Computable data refinements by quotients and parametricity[1]

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Motivation

Verifying computer algebra algorithms

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- Computer algebra algorithms can help automate proofs
- Formal proofs bridge the gap between paper correctness proofs and real-life implementations
- Proof assistants can provide independent verification of results obtained by computer algebra programs (e.g. $\zeta(3)$ is irrational, computation of homology groups)

Context

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Specificity of computer algebra programs:

- Computer algebra algorithms can have complex specifications
- Efficiency matters!

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Specificity of computer algebra programs:

- Computer algebra algorithms can have complex specifications
- Efficiency matters!

Problem: these aspects are often in tension
We suggest a methodology based on refinements to achieve separation of concerns

Separation of concerns

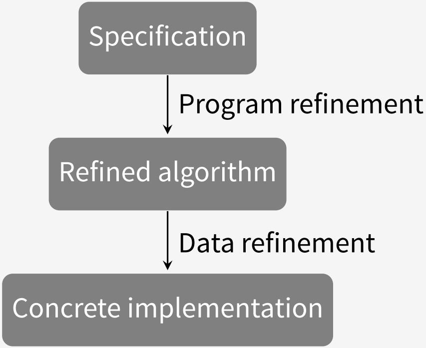
We know that a program must be correct and we can study it from that viewpoint only; we also know that it should be efficient and we can study its efficiency on another day, so to speak. [...] But nothing is gained – on the contrary! – by tackling these various aspects simultaneously. It is what I sometimes have called "the separation of concerns"

Dijkstra, Edsger W.
"On the role of scientific thought" (1982)

Program and data refinements

We distinguish two kinds of refinements:

- Program refinement: improving the algorithmics
- Data refinement: switching to more efficient data representation



Isomorphic structures

First example: natural numbers

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Pb: this goes against the "small scale reflection" approach (following SSReflect)

Partial operators

Second example: polynomials in SSReflect

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Variable R : ringType.
Record polynomial :=
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Operators over `(seq R)` are partially specified as refinements of their counterparts from `(polynomial R)`.

Quotient

Third example: rational numbers

```
Record rat : Set := Rat {
  valq : (int * int) ;
  _ : (0 < valq.2) && coprime `|valq.1| `|valq.2|
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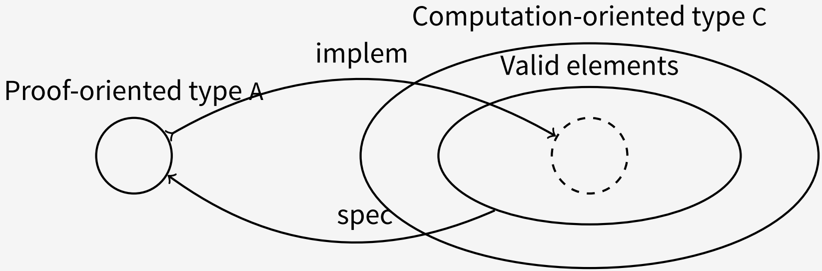
- This invariant is costly to maintain during computations

We would like to relax the constraint and express that `rat` is isomorphic to a quotient of a subset of pairs of integers.

Interface for refinements



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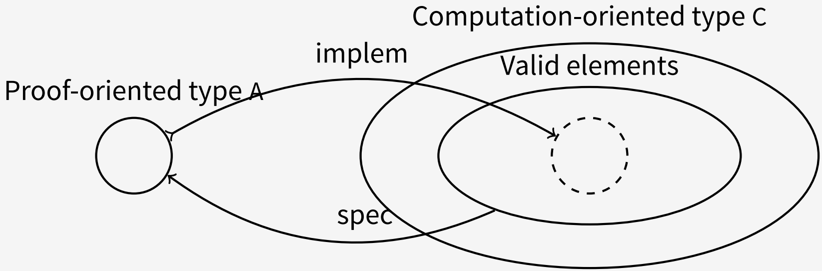


