Types in Proof Mining

Ulrich Kohlenbach

Technische Universität Darmstadt, Fachbereich Mathematik
Schlossgartenstraße 7, 64289 Darmstadt, Germany
kohlenbach@mathematik.tu-darmstadt.de

Abstract

During the last 20 years a new applied form of proof theory (sometimes referred to as 'proof mining') has been developed which uses proof-theoretic transformations to extract hidden quantitative and computational information from given (prima facie ineffective) proofs ([6]). The historical roots of this development go back to the 50's and the pioneering work on 'unwinding of proof' done by G. Kreisel ([10]). The modern 'proof mining' paradigm has been most systematically pursued in the context of nonlinear analysis, ergodic theory and fixed point theory in recent years. The main proof-theoretic techniques used are extensions and novel forms of functional interpretation that are based on Gödel's famous 1958 'Dialectica'-interpretation ([1, 7]). All these interpretations are formulated in languages of functionals of finite types.

Until 2000, applications of proof mining in analysis mainly concerned the context of Polish spaces and continuous functions between them which can be represented by the Baire space $\mathbb{N}^{\mathbb{N}}$ and functionals $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ and so finite types over the base type $\mathbb{N}$ for natural numbers were sufficient.

Starting with 2001 ([4]) a number of applications in metric fixed point theory emerged which deal with theorems that hold in general classes of spaces such as all Banach spaces or all Hilbert spaces etc. In these applications, effective bounds of an amazing uniformity were extracted in the sense that the bounds did not depend on metrically bounded parameters which in the context of Polish space usually can only be expected under strong compactness assumptions. In fact, it is the very feature of the proofs in question not to use any separability assumptions on the spaces in question which makes this possible.

Starting in 2005, general so-called logical metatheorems have been developed which explain these applications and paved the way for numerous new applications ([5, 3, 6, 2]). In order to faithfully reflect the absence of any use of separability, this requires a formal framework different from the usual ones used in proof theory, constructive mathematics, reverse mathematics or computable analysis which usually represent (complete) metric or Banach spaces as the completion of a countable dense subset. In our framework, we instead add abstract metric structures X (or $X_1, \dots, X_n$) as new base types to the language together with the appropriate constants (such as a (pseudo-)metric d_X) and the appropriate axioms and consider all finite types over $\mathbb{N}, X_1, \dots, X_n$. Equality for such a new base type X with pseudo-metric d_X is a defined notion $x =_X y := d_X(x, y) =_{\mathbb{R}} 0$ reflecting that we work over the metric space induced by the pseudo-metric d_X. Here real numbers are represented via the usual Cauchy representation and $=_{\mathbb{R}} \in \Pi^0_1$ is the corresponding equivalence relation on the space $\mathbb{N}^{\mathbb{N}}$ of Cauchy names. It is the interplay between the world of abstract structures X and the world of represented concrete Polish spaces such as $\mathbb{R}$ which requires considerable care.

As well-known from functional interpretation, the ability of the latter to unwind proofs based on classical logic (by satisfying the so-called Markov principle) make it necessary to put a severe restriction on the use of the axiom of extensionality (i.e. the axiom stating that functionals

respect the extensionally defined equality) which has to be replaced by a weak extentsionality rule. Whereas this is not a real restriction in the context of Polish spaces (due to the well-known elimination-of-extensionality method of Gandy and Luckhardt), this is rather different in our context where already the extensionality of objects f of type $X \to X$ is too strong to be included as an axiom. While in many cases, this extensionality follows from assumptions on f (such as being Lipschitzian), one has to rely on weak extensionality in other cases.

Although the theorems from mathematics to be studied always only use types of very low degree, the proof-transformations based on functional interpretations make use of the whole hierarchy of finite types. Moreover, it is via these types that the uniform bounding data to be extracted from the proof are being controlled throughout the proof via novel forms of majorization. Let ρ be a finite type over $\mathbb{N}, X_1, \ldots, X_n$ and $\widehat{\rho}$ be the result of replacing all occurrences of $X_1, \ldots, X_n$ in ρ by $\mathbb{N}$. Then the objects serving as majorants for objects of type ρ are of type $\widehat{\rho}$ and all the computations take place on these majorants, i.e. on objects which have a finite type over $\mathbb{N}$ so that usual computability theory applies irrespectively of whether the structures $X_1, \ldots, X_n$ carry any computability notion.

We will give a survey on these developments and - as time permits - present some recent applications to nonlinear ergodic theory dealing with explicit rates of asymptotic regularity and metastability as well as algorithmic learning information ([8, 9, 11]).

References

[1] Jeremy Avigad and Solomon Feferman. Gödel's functional ('Dialectica') interpretation. In Samuel R. Buss, editor, *Handbook of Proof Theory*, pages 337–405. Elsevier, 1998.

[2] Eyvind Martol Briseid. Logical aspects of rates of convergence in metric spaces. *J. Symb. Log.*, 74:1401–1428, 2009.

[3] Philipp Gerhardy and Ulrich Kohlenbach. General logical metatheorems for functional analysis. *Trans. Amer. Math. Soc.*, 360:2615–2660, 2008.

[4] Ulrich Kohlenbach. A quantitative version of a theorem due to Borwein-Reich-Shafrir. *Numer.Funct.Anal.Optimiz.*, 22:641–656, 2001.

[5] Ulrich Kohlenbach. Some logical metatheorems with applications in functional analysis. *Trans. Amer. Math. Soc.*, 357:89–128, 2005.

[6] Ulrich Kohlenbach. *Applied Proof Theory: Proof Interpretations and their Use in Mathematics.* Springer Monographs in Mathematics. Springer Verlag, 2008.

[7] Ulrich Kohlenbach. Gödel's functional interpretation and its use in current mathematics. In M. et al. Baaz, editor, *Kurt Gödel and the Foundations of Mathematics. Horizons of Truth*, pages 223–267. Cambridge University Press, New York, 2011.

[8] Ulrich Kohlenbach and Laurenţiu Leuştean. Effective metastability for Halpern iterates in CAT(0) space. *Advances in Mathematics*, 231:2526–2556, 2012.

[9] Ulrich Kohlenbach and Laurenţiu Leuştean. On the computational content of convergence proofs via Banach limitd. *Philosophical Transactions of the Royal Society A*, 370:3449–3463, 2012.

[10] Angus Macinytre. The mathematical significance of proof theory. *Philosophical Transactions of the Royal Society A*, 363:2419–2435, 2005.

[11] Pavol Safarik. A quantitative nonlinear strong ergodic theorem for Hilbert spaces. *J. Math. Analysis Appl.*, 391:26–37, 2012.