Realizability for Peano Arithmetic with Winning Conditions in HON Games

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ENS de Lyon

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Introduction

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- Adequacy and extraction for Peano arithmetic
  - Interpretation of system T terms corresponding to CPS translated proofs are indeed realizers
  - Extraction in the spirit of Friedman's trick

Realizability

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Depends crucially on the choice of $\mathcal{L}$

Our framework

Kleene's original version:

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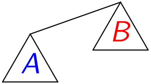
- $\mathcal{M} =$ strategies of the HON game semantics

  Motivations: nice framework for $\left|\begin{array}{l}\text{references}\\ \text{control operators}\end{array}\right.$

Arenas

$A \to B \rightsquigarrow$



$\mathbb{U} \rightsquigarrow$

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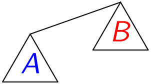
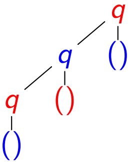


$\mathbb{U} \rightsquigarrow$

$(\mathbb{U} \to \mathbb{U}) \to \mathbb{U} \rightsquigarrow$



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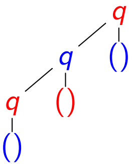

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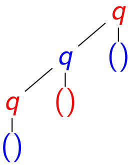

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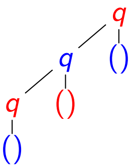

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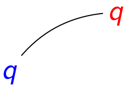

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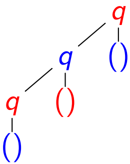

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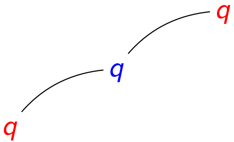

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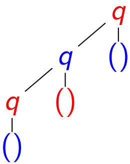

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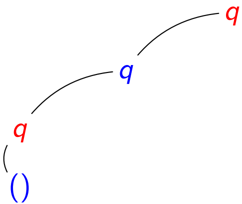

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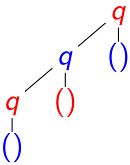

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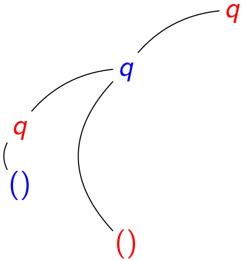

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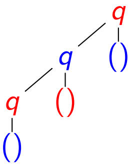

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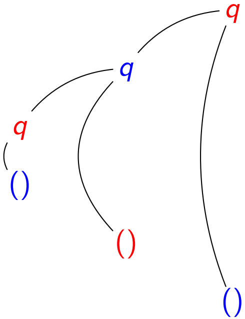

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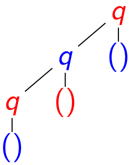

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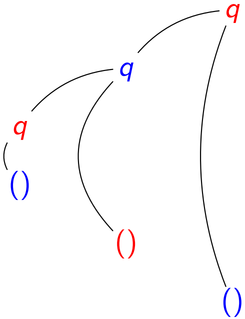

$\lambda F.F()$

Arenas

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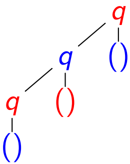

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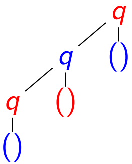
$(\mathbb{U} \to \mathbb{U}) \to \mathbb{U} \leadsto$



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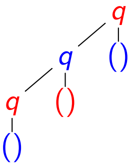

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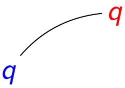

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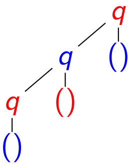

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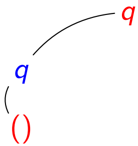

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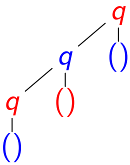

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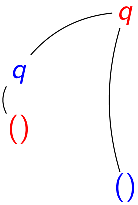

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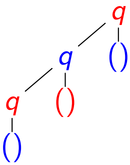

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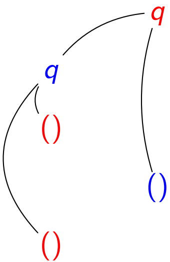

