Are streamless sets Noetherian?

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Finiteness

Constructively, there are at least four definitions of a set A of natural numbers being finite.

(i) The set A is given by a list. (Enumerated sets)

(ii) There exists a bound such that any list over A contains duplicates whenever its length exceeds the bound. (Size-bounded sets)

(iii) The root of the tree of duplicate-free lists over A is inductively accessible. (Noetherian sets)

(iv) Every stream over A has a duplicate. (Streamless sets)

Enumerated sets

A set $A \subseteq \text{nat}$ is *enumerated*, *enum A*, if all its elements can be listed, or

$$\frac{\forall x : A.\ \mathit{false}}{\mathit{enum}\ A} \qquad \frac{x : A \quad \mathit{enum}\ (A \setminus \{x\})}{\mathit{enum}\ A}$$

A proof of *enum A* is essentially an exhaustive duplicate-free list of elements of A.

Size-bounded sets

A set $A \subseteq \mathsf{nat}$ is *size-bounded by n* if any duplicate-free list over A is of length of less than n.

$$\frac{\forall x : A.\ \mathit{bounded}_n\ (A \setminus \{x\})}{\mathit{bounded}_{n+1}\ A}$$

A set A is *size-bounded* if there exists n such that $\mathit{bounded}_n\ A$.

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Noetherian sets

A set A is *Noetherian*, *Noet* A, if, for all $x \in A$, $A \backslash \{x\}$ is Noetherian. Formally,

$$\frac{\forall x \in A.\, Noet\ (A\backslash\{x\})}{Noet\ A}$$

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Streamless sets

A set $A \subseteq \mathsf{nat}$ is streamless if every stream over A has duplicates.

$$\forall f : \mathsf{nat} \to A . \exists n . \exists m > n . f(n) = f(m)$$

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Noetherian sets (revisited)

Let $A : Set$ and $R : A \to A \to Prop$.
For $x : A$ and $l : A^*$, we say x R-belongs to l, written $x \in_R l$, if l contains an element to which x is related by R. Or,

$$\frac{R\,x\,y}{x \in_R y :: l} \qquad \frac{x \in_R l}{x \in_R y :: l}$$

A list $l : A^*$ is R-good, written $good_R\, l$, if there exists $n < len(l)$ and $m < n$ such that $R\, l(m)\, l(n)$. Or,

$$\frac{x \in_R l}{good_R\, x :: l} \qquad \frac{good_R\, l}{good_R\, x :: l}$$

Noetherian sets (revisited)

A relation $R : A \to A \to \mathit{Prop}$ on a set A is *streamless* if every stream α over A has a prefix which is R-good.

Given a relation $R : A \to A \to \mathit{Prop}$, we define R-*accessibility* of a list $l : A^*$, written $\mathit{Acc}_R\, l$, inductively by

$$\frac{\mathit{good}_R\, l}{\mathit{Acc}_R\, l} \qquad \frac{\forall a : A.\, \mathit{Acc}_R\,(a : l)}{\mathit{Acc}_R\, l}$$

so that l is R-accessible if either l is R-good, or, for all $a : A$, $a :: l$ is R-accessible.

We say a relation $R : A \to A \to \mathit{Prop}$ is Noetherian, if an empty list $\langle\rangle$ is R-accessible, i.e., $\mathit{Acc}_R\, \langle\rangle$.

Abstracting from the Halting set

Given a predicate $H : \mathrm{nat} \to \mathit{Prop}$ on natural numbers, we define a predicate $P_H : \mathrm{nat} \to \mathit{Prop}$ inductively by

$$\frac{}{P_H\,0}\ \mathrm{P_0} \qquad \frac{P_H\,n \quad (n \in H \vee \neg n \in H)}{P_H\,(n+1)}\ \mathrm{P_S}$$

so that if $P_H\,n$ holds, we have a proof for $H\,m \vee \neg H\,m$ for all $m < n$.

Lemma

For any n*,* $P_H\,n$ *implies* $\neg\neg P_H\,(n+1)$*.*

Corollary

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Lemma
For any n, $P_H\, n$ implies $\neg\neg P_H\,(n+1)$.

Corollary
For any n, $\neg\neg P_H\, n$.

$\approx_{P_H}$ is not Noetherian

Define a relation

$$\approx_{P_H} : (\Sigma n : nat.\, P_H\, n) \rightarrow (\Sigma n : nat.\, P_H\, n) \rightarrow Prop$$

such that $(n, h_n) \approx_{P_H} (m, h_m)$ iff $n = m$.

Lemma
For any $l : (\Sigma n : nat.\, P_H\, n)^*$, $Acc_{\approx_{P_H}}\, l$ implies $good_{\approx_{P_H}}\, l$.

Corollary
$\neg Acc_{\approx_{P_H}} \langle\rangle$.

MP $\vdash \approx_{P_H}$ is streamless

Lemma

Assume that it is absurd that H is decidable, namely, $\neg(\forall n.\, n \in H \vee \neg n \in H)$. *Assuming Markov's Principle,* $\approx_{P_H}$ *is streamless.*

What we obtain:

In the presence of an undecidable set and Markov's Principle, there is a streamless set which is not provably Noetherian.

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Realizability model

- Construct a domain model for type theory based on untyped lambda calculus extended by constants, following the approach of Coquand and Spiwack.
- Turn the domain model into a realizability model where the terms of the extended lambda calculus are the realizers.
- In this model, MP is realizable, and we can also construct an undecidable set.
- This way, we obtain that

$$MP \to \forall H : \mathrm{nat} \to Prop. \neg\neg\forall n. H\, n \vee \neg H\, n$$

unprovable in type theory.
- We learn that streamless implies Noetherian is unprovable in type theory.