Types in Proof Mining

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Extractive Proof Theory (G. Kreisel): New results by logical analysis of proofs

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Naive Attempt: try to extract an explicit computable function realizing (or bounding) '$\exists y$': $\forall \mathbf{x} \in \mathbb{N}\, \mathbf{F}(\mathbf{x}, \mathbf{f}(\mathbf{x}))$.

Naive attempt fails

Let (a_n) be a nonincreasing sequence in $[0,1]$. Then, clearly, (a_n) is convergent and so a Cauchy sequence which we write as:

$$\mathbf{(1)}\ \forall k \in \mathbb{N} \exists n \in \mathbb{N} \forall m \in \mathbb{N} \forall i, j \in [n; n+m]\ (|a_i - a_j| \leq 2^{-k}),$$

where $[n; n+m] := \{n, n+1, \ldots, n+m\}$.

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By E. Specker 1949 there exist **computable** such sequences (a_n) even in $\mathbb{Q} \cap [0,1]$ **without computable bound** on '$\exists n$' in (1).

Consider the (partial) Herbrand normal form of (1) :

$$\mathbf{(2)}\ \forall k \in \mathbb{N} \forall g \in \mathbb{N}^{\mathbb{N}} \exists n \in \mathbb{N} \forall i, j \in [n; n + g(n)]\ (|a_i - a_j| \leq 2^{-k}).$$

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Proposition

Let (a_n) be any nonincreasing sequence in $[0, 1]$ then

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where

$$\Phi^*(g, k) := \tilde{g}^{(2^k - 1)}(0) \text{ with } \tilde{g}(n) := n + g(n).$$

Moreover, there exists an $i < 2^k$ such that n can be taken as $\tilde{g}^{(i)}(0)$.

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$$\forall \mathbf{k} \in \mathbb{N} \forall \mathbf{g} \in \mathbb{N}^{\mathbb{N}} \exists \mathbf{n} \leq \mathbf{\Phi}^*(\mathbf{g}, \mathbf{k}) \forall \mathbf{i}, \mathbf{j} \in [\mathbf{n}; \mathbf{n} + \mathbf{g}(\mathbf{n})]\ (|\mathbf{a_i} - \mathbf{a_j}| \leq 2^{-\mathbf{k}}),$$

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Moreover, there exists an $i < 2^k$ such that n can be taken as $\tilde{g}^{(i)}(0)$.

Generalized (variable length) Herbrand disjunction!

An Example from Ergodic Theory

X **Hilbert space**, $f : X \to X$ **linear** and $\|f(x)\| \leq \|x\|$ for all $x \in X$.

$$\mathbf{A_n(x) := \frac{1}{n+1} S_n(x),} \text{ where } \mathbf{S_n(x) := \sum_{i=0}^{n} f^i(x) \quad (n \geq 0).}$$

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For every $x \in X$, the sequence $(A_n(x))_n$ converges.

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Theorem (Garrett Birkhoff 1939)

Mean Ergodic Theorem holds for uniformly convex Banach spaces.

Based on a logical metatheorem to be discussed below:

Theorem (K./Leuştean, Ergodic Theor. Dynam. Syst. 2009)

X uniformly convex Banach space,

η a modulus of uniform convexity and $f : X \to X$ as above, $b > 0$.

Then for all $x \in X$ with $\|x\| \leq b$, all $\varepsilon > 0$, all $g : \mathbb{N} \to \mathbb{N}$:

$$\exists \mathbf{n} \leq \boldsymbol{\Phi}(\varepsilon, \mathbf{g}, \mathbf{b}, \eta)\, \forall \mathbf{i}, \mathbf{j} \in [\mathbf{n}; \mathbf{n} + \mathbf{g}(\mathbf{n})] \left(\|\mathbf{A_i}(\mathbf{x}) - \mathbf{A_j}(\mathbf{x})\| < \varepsilon\right),$$

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$$\mathbf{h}, \tilde{\mathbf{h}} : \mathbb{N} \to \mathbb{N}, \ \mathbf{h}(\mathbf{n}) := \mathbf{2}(\mathbf{Mn} + \mathbf{g}(\mathbf{Mn})), \quad \tilde{\mathbf{h}}(\mathbf{n}) := \max_{\mathbf{i} \leq \mathbf{n}} \mathbf{h}(\mathbf{i}).$$

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Special Hilbert case: treated prior by Avigad/Gerhardy/Towsner (again based on logical metatheorem).

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Related problem: bad behavior w.r.t. modus ponens!

