Higher Inductive Types in Homotopy Type Theory

Steve Awodey[1*]

Institute for Advanced Study
School of Mathematics
Princeton, NJ 08540 USA

Carnegie Mellon University
Department of Philosophy
Pittsburgh, PA 15213 USA

Homotopy Type Theory (HoTT) refers to the homotopical interpretation [1] of Martin-Löf's intensional, constructive type theory (MLTT) [5], together with several new principles motivated by that interpretation. Voevodsky's *Univalent Foundations* program [6] is a conception for a new foundation for mathematics, based on HoTT and implemented in a proof assistant like Coq [2].

Among the new principles to be added to MLTT are the Univalence Axiom [4], and the so-called *higher inductive types* (HITs), a new idea due to Lumsdaine and Shulman which allows for the introduction of some basic spaces and constructions from homotopy theory. For example, the n-dimensional spheres S^n can be implemented as HITs, in a way analogous to the implementation of the natural numbers as a conventional inductive type. Other examples include the unit interval; truncations, such as bracket-types $[A]$; and quotients by equivalent relations or groupoids. The combination of univalence and HITs is turning out to be a very powerful and workable system for the formalization of homotopy theory, with the recently given, formally verified proofs of some fundamental results, such as determinations of various of the homotopy groups of spheres by Brunerie and Licata. See [3] for much work in progress

After briefly reviewing the foregoing developments, I will give an impredicative encoding of certain HITs on the basis of a new representation theorem, which states that every type of a particular kind is equivalent to its double dual in the space of coherent natural transformations. A realizability model is also provided, establishing the consistency of impredicative HoTT and its extension by HITs.

References

[1] S. Awodey and M. A. Warren, *Homotopy theoretic models of identity types*, Mathematical Proceedings of the Cambridge Philosophical Society 146 (2009), no. 1, 45-55.

[2] The Coq proof assistant, `http://coq.inria.fr`

[3] The IAS Univalent Foundations wiki, `http://uf-ias-2012.wikispaces.com`.

[4] C. Kapulkin, P. Lumsdaine, V. Voevodsky: *The simplicial model of univalent foundations*, on the arXiv as `arXiv:math/1211.2851`, 2012.

[5] P. Martin-Löf, *An intuitionistic theory of types*, Twenty-five years of constructive type theory (Venice, 1995), Oxford Univ. Press, New York, 1998, pp. 127-172.

[6] V. Voevodsky, *Univalent foundations project*, Modified version of an NSF grant application, `http://www.math.ias.edu/~vladimir/Site3/Univalent_Foundations.html`, 2010.

[*] NSF Grant DMS-1001191 and Air Force OSR Grant 11NL035