The formalization of Bishop's theory of sets and functions revisited

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The problem of the formalization of Bishop's informal system of constructive mathematics BISH (see [2], [3]) motivated the introduction of many and different formal systems in the past. Martin-Löf's type theory MLTT (see e.g., [5]), Feferman's system of explicit mathematics T_0 (see e.g., [4]), Myhill's constructive theory of sets and functions CST in [6], or Aczel's constructive set theory CZF (see [1]), are some of them.

In the first part of our proposed talk we plan to explain briefly how all these systems have features that are "unexpected" from the point of view of BISH. Of course, these features are due to the fact that, soon after their development, these systems took directions different from the original source of BISH. In MLTT, for example, there are universes, the type of natural numbers is defined, equality of elements of a type is special to each type but the J-rule, the main proof tool of MLTT, applies to all types. Moreover, the function type is defined inductively and function extensionality is unprovable in the intensional version of MLTT. In BISH there is no universe, the set of naturals is undefined, the equality of the elements of a set has no uniform character, the set of functions $\mathbb{F}(A, B)$ from A to B is not defined explicitly, and function extensionality is the defined equality of $\mathbb{F}(A, B)$.

In the second part of our proposed talk we plan to present our current approach to the formalization of Bishop's informal theory of sets and functions BSFT. Our approach is connected to Myhill's theory CST, maybe the formal system closest to BISH, but also quite different from it. We examine whether the "global" extensionality axiom, the axioms for the domain and the codomain of a given function, the full exponentiation axiom, the union axiom, which are found in CST, reflect the practice of BISH. Bishop defined equality only for subsets of a given set, and a function is a priori given with its domain and codomain. Moreover, in BISH we find assignment routines that are used to define families of subsets of a given set and families of arbitrary sets. These function-like objects, probably, need to be interpreted as new primitive objects.

The aim of our study, which is work in progress, is to determine a spartan formal theory of set-like and function-like objects that will reflect the practice of Bishop's constructive mathematics while avoiding serious unexpected features with respect to it.

References

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