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Our approach is based on novel forms and extensions of:

K. Gödel's functional interpretation!

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HA$^\omega$ has λ-abstraction $(\lambda x^\rho . t[x]^\tau)(s^\rho) =_\tau t[s/x]$ and **primitive recursion in all finite types** (Hilbert 1926, Gödel 1958): for $x \in \mathbb{N}$

$$\mathbf{R}_\rho(\mathbf{0}, \mathbf{y}, \mathbf{z}) =_\rho \mathbf{y},\ \mathbf{R}_\rho(\mathbf{x}+\mathbf{1}, \mathbf{y}, \mathbf{z}) =_\rho \mathbf{z}(\mathbf{R}_\rho \mathbf{xyz}, \mathbf{x}),$$

where $=_\rho$ is defined as pointwise (extensional) equality (with a weak extensionality rule; see later).

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$$\mathbf{R}_\rho(0, y, z) =_\rho y, \ \mathbf{R}_\rho(x+1, y, z) =_\rho z(\mathbf{R}_\rho xyz, x),$$

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$\mathbf{PA}^\omega = \mathbf{HA}^\omega + (A \vee \neg A)$.

Towards proofs based on classical logic

Entrance door for classical logic: Markov's principle $\mathbf{M}^\omega$!

$$\mathbf{M}^\omega : \neg\neg\exists x^\rho A_{qf}(x) \to \exists x^\rho A_{qf}(x), \quad A_{qf} \text{ quantifier-free.}$$

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For $\rho \neq \mathbb{N}$: not even unbounded search possible!

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G **extracts** from a given proof p

$$\mathbf{p} \vdash \forall \mathbf{x}\, \exists \mathbf{y}\, \mathbf{A}_{\mathbf{qf}}(\mathbf{x}, \mathbf{y})$$

an explicit effective functional Φ realizing A^G, i.e.

$$\forall \mathbf{x}\, \mathbf{A}_{\mathbf{qf}}(\mathbf{x}, \boldsymbol{\Phi}(\mathbf{x})).$$

Functional interpretation in five minutes

Gödel's **functional interpretation** G is a map

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- $\mathbf{A} \leftrightarrow \mathbf{A}^\mathbf{G}$ by **quantifier-free choice** in all types

$$\mathbf{QF\text{-}AC} : \ \forall \underline{\mathbf{a}} \exists \underline{\mathbf{b}} \, \mathbf{F}_{\mathbf{qf}}(\underline{\mathbf{a}}, \underline{\mathbf{b}}) \longrightarrow \exists \underline{\mathbf{B}} \forall \underline{\mathbf{a}} \, \mathbf{F}_{\mathbf{qf}}(\underline{\mathbf{a}}, \underline{\mathbf{B}}(\underline{\mathbf{a}})).$$

Functional interpretation in five minutes

Gödel's **functional interpretation** G is a map

$$\mathbf{G} : \mathbf{Form}(\mathrm{PA}^\omega) \longrightarrow \mathbf{Form}(\mathrm{PA}^\omega), \ \mathbf{A} \mapsto \mathbf{A}^\mathbf{G}$$

such that

- $\mathbf{A}^\mathbf{G} \equiv \forall \underline{\mathbf{x}} \exists \underline{\mathbf{y}} \, \mathbf{A}_\mathbf{G}(\underline{\mathbf{x}}, \underline{\mathbf{y}})$, where $\mathbf{A}_\mathbf{G}$ is **quantifier-free**,
- For $\mathbf{A} \equiv \forall \underline{\mathbf{x}} \exists \underline{\mathbf{y}} \, \mathbf{A}_{\mathbf{qf}}(\underline{\mathbf{x}}, \underline{\mathbf{y}})$ one has $\mathbf{A}^\mathbf{G} \equiv \mathbf{A}$.
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- $\mathbf{A^G \equiv \forall \underline{x} \exists \underline{y}\, A_G(\underline{x}, \underline{y})}$, where $\mathbf{A_G}$ is **quantifier-free**,
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- $\underline{\mathbf{x}}, \underline{\mathbf{y}}$ are tuples of **functionals of finite type.**
- $\mathbf{N :=}$ Krivine's negative transl. $\overset{\text{Streicher/K.}}{\Rightarrow}$ $\mathbf{G}$ = Shoenfield Variant!

$$A^G \equiv \forall u \exists x\, A_G(u, x), \quad B^G \equiv \forall v \exists y\, B_G(v, y).$$

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- It also scales up all the way to full countable and even dependent choice (including full 2nd order arithmetic), where then Φ is bar recursive (and holds in $\mathcal{M}^\omega$ or $\mathcal{C}^\omega$): Spector 1962.

