A Learning-Based model of Polymorphism (ongoing research)

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Possible research goals

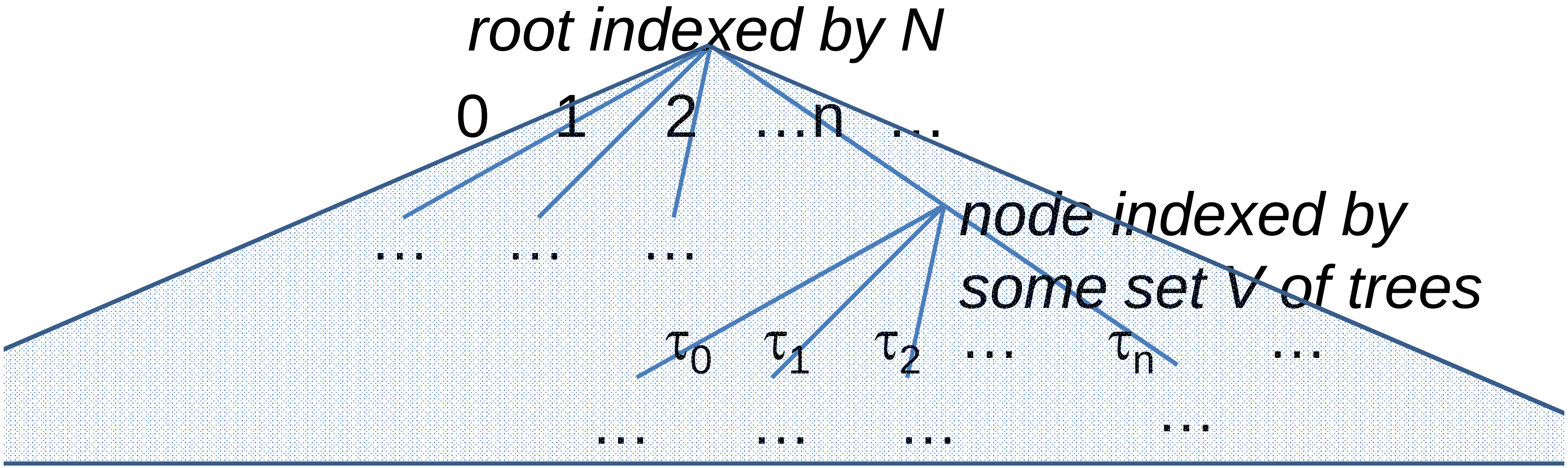
- **Proof Theory.** To find a proof-theoretical analysis of second order arithmetic (a recursive notation system for all provably well-founded ordinals of second order arithmetic).
- **Type Theory.** To find a model of system F in which terms and types are well-founded trees, defined in a ***stratified*** way.
- **Learning Theory**. To introduce a model in which a map may learn a ***correct*** output for input values outside its original domain.

First order predicative Type Theory

- We may easily define a model U of first order type theory in which terms and types are trees defined in a stratified way.
- Basic types like **Bool, N, List** and their terms have a direct interpretation.
- A type **A->B** is interpreted by a binary tree [A->B] whose immediate subtrees are the interpretations [A], [B] of A, B. Then we define a membership relation (.) $\in$[A->B].
- A term **t:A->B** is interpreted by a tree [t] with the children of the root indexed by a$\in$[A], and immediate subtrees [t](a) for any a$\in$[A].
- This interpretation U is ***stratified*** because uses previously defined objects only. It describes a term as a total computation. We may define the interpretation using the subset Ω of well-founded trees of U.

The universe U of a stratified interpretation

- U is a set of trees, such that the index set of branchings of every node is either a basic type or some set V of ***previously defined*** trees:



