# NONSTANDARD ANALYSIS, COMPUTABILITY THEORY, AND METASTABILITY

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Abstract. We discuss a new connection between Nonstandard Analysis and computability theory, pioneered in [1], based on the following two intimately related topics.

(T.1) A basic property of *Cantor space* $2^{\mathbb{N}}$ is *Heine-Borel compactness*: Any open cover of $2^{\mathbb{N}}$, has a *finite* sub-cover. A natural question is: *How hard is it to compute such a finite sub-cover*? We make this precise by analysing functionals that given $g : 2^{\mathbb{N}} \to \mathbb{N}$, output $\langle f_0, \dots, f_n \rangle$ in $2^{\mathbb{N}}$ such that the neighbourhoods defined from $\overline{f_i} g(f_i)$ for $i \leq n$ cover $2^{\mathbb{N}}$. The *special* and *weak* fan functionals are central objects in this study and exhibit *extreme* computational hardness.

(T.2) A basic property of $2^{\mathbb{N}}$ in *Nonstandard Analysis* is Abraham Robinson's *nonstandard compactness*, i.e. that every binary sequence is 'infinitely close' to a *standard* binary sequence. We analyse the strength of this nonstandard compactness property in the spirit of *Reverse Mathematics*, which turns out to be intimately related to the computational properties of the special and weak fan functionals.

We connect the topics (T.1) and (T.2) to mainstream mathematics by deriving the special fan functional from *slight* variations of Tao's notion of *metastability* ([2]). Based on the latter observation, we establish that many mathematical theorems naturally have 'metastable versions' which involve functionals of extreme computational hardness. We also discuss exceptions, like the *infinite pigeon hole principle*, whose metastable versions stay within Gödel's $T$.

## References

[1] Dag Normann and Sam Sanders, *Nonstandard Analysis, Computability Theory, and their connections*, Submitted, Available from arXiv: `https://arxiv.org/abs/1702.06556` (2017).

[2] Terence Tao, *Structure and randomness*, American Mathematical Society, Providence, RI, 2008. Pages from year one of a mathematical blog.

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