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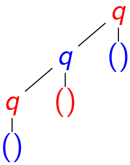

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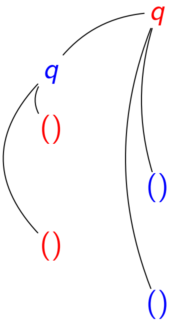

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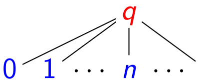
Beyond λ-terms: references

$\mathbb{N} \rightsquigarrow$



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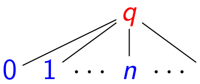

$$(\mathbb{U} \to \mathbb{U}) \to \mathbb{N}$$

$\lambda F.\text{new a:=0 in } F(\lambda y.\text{a:=!a+1; } y)(\lambda\ \text{!a})$

Beyond λ-terms: references

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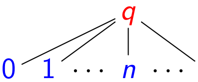

q

$0 \quad 1 \quad \cdots \quad n \quad \cdots$

$$\begin{array}{ccccc}(\mathbb{U} & \to & \mathbb{U}) & \to & \mathbb{N}\\ &&&& q\end{array}$$

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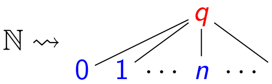

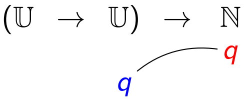

a=0

λF.new a:=0 in $F(\lambda y$.a:=!a+1; $y)(\lambda$!a)

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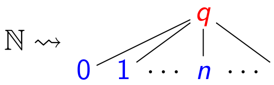

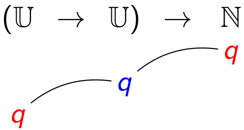

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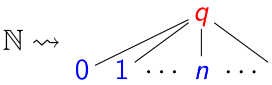

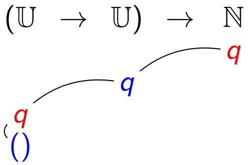

a=1

λF.new a:=0 in F(λy.a:=!a+1; y)(λ !a)

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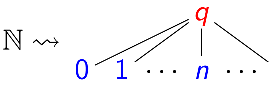

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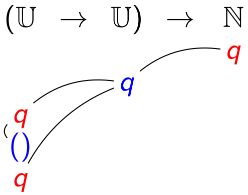
q

$0 \quad 1 \quad \cdots \quad n \quad \cdots$



$(\mathbb{U} \to \mathbb{U}) \to \mathbb{N}$

q

q

q

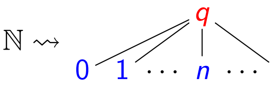
$()$

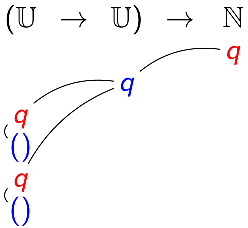
q

a=1

λF.new a:=0 in $F(\lambda y$.a:=!a+1; $y)(\lambda$!a)

Beyond λ-terms: references

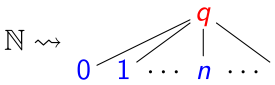


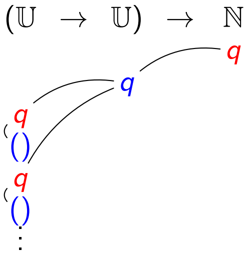


a=2

λF.new a:=0 in $F(\lambda y$.a:=!a+1; $y)(\lambda$!a)

Beyond λ-terms: references





a=n-1

λF.new a:=0 in F(λy.a:=!a+1; y)(λ !a)

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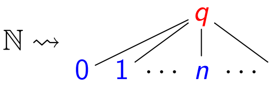

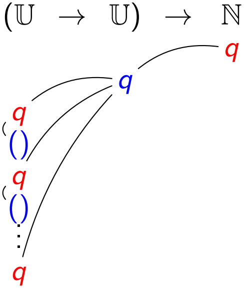

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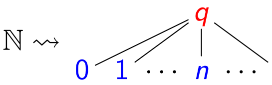

