1 &: A \to (\text{Stream} + A)
\end{align*}
\]

we get \( g : A \to \text{Stream} \) s.t.

\[
\begin{align*}
\text{head} \ (g \ a) &= f_0 \ a \\
\text{tail} \ (g \ a) &= s & \text{if} & f_1 \ a = \text{inl} \ s \\
\text{tail} \ (g \ a) &= g \ a' & \text{if} & f_1 \ a = \text{inr} \ a'
\end{align*}
\]
Definition of cons by Corecursion

\[
\begin{align*}
\text{head}(g \ a) &= f_0 \ a \\
\text{tail}(g \ a) &= s \quad \text{if} \quad f_1 \ a = \text{inl} \ s \\
\text{tail}(g \ a) &= g \ a' \quad \text{if} \quad f_1 \ a = \text{inr} \ a'
\end{align*}
\]

\[
\begin{align*}
\text{cons} : \mathbb{N} \to \text{Stream} \to \text{Stream} \\
\text{head} \ (\text{cons} \ n \ s) &= n \\
\text{tail} \ (\text{cons} \ n \ s) &= s
\end{align*}
\]
Nested Corecursion

\[
\text{stutter} : \mathbb{N} \rightarrow \text{Stream} \\
\text{head} \ (\text{stutter} \ n) = n \\
\text{head} \ (\text{tail} \ (\text{stutter} \ n)) = n \\
\text{tail} \ (\text{tail} \ (\text{stutter} \ n)) = \text{stutter} \ (n + 1)
\]

Even more general schemata can be defined.
Definition of Coalgebras by Observations

- We see now that elements of coalgebras are defined by their observations:
  An element \( s \) of \( \text{Stream} \) is given by defining

\[
\begin{align*}
  \text{head} & \quad s \quad : \quad \mathbb{N} \\
  \text{tail} & \quad s \quad : \quad \text{Stream}
\end{align*}
\]

- This generalises the function type.
  Functions \( f : A \to B \) are as well determined by observations, namely by defining

\[
f \ a \ : \ B
\]

- An \( f : A \to B \) is any program which applied to \( a : A \) returns some \( b : B \).

- **Inductive data types** are defined by **construction coalgebraic data types** and **functions** by **observations**.
Objects in Object-Oriented Programming are types which are defined by their observations.

Therefore objects are coalgebraic types by nature.
Weakly Final Coalgebra

- Equality for final coalgebras is undecidable:
  Two streams
  \[ s = (a_0, a_1, a_2, \ldots) \]
  \[ t = (b_0, b_1, b_2, \ldots) \]
  are equal iff \( a_i = b_i \) for all \( i \).

- Even the weak assumption
  \[ \forall s. \exists n, s'. s = \text{cons} \ n \ s' \]
  results in an undecidable equality.

- Weakly final coalgebras obtained by omitting uniqueness of \( g \) in diagram for coalgebras.

- However, one can extend schema of coiteration as above, and still preserve decidability of equality.
  - Those schemata are usually not derivable in weakly final coalgebras.
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We can define now functions by patterns and copatterns.

Example define stream:
\[
f \ n = \n, n, n - 1, n - 1, \ldots 0, 0, N, N, N - 1, N - 1, \ldots 0, 0, N, N, N - 1, N - 1,
\]
Patterns and Copatterns

\[ f \ n = n, \ n, \ n-1, \ n-1, \ldots 0, 0, \ N, \ N, \ N-1, \ N-1, \ldots 0, 0, \ N, \ N, \ N-1, \ N-1, \ldots \]

\[ f : \mathbb{N} \rightarrow \text{Stream} \]

\[ f = ? \]
Patterns and Copatterns

\[ f(n) = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1, \ldots \]

\[ f : \mathbb{N} \rightarrow \text{Stream} \]

Pattern match on \( f : \mathbb{N} \rightarrow \text{Stream} \):

\[ f(n) = ? \]
$$f \ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1,$$

$$f : \mathbb{N} \rightarrow \text{Stream}$$

$$f \ n = ?$$

**Copattern matching** on \( f \ n : \text{Stream} :$$

$$f : \mathbb{N} \rightarrow \text{Stream}$$

$$\text{head} \ (f \ n) = ?$$

$$\text{tail} \ (f \ n) = ?$$
Patterns and Copatterns

\[ f \ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1, \ldots \]

\[ f : \mathbb{N} \to \text{Stream} \]
\[ \text{head} (f \ n) = ? \]
\[ \text{tail} (f \ n) = ? \]