```
Class refinement A C := Refinement {
  implem : A -> C;
  spec : C -> option A;
  implemK : forall x : A, spec (implem x) = Some x
}.
```

Expressing a refinement

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Class refines {A C} `{refinement A C} (a : A) (b : C) :=
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Addition over N refines the one over `nat`:

```
Lemma refines_add (m n : nat) (u v : N) : refines m u ->
  refines n v -> refines (addn m n) (N.add u v)
```

Adding new rules to the game

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- Generic programming: only one description of the algorithm, then specialized for proofs or computations
- Compositionality: refining `(polynomial R)` to `(seq R)`, what is `R`?
- Automating correctness proofs when changing representations

Generic programming: addition over rationals

Generic datatype

```
Definition Q Z := (Z * Z).
```

Generic operations

```
Definition addQ Z ‘{add Z} ‘{mul Z} : add (Q Z) :=
  fun x y => (x.1 * y.2 + y.1 * x.2, x.2 * y.2).
```

To prove correctness of `addQ`, operators `(+ : add Z)` and `(* : mul Z)` are instanciated to proof-oriented definitions.
When computing, these operators are instanciated to more efficient ones.

Compositionality

Correctness of addQ

```
Definition addQ Z ‘{add Z} ‘{mul Z} : add (Q Z) :=
  fun x y => (x.1 * y.2 + y.1 * x.2, x.2 * y.2).

Lemma refines_addQ Z ‘{refinement int Z, add Z, mul Z} :
  [...] ->
  forall (x y : rat) (u v : Q Z), refines m u ->
  refines n v -> refines (addq m n) (addQ u v).
```

This will be provable as soon as the addition and multiplication over `Z` refines the ones over `int`.
Hence, refinements are composable: for any `Z` refining `int`, `(Q Z)` refines `rat` (with associated operators).

Automation

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Correctness of addQ

```
Class param A B (R : A -> B -> Prop) (m : A) (n : B) :=
  param_rel : R m n.

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Lemma refines_addQ Z `{refinement int Z, add Z, mul Z} :
  param (refines ==> refines ==> refines) addz (+) ->
  param (refines ==> refines ==> refines) mulz (*) ->
  param (refines ==> refines ==> refines) addq addQ
```

Automation

Correctness of addQ

```
Class param A B (R : A -> B -> Prop) (m : A) (n : B) :=
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Lemma refines_addQ_int :
  param (refines ==> refines ==> refines)
    addq (@addQ int addz mulz)

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```

- We prove `refines_addQ_int` manually
- Then we deduce `refines_addQ` by meta-programmming

Parametricity of addQ

```
  Z : Type
  _ : refinement int Z
  addZ : add Z
  mulZ : mul Z
  _ : param (refines ==> refines ==> refines) addz (+)
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=====================================================
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```

Parametricity of addQ

```
   Z : Type
   _ : refinement int Z
   addZ : add Z
   _ : param (refines ==> refines ==> refines) addz (+)
=============================================================
param ((refines ==> refines ==> refines) ==>
      refines ==> refines ==> refines)
  (@addQ int addz) (@addQ Z (+))
```

Parametricity of addQ

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  Z : Type
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=============================================================
param ((refines ==> refines ==> refines) ==>
      (refines ==> refines ==> refines) ==>
       refines ==> refines ==> refines)
  (@addQ int) (@addQ Z)
```

Conclusion and ongoing work

The approach we described:

- Reconciles convenient proofs with efficient computations
- Provides a mechanism to smoothly switch from one world to the other
- Avoids duplication of code

We are currently:

- Applying it to a variety of data structures (polynomials, matrices) supporting algorithms we had previously verified: Karatsuba's polynomial multiplication, Strassen's matrix product, Sasaki-Murao algorithm
- Polishing technical details to improve performance of proof search and efficiency of the generic code

Thanks!