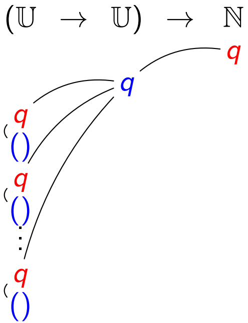

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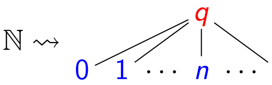

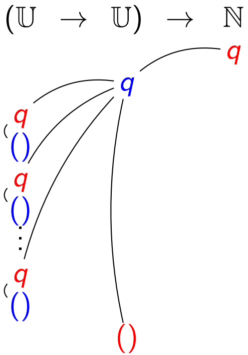

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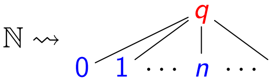

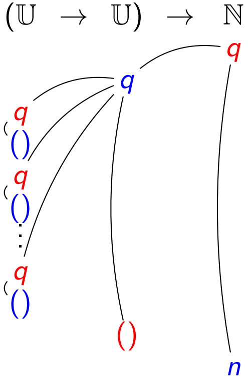

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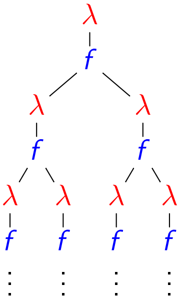
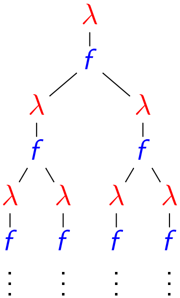
Infinitary model

Infinitary model



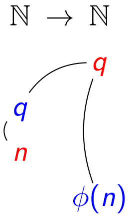
$Y(\lambda x.fxx)$

Infinitary model



$Y(\lambda x.fxx)$

$\phi : \mathbb{N} \to \mathbb{N}$ any function


$\mathbb{N} \to \mathbb{N}$

q

q

n

$\phi(n)$

1. Introduction
2. HON game semantics
3. Winning conditions
4. Realizability

Introduction

How to define a realizability relation in presence of references and infinite behaviours

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Our way:

- Realizability defined at the low-level of *interactions*
- Then lifted to the level of strategies

Winning conditions

$\mathcal{A}$

$\mathcal{A}$: arena

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$(\mathcal{A}, \mathcal{W}_{\mathcal{A}})$

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$\mathcal{W}_{\mathcal{A}}$: set of threads (interactions) on $\mathcal{A}$

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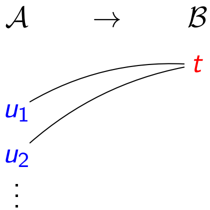

$\mathcal{A} \quad \to \quad \mathcal{B}$

t

u_1

u_2

$\vdots$

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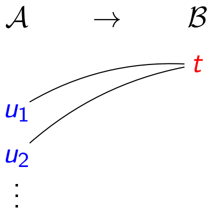

$t \in \mathcal{W}_{\mathcal{A} \to \mathcal{B}}$ if:

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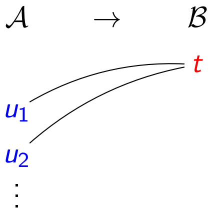

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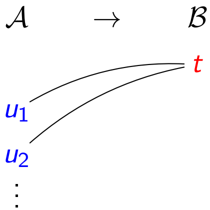


$t \in \mathcal{W}_{\mathcal{A} \to \mathcal{B}}$ if:

whenever $u_1, u_2, \ldots \in \mathcal{W}_\mathcal{A}$, $t \in \mathcal{W}_\mathcal{B}$

A winning thread

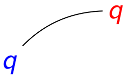
$$((A^* \rightarrow B^*) \rightarrow A^*) \rightarrow A^*$$

A winning thread

$$
\begin{array}{ccccccc}
((A^* & \rightarrow & B^*) & \rightarrow & A^*) & \rightarrow & A^* \\
 & & & & & & q
\end{array}
$$

A winning thread

$$((A^* \to B^*) \to A^*) \to A^*$$



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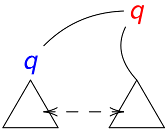