Connection to no-counterexample interpretation

Let A be a prenex (arithmetical) formula and $A^S, A^G, A^{n.c.i}$ its Skolem, G and $n.c.i.$ interpretations resp., then

$$\mathrm{HA}^{\boldsymbol{\omega}} \vdash \mathbf{A^S} \longrightarrow \mathbf{A^G} \longrightarrow \mathbf{A^{n.c.i}},$$

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- A^G **just right**:

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Majorizability

The functionals occurring in functional interpretation (such as the primitive recursive ones from PA^ω but also the bar recursive ones) have a striking mathematical structure property:

Definition (W.A. Howard 1973)

$$\begin{cases} \mathbf{x^* \gtrsim_{\mathbb{N}} x :\equiv x^* \geq x,} \\ \mathbf{x^* \gtrsim_{\rho\to\tau} x :\equiv \forall y^*, y(y^* \gtrsim_\rho y \to x^*(y^*) \gtrsim_\tau x(y)).} \end{cases}$$

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Monotone functional interpretation MD (K.96) directly extracts majorants for functionals satisfying D.

- Provides **uniform** bounds.
- Applied to principles such as the **binary König's Lemma WKL** which do not have a computable D-interpretation.

A logical metatheorem for concrete spaces P, K

P Polish, K compact metric space, $A_\exists$ existential, $=_X, =_K$-extensional.
BA:= basic arithmetic, e.g. PA^ω+QF-AC, HBC Heine/Borel compactness (via WKL).

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Important: $\Phi(f_x)$ does not depend on $y \in K$ but on representation f_x of x!

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- In the mean ergodic theorem, X was a totally general Hilbert space and the independence of $x \in X$ (and f) only depended on $b \geq \|x\|$.
- Crucially used for this that the proof treats X as abstract structure that is not represented as separable space.

Formal systems for analysis with abstract spaces X

Types: (i) $\mathbb{N}, X$ are types, (ii) with ρ, τ also $\rho \to \tau$ is a type.

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$\mathcal{A}^{\omega,X}$**:=PA**$^{\omega,X}$+**DC**, where

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$\mathcal{A}^{\omega}[X, d, \ldots]$ results by adding constants $d_X, \ldots$ with axioms expressing that $(X, d, \ldots)$ is a nonempty metric, hyperbolic ... space.

A warning concerning equality

Extensionality rule (**only!**):

$$\frac{s =_\rho t}{r(s) =_\tau r(t)},$$

where only $x =_{\mathbb{N}} y$ primitive equality predicate but for $\rho \to \tau$

$$x^X =_X y^X :\equiv d_X(x,y) =_{\mathbb{R}} 0_{\mathbb{R}},$$
$$x =_{\rho\to\tau} y :\equiv \forall v^\rho (s(v) =_\tau t(v)).$$

A novel form of majorization

y, x functionals of types $\rho, \widehat{\rho} := \rho[\mathbb{N}/X]$ and a^X of type X:

$$\mathbf{x}^{\mathbb{N}} \gtrsim^{\mathbf{a}}_{\mathbb{N}} \mathbf{y}^{\mathbb{N}} :\equiv \mathbf{x} \geq \mathbf{y}$$
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Example:

$$\mathbf{f^* \gtrsim^a_{X \to X} f \equiv \forall n \in \mathbb{N}, x \in X[n \geq d(a, x) \to f^*(n) \geq d(a, f(x))].}$$

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$f : X \to X$ is **nonexpansive (n.e.)** if $\mathbf{d}(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{y})) \leq \mathbf{d}(\mathbf{x}, \mathbf{y})$.

Then $\lambda n.n + b \gtrsim^{a}_{X \to X} f$, if $d(a, f(a)) \leq b$.

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Normed linear case: $a := 0_X$.

Treatment of several metric structures $X_1, X_2, \ldots, X_n$

- Instead of one base type X one can also have several of which some are metric spaces, other normed spaces etc. together with the **product spaces** (where the majorants depend on the product metric chosen but not the majorizability).

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  Nonexpansive maps $f : X \to Y$ are (a, b)-majorized by Id $(a \in X, b \in Y)$.
- **Convex subsets** $C \subseteq X$ can be added as new types which are related with X via isometric isomorphic embeddings. For uniformly convex Banach spaces X and closed convex C one can add (easy to majorize) **metric projection operators** characterized by universal axioms (Master Thesis D. Günzel 2013).