Pattern matching on the first \( n : \mathbb{N} \):

\[ f : \mathbb{N} \to \text{Stream} \]
\[ \text{head} (f \ 0) = ? \]
\[ \text{head} (f (S \ n)) = ? \]
\[ \text{tail} (f \ n) = ? \]
Patterns and Copatterns

\[ f \ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1, \ldots \]

\[ f : \mathbb{N} \rightarrow \text{Stream} \]
\[ \text{head} (f \ 0) = ? \]
\[ \text{head} (f \ (S \ n)) = ? \]
\[ \text{tail} (f \ n) = ? \]

Pattern matching on second \( n : \mathbb{N} \):

\[ f : \mathbb{N} \rightarrow \text{Stream} \]
\[ \text{head} (f \ 0) = ? \]
\[ \text{head} (f \ (S \ n)) = ? \]
\[ \text{tail} (f \ 0) = ? \]
\[ \text{tail} (f \ (S \ n)) = ? \]
Patterns and Copatterns

\[ f \ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1, \ldots \]

\[ f : \mathbb{N} \rightarrow \text{Stream} \]

\[ \text{head} (f \ 0) = ? \]
\[ \text{head} (f \ (S \ n)) = ? \]
\[ \text{tail} (f \ 0) = ? \]
\[ \text{tail} (f \ (S \ n)) = ? \]

**Copattern matching** on \( \text{tail} (f \ 0) : \text{Stream} \)

\[ f : \mathbb{N} \rightarrow \text{Stream} \]

\[ \text{head} \ (f \ 0) = ? \]
\[ \text{head} \ (f \ (S \ n)) = ? \]
\[ \text{head} \ (\text{tail} \ (f \ 0)) = ? \]
\[ \text{tail} \ (\text{tail} \ (f \ 0)) = ? \]
\[ \text{tail} \ (f \ (S \ n)) = ? \]
Patterns and Copatterns

\[ f : \mathbb{N} \rightarrow \text{Stream} \]

head \((f \ 0) = ? \)
head \((f \ (S \ n)) = ? \)
head \((\text{tail} \ (f \ 0)) = ? \)
tail \((\text{tail} \ (f \ 0)) = ? \)
tail \((f \ (S \ n)) = ? \)

**Copattern matching** on \( \text{tail} \ (f \ (S \ n)) : \text{Stream} \):

\[ f : \mathbb{N} \rightarrow \text{Stream} \]

head \((f \ 0) = ? \)
head \((f \ (S \ n)) = ? \)
head \((\text{tail} \ (f \ 0)) = ? \)
tail \((\text{tail} \ (f \ 0)) = ? \)
hthead \((\text{tail} \ (f \ (S \ n))) = ? \)
tail \((\text{tail} \ (f \ (S \ n))) = ? \)
We resolve the goals:

\[
\begin{align*}
&f : \mathbb{N} \rightarrow \text{Stream} \\
&\text{head } (f \ 0) = 0 \\
&\text{head } (\text{tail } (f \ 0)) = 0 \\
&\text{tail } (\text{tail } (f \ 0)) = f \ \mathbb{N} \\
&\text{head } (f \ (S \ n)) = S \ n \\
&\text{head } (\text{tail } (f \ (S \ n))) = S \ n \\
&\text{tail } (\text{tail } (f \ (S \ n))) = f \ n
\end{align*}
\]
Results of paper in POPL (2013)

- Development of a recursive simply typed calculus (no termination check).
- Allows to derive schemata for pattern/copattern matching.
- Proof that subject reduction holds.

\[ t : A, \quad t \rightarrow t' \text{ implies } t' : A \]
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Fibonacci Numbers

Efficient Haskell version adapted to our codata notation:

```
codata Stream : Set where  
    cons : \N -> Stream -> Stream

tail : Stream -> Stream
    tail (cons n l) = l

addStream : Stream -> Stream -> Stream
    addStream (cons n l) (cons n' l') = cons (n + n') (addStream l l')

fib : Stream
    fib = cons 1 (cons 1 (addStream fib (tail fib)))
```