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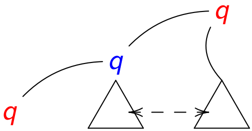

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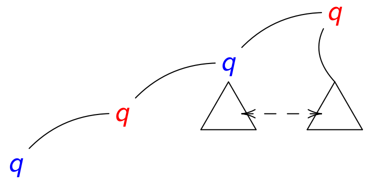

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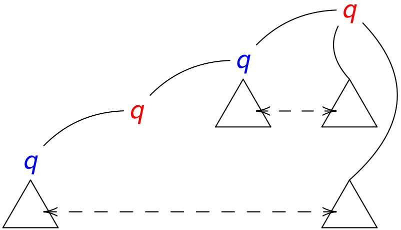

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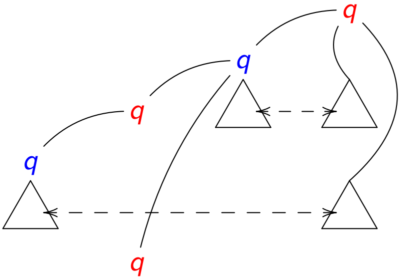

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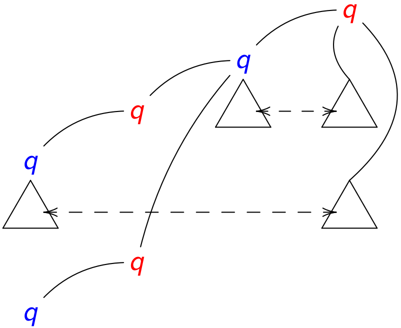

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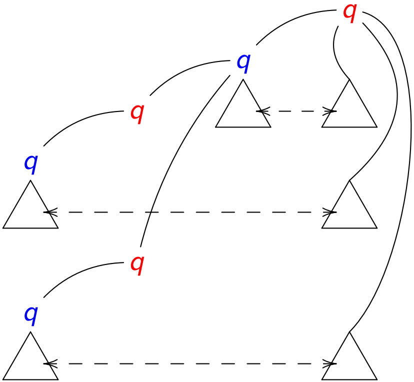

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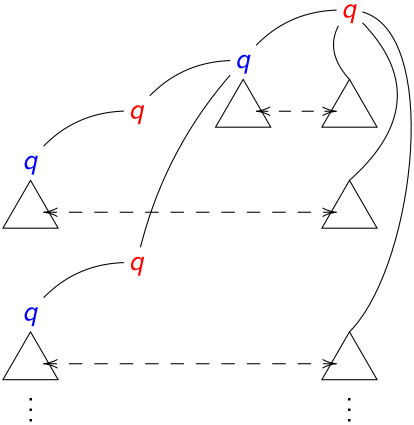

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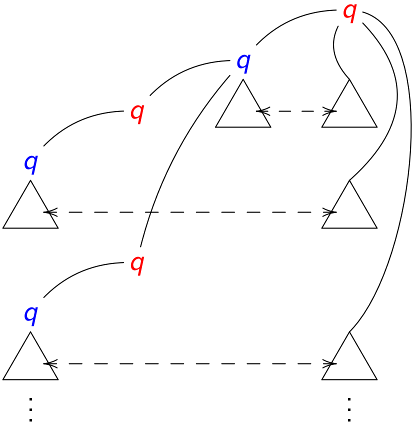
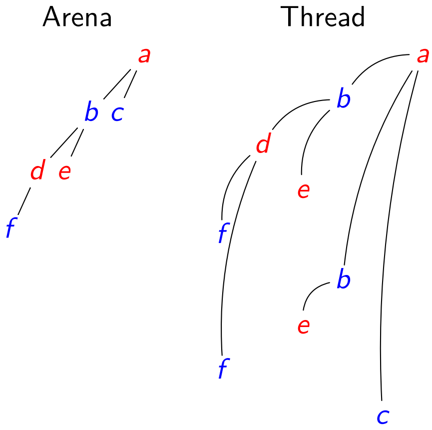