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- **Structures** which are not sufficiently uniform **get uniformized**: strict convex $\rightarrow$ uniformly convex; uniformly Gâteaux differentiable norm $\rightarrow$ uniformly smooth; separability $\rightarrow$ total boundedness (of bounded substructures) etc.

Small types (over $\mathbb{N}, X$): $\mathbb{N}$, $\mathbb{N} \to \mathbb{N}$, X, $\mathbb{N} \to X$, $X \to X$.

Theorem (K., Trans.AMS 2005, Gerhardy/K., Trans.AMS 2008)

Let P, K be Polish resp. compact metric spaces, $A_\exists$ $\exists$-formula, $\underline{\tau}$ small. If $\mathcal{A}^\omega[X, d, W]$ **proves**

$$\forall \mathbf{x} \in \mathbf{P} \forall \mathbf{y} \in \mathbf{K} \forall \underline{\mathbf{z}}^{\underline{\tau}} \exists \mathbf{v}^{\mathbb{N}} \mathbf{A}_\exists(\mathbf{x}, \mathbf{y}, \underline{\mathbf{z}}, \mathbf{v}),$$

then one can extract a **computable** $\Phi : \mathbb{N}^{\mathbb{N}} \times \underline{\mathbb{N}}^{(\mathbb{N})} \to \mathbb{N}$ s.t. the following holds in every nonempty hyperbolic space: for all representatives $r_x \in \mathbb{N}^{\mathbb{N}}$ of $x \in P$ and all $\underline{z}^{\underline{\tau}}$ and $\underline{z}^* \in \mathbb{N}^{(\mathbb{N})}$ s.t. $\exists a \in X(\underline{z}^* \gtrsim^a_{\underline{\tau}} \underline{z})$:

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Small types (over $\mathbb{N}, X$): $\mathbb{N}$, $\mathbb{N} \to \mathbb{N}$, X, $\mathbb{N} \to X$, $X \to X$.

Theorem (K., Trans.AMS 2005, Gerhardy/K.,Trans.AMS 2008)

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Explains the unwinding of Birkhoff's proof above.

Elements of the proof

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- The interpretation of d_X or $\|\cdot\|_X$ uses an **ineffective discontinuous** (but **trivially majorizable**) operator $\circ$ to select a canonical name for the resulting real.
- The constants related to X are all **majorized by simple terms not involving X**.

Warning on extensionality

Full extensionality is in conflict with the metatheorem:

$$\forall f^{X\to X}, x^X, y^X (x =_X y \to f(x) =_X f(y))$$

yields with Markov's principle

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and hence with metatheorem for bounded metric spaces X:

All functions $f : X \to X$ are equicontinuous

(and the modulus only depends on the bound b of the metric).

Baillon's nonlinear ergodic theorem

Theorem (J.-B. Baillon 1975): X Hilbert space, $C \subset X$ bounded closed and convex, $U : C \to C$ nonexpansive. Then for every $u_0 \in C$, the sequence of Cesàro means (u_n)

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- **Strong convergence in general fails** (counterexample by Baillon).
- In important **special cases** (see talk on Friday) **strong convergence** can be established.

A quantitative ‘metastable’ version of Baillon’s theorem

Theorem (K., Comm.Contemp.Math.2012)

Logical analysis of the proof of Baillon’s theorem due to Brézis-Browder yields a primitive recursive functional φ such that for a bar-recursive bound Ω^* extracted from the weak compactness proof and $\varepsilon > 0$, $g : \mathbb{N} \to \mathbb{N}$, we get $\varphi(\Omega^*, \varepsilon, b, g)$ as a bound on the metastable version of the weak Cauchy property of the Cesàro means (u_n), i.e.

$$\forall \mathbf{w} \in \mathbf{B_1(0)}\, \exists \mathbf{n} \leq \varphi(\mathbf{\Omega^*}, \varepsilon, \mathbf{b}, \mathbf{g})\, \forall \mathbf{i}, \mathbf{j} \in [\mathbf{n}; \mathbf{n}+\mathbf{g}(\mathbf{n})]\, \left(|\langle \mathbf{u_i} - \mathbf{u_j}, \mathbf{w} \rangle| < \varepsilon\right).$$

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Based on the detailed construction of Ω^* and results of W.A. Howard it follows that $\Phi(\varepsilon, b, g) := \varphi(\Omega^*, \varepsilon, b, g)$ is definable in Gödel's T_4 (note that Φ has type level 2).