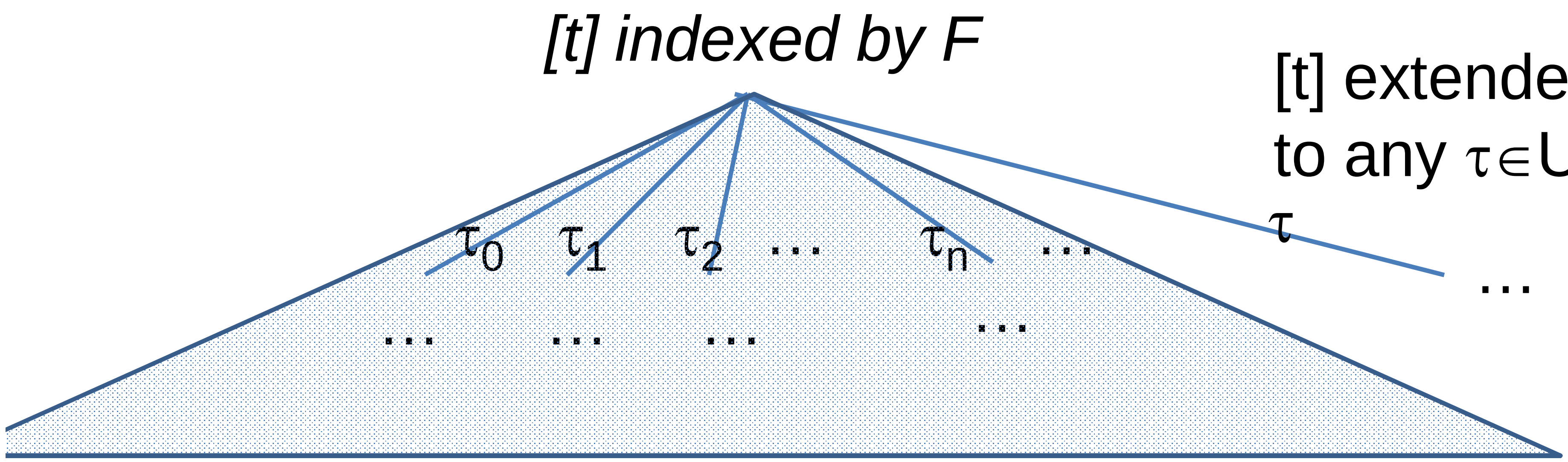
Above: a generic tree $\tau \in U$. We denote with Ω the subset of well-founded trees of U.

Interpreting second order quantifications

- It is well-known why Ω fails to interpret types of the form $\forall\alpha.A$: $[\forall\alpha.A]$ should be a tree whose branching is indexed over all tree-interpretations of types, ***including the very tree*** $[\forall\alpha.A]$ ***we are defining*** and which is not yet available.
- It is well-known too that this obstacle does not prevent to interpret terms as ***partial*** computations and ***ill-founded*** trees of U: it is enough to define $[\forall\alpha.A]$ as a tree whose branching is the set F of all ***finite*** trees, and consider only Scott continuous maps, using only a finite part of the input to produce a finite part of the output.
- From the behavior of some $[t]\in[\forall\alpha.A]$ over all finite trees of F we may uniquely define a Scott continuous map over all trees of U, and over all interpretations [T] of some type T.

The interpretation of $[t]\in[\forall\alpha.A]$ in a Scott Domain of trees

- [t] is a tree whose index set is the set F of all finite trees, and ***by continuity it has a unique extension*** to a map over the set U of all trees of the model.



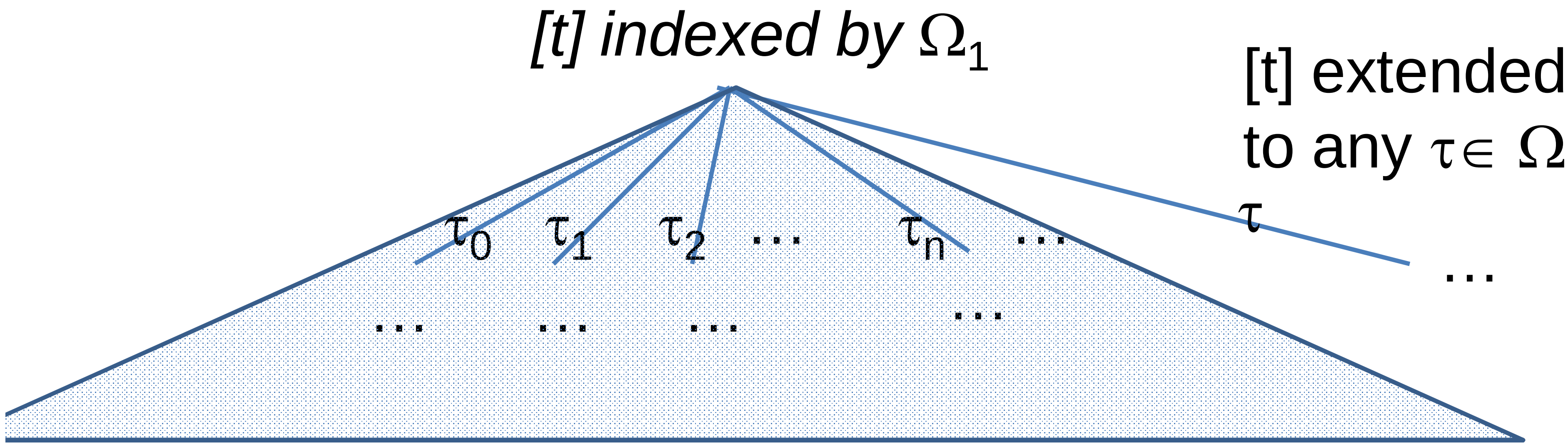
The set F of finite trees is a kind of ***threshold*** for the unicity of the extension: if we know the behavior of [t] over F we know the behavior of [t] over U