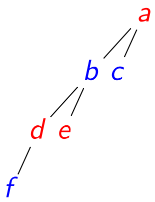
threads defining the strategy cc

P-subthread, O-subthread

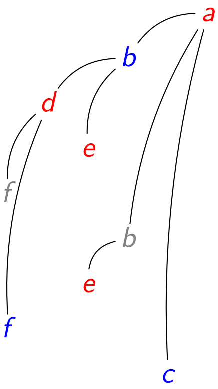


P-subthread, *O*-subthread

Arena



Thread



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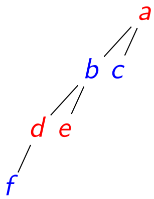

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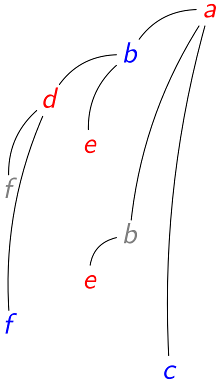

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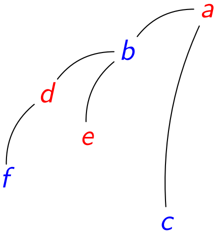

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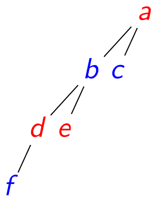

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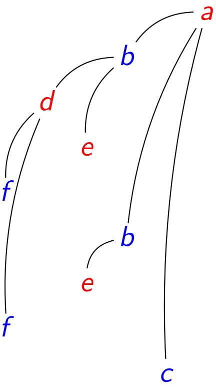

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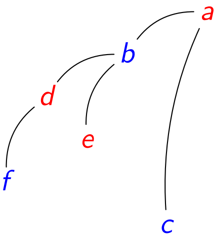

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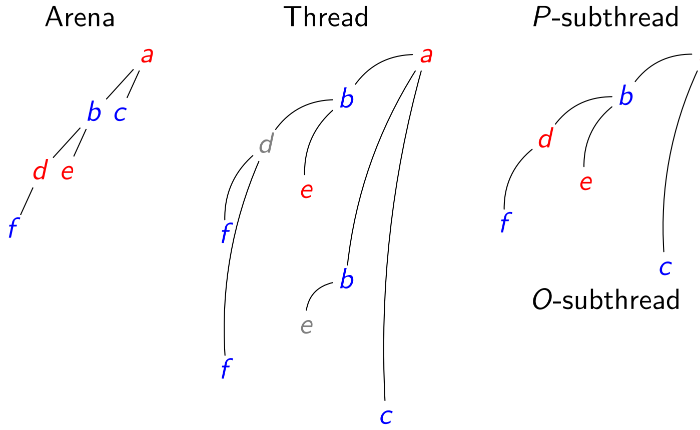

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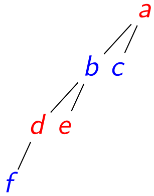

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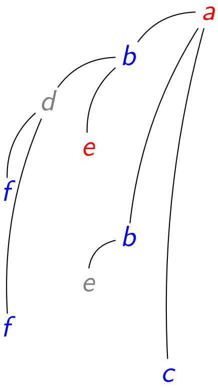

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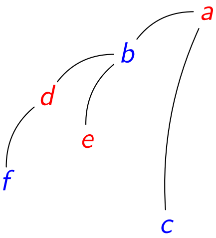
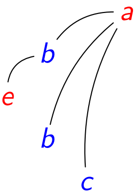


O-subthread



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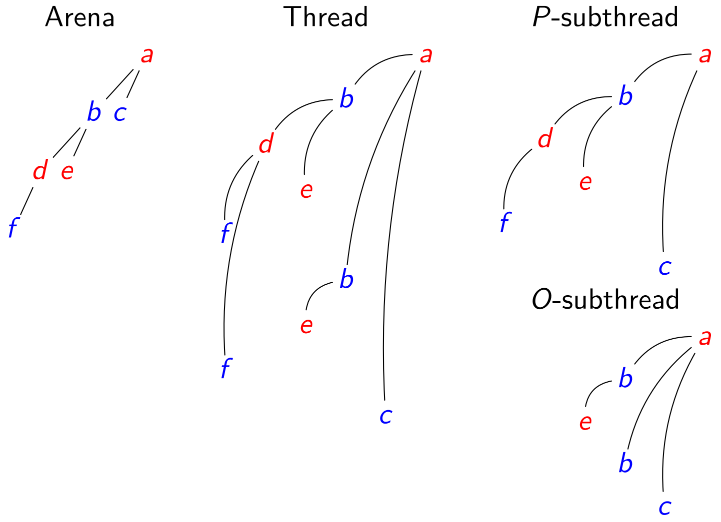