A strong nonlinear ergodic theorem

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Theorem [R. Wittmann 1990]: Let $C \subseteq X$ be an **arbitrary** subset and $f : C \to C$ s.t.

$$\forall \mathbf{u}, \mathbf{v} \in \mathbf{C}\ (\|\mathbf{f}(\mathbf{u}) + \mathbf{f}(\mathbf{v})\| \leq \|\mathbf{u} + \mathbf{v}\|).$$

Then the sequence of Cesàro means (x_n) converges strongly.

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Then the sequence of Cesàro means (x_n) converges strongly.

- Holds for C closed under $v \mapsto -v$ and nonexpansive, odd f.
- The condition above does **not even** imply that f is **continuous**. But f has a **trivial majorant** $f^* := Id$ $(v := u)$.
  Hence: Metatheorem applicable!

Theorem (P. Safarik, J. Math. Anal. Appl. 2012)

$$\forall k \in \mathbb{N}\, \forall g \in \mathbb{N}^{\mathbb{N}}\, \exists m \leq \Phi(k, b, g^M)\, (\|x_m - x_{m+g(m)}\| \leq 2^{-k}),$$

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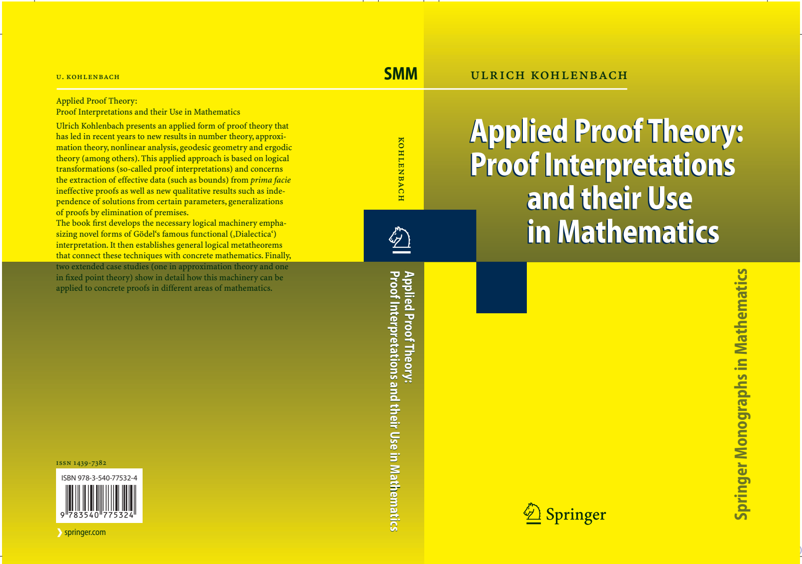
where (for $b \geq \|x_0\|$)

$$\Phi(k, b, g) := (N(2k+7, g) + P(2k+7, g)) \cdot b \cdot 2^{2k+8} + 1,$$
$$P(k, g) := P_0(k, F(k, g, N(k, g))),$$
$$F(k, g, n)(p) := p + n + \tilde{g}((n+p) \cdot b \cdot 2^{k+1}),$$
$$L(k, g)(n) := n + P_0(k, F(k, g, n)) + \tilde{g}((n + P_0(k, F(k, g, n))) \cdot b \cdot 2^{k+1}),$$
$$N(k, g) := (L(k, g))^{(b^2 2^{k+2})}(0),$$
$$P_0(k, f) := \tilde{f}^{(b^2 2^k)}(0),\ \tilde{f}(n) := n + f(n),\ f^M(n) := \max_{i \leq n+1} f(i).$$



U. KOHLENBACH

Applied Proof Theory:
Proof Interpretations and their Use in Mathematics

Ulrich Kohlenbach presents an applied form of proof theory that has led in recent years to new results in number theory, approximation theory, nonlinear analysis, geodesic geometry and ergodic theory (among others). This applied approach is based on logical transformations (so-called proof interpretations) and concerns the extraction of effective data (such as bounds) from *prima facie* ineffective proofs as well as new qualitative results such as independence of solutions from certain parameters, generalizations of proofs by elimination of premises.
The book first develops the necessary logical machinery emphasizing novel forms of Gödel's famous functional (,Dialectica') interpretation. It then establishes general logical metatheorems that connect these techniques with concrete mathematics. Finally, two extended case studies (one in approximation theory and one in fixed point theory) show in detail how this machinery can be applied to concrete proofs in different areas of mathematics.

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