Requires lazy evaluation.
Fibonacci Numbers using Coalgebras

\[
\begin{align*}
\text{coalg } \text{Stream} : \text{Set} & \text{ where } \\
\text{head} & : \text{Stream } \to \mathbb{N} \\
\text{tail} & : \text{Stream } \to \text{Stream} \\
\text{addStream} & : \text{Stream } \to \text{Stream } \to \text{Stream} \\
\text{head} (\text{addStream } l l') & = \text{head } l + \text{head } l' \\
\text{tail} (\text{addStream } l l') & = \text{addStream} (\text{tail } l) (\text{tail } l') \\
\text{fib} & : \text{Stream} \\
\text{head fib} & = 1 \\
\text{head} (\text{tail fib}) & = 1 \\
\text{tail} (\text{tail fib}) & = \text{addStream fib} (\text{tail fib}) \\
\end{align*}
\]

No laziness required. Requires full corecursion (but terminates).
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Having more than one constructor in algebras correspond to disjoint union:

```haskell
data ℕ : Set where
  0 : ℕ
  S : ℕ → ℕ
```

corresponds to

```haskell
data ℕ : Set where
  intro : (1 + ℕ) → ℕ
```
Dual of disjoint union is products, and therefore multiple destructors correspond to product:

\[
\text{coalg Stream : Set where}
\begin{align*}
\text{head} & : \text{Stream} \rightarrow \mathbb{N} \\
\text{tail} & : \text{Stream} \rightarrow \text{Stream}
\end{align*}
\]

corresponds to

\[
\text{coalg Stream : Set where}
\begin{align*}
\text{case} & : \text{Stream} \rightarrow (\mathbb{N} \times \text{Stream})
\end{align*}
\]
Consider

\[
\text{codata coList : Set where} \\
\quad \text{nil : coList} \\
\quad \text{cons : } \mathbb{N} \to \text{coList} \to \text{coList}
\]

Cannot be simulated by a coalgebra with several destructors.
Simulating Codata Types by Simultaneous Algebras/Coalgebras

Represent Codata as follows

```plaintext
mutual
coalg coList : Set where
  unfold : coList → coListShape
data coListShape : Set where
  nil : coListShape
  cons : ℕ → coList → coListShape
```
Definition of Append

\[\text{append} : \text{coList} \rightarrow \text{coList} \rightarrow \text{coList}\]
\[\text{append } l \ l' =?\]
Definition of Append

append : coList → coList → coList
append l l' =?

We copattern match on append l l' : coList:

append : coList → coList → coList
unfold (append l l') =?
Definition of Append

\[
\text{append} : \text{coList} \rightarrow \text{coList} \rightarrow \text{coList}
\]

\[
\text{unfold} (\text{append } l \ l') = ?
\]

We cannot pattern match on \( l \).
But we can do so on \((\text{unfold } l)\):

\[
\text{append} : \text{coList} \rightarrow \text{coList} \rightarrow \text{coList}
\]

\[
\text{unfold} (\text{append } l \ l') =
\]

\[
\text{case (unfold } l\text{) of}
\]

\[
\text{nil} \quad \rightarrow \quad ?
\]

\[
(\text{cons } n \ l) \quad \rightarrow \quad ?
\]
Definition of Append

append : coList → coList → coList
unfold (append l l′) =
  \text{case } (unfold l) \text{ of }
  \text{nil} \rightarrow ?
  (\text{cons } n \ l) \rightarrow ?

We resolve the goals:

append : coList → coList → coList
unfold (append l l′) =
  \text{case } (unfold l) \text{ of }
  \text{nil} \rightarrow \text{ unfold } l′
  (\text{cons } n \ l) \rightarrow \text{ cons } n \ (\text{append } l \ l′)
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- Symmetry between
  - algebras and coalgebras,
  - iteration and coiteration,
  - recursion and corecursion,
  - patterns and copatterns.
- Final algebras are defined by construction, coalgebras and function types by observation.
- Codata construct assumes every element is introduced by a constructor, which results in
  - either undecidable equality
  - or requires sophisticated restrictions on reduction rule which are difficult to get right.
    - Problem of subreduction in Coq.
    - Too restrictive elimination principle in Agda.
- Weakly final coalgebras solve this problem, by having reduction rules which can always be applied independent of context.
More details can be found in my proper talk tomorrow.

- Assumption $\forall s : \text{Stream}. \exists n, s'. s = \text{cons } n \ s'$ results in undecidable equality.
- How to replace copattern matching by combinators.