Winning strategy

σ strategy on $\mathcal{A}$

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Definition (Augmented play)

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$s \in \sigma$ or s maximal (odd or infinite) play of σ

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σ winning on $(\mathcal{A}, \mathcal{W}_{\mathcal{A}})$:

t augmented thread of $\sigma \Rightarrow t \in \mathcal{W}_{\mathcal{A}}$

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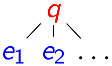
$\mathcal{L}$ first-order language, E countable $\mathcal{L}$-structure, arena $\mathcal{E} \equiv$ 

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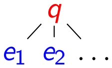
$\mathcal{L}$ first-order language, E countable $\mathcal{L}$-structure, arena $\mathcal{E} \equiv$ 

formula	arena	winning condition
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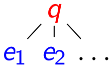

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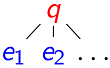

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$\forall x A$	A^*	$\bigcap_{e \in \mathbb{E}} \mathcal{W}_{A[e/x]}$

The realizability relation

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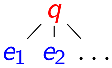

formula	arena	winning condition
A	A^*	$\mathcal{W}_A$
$A \Rightarrow B$	$A^* \to B^*$	$\mathcal{W}_{A \to B}$
$\forall x A$	A^*	$\bigcap_{e \in \mathbb{E}} \mathcal{W}_{A[e/x]}$

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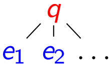
$\mathcal{L}$ first-order language, E countable $\mathcal{L}$-structure, arena $\mathcal{E} \equiv$ 

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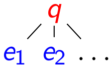
$\mathcal{L}$ first-order language, E countable $\mathcal{L}$-structure, arena $\mathcal{E} \equiv$ 

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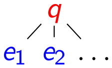

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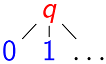
$$id_{\mathcal{E}} \Vdash \forall x(x = x) \qquad id_{A^* \to \mathcal{E}} \Vdash \forall x \forall y((A[x] \Rightarrow \bot) \Rightarrow A[y] \Rightarrow x \neq y)$$

Peano

$E = \mathbb{N}$, associated arena



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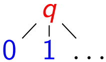

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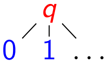
$E = \mathbb{N}$, associated arena 

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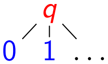
$E = \mathbb{N}$, associated arena 

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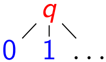

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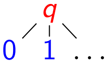
$E = \mathbb{N}$, associated arena 

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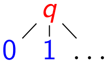
$E = \mathbb{N}$, associated arena 

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Theorem (Adequacy for Peano arithmetic)

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$\perp\!\!\!\perp \subseteq \mathbb{N}$ (à la Friedman)

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- $+$ some adaptation for equality

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Theorem (Extraction)

If π is a proof of $PA \vdash \forall x \exists y A[x, y]$ with A atomic,
$\pi \rightsquigarrow \sigma$ s.t. for any n, $\sigma(\llbracket n \rrbracket) = \llbracket f(n) \rrbracket$

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No secret: a CPS translation (c.f.: Reus & Streicher)

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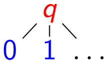

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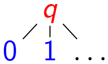

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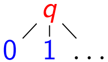

is $R^{(R^{\mathbf{N}})}$ where:

$$R \equiv \left\{ \bullet \right\} \qquad \mathbf{N} \equiv \{ \emptyset \mid n \in \mathbb{N} \}$$

Conclusion

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- Realizability for classical logic in presence of references

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