Toward a stratified interpretation of second order quantifications

- The interpretation with Scott continuous maps interpret types as elements of U (as possibly ill-founded trees), not as elements of Ω (well-founded trees), and terms as possibly non-terminating computations.
- Our goal is: by adapting Girard's interpretation of β-logic, we would like to interpret in Ω terms and types with non-nested quantifiers, using and ***terminating*** computation only.
- $[\forall\alpha.A]$ is interpreted as a tree whose branching is over the set Ω_1 of ***all at most countable well-founded trees***.
- We would like to prove: ***every map $\Omega_1 \rightarrow \Omega$ has a unique extension to a map :$\Omega \rightarrow \Omega$.***

The interpretation of $[t] \in [\forall\alpha.A]$ using branchings over Ω_1

- [t] is a map from the set Ω_1 of all at most countable well-founded trees to the set Ω of well-founded trees, we would like to extend it to a map : $\Omega \to \Omega$



We claim the set Ω_1 is a kind of ***threshold*** for being sure that the unique extension of a map map $:\Omega_1 \to \Omega$ to a map $:\Omega \to U$ is a map $:\Omega \to \Omega$.

How a polymorphic program may use a input type

- A term $[t] \in [\forall\alpha.(A\text{->}B)]$ may be instanced over any interpretation [T] of a type, but if [T] is not known when $[\forall\alpha.(A\text{->}B)]$ is defined, then the use of [T] in [t] is very reduced.
- Indeed, if during the computation of [t]([T],[a]) we find some tree indexed by [T], we have no way to select some child of the tree, because we do not know "how" the elements $x \in [T]$ are defined.
- There is only a trivial way to use [T]: ***it is to produce a tree indexed by [T].*** For any map [t] interpreting a term of system F there is some ϕ such that for all "unknown" [T] we have $$[t](\wedge_{x \in [T]}\tau_x) = \wedge_{x \in [T]}\phi(\tau_x)$$

Why the restriction to Ω_1 should work (proof sketch)

- ***Due to the limited used of the argument [T],*** any countable restriction of [t]([T],[a]) of of [T], [a]$\in\Omega$, is obtained from some restriction [T_0], [a_0]$\in\Omega_1$ of [T], [a], and [T_0], [a_0] belong to the branching of the tree [$\forall\alpha$.(A->B)] .
- By assumption if [T_0], [a_0]$\in\Omega_1$ then [t]([T_0],[a_0])$\in\Omega$.
- Thus, all countable restrictions [t]([T_0],[a_0]) of [t]([T])([a]) are in Ω. Since ***a tree is well-founded if and only if all its countable restrictions are,*** we conclude that [t]([T])([a])$\in\Omega$.

Summing up

- We sketched how to interpret any term $t:\forall\alpha.(A\text{->}B)$ by some continuous map [t]: $[\forall\alpha.(A\text{->}B)]$ ***which may only recopy a branching over some not yet defined type***.
- We may extend [t] to the domain U by continuity.
- ***Due to the restricted behavior of*** [t], we may prove that if [t] preserves well-foundation over the set Ω_1, then it preserves well-foundation over the set Ω.
- We inductively define Ω_n as the set of all well-founded trees with branching either over N, or over Ω_i for some i<n. ***We hope to interpret degree n quantifiers of system F similary***, by taking Ω_n as domain in which to test the fact that the map preserves well-foundation.

Talk given at Manifacture des Tabacs, Toulouse