Typoids in Univalent Type Theory

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Abstract

Bishop's notion of set, introduced in [2], is interpreted in Martin-Löf's type theory (see [4] and [5]) through the notion of setoid (see e.g., [6] and [1]). A setoid is a type A paired with an equivalence relation $\simeq_A\colon A \to A \to \mathcal{U}$ on A such that $x \simeq_A y$ is a proposition, for every $x, y : A$. The morphisms between setoids are the functions between them that respect the corresponding equivalences. The resulting category of setoids is cartesian closed. Considering $x \simeq_A y$ to be any type we study the notions of typoid and typoid function between typoids.

A *typoid* is a structure $\mathcal{A} \equiv (A, \simeq_A, \mathtt{eqv}_A, *_A, {}^{-1_A}, \cong_A)$, where $A : \mathcal{U}$ and $\simeq_A\colon \prod_{x,y:A} \mathcal{U}$ is an equivalence relation on A such that $\mathtt{eqv}_A$ witnesses reflexivity, $*_A$ transivity and ${}^{-1_A}$ symmetry. Moreover, $\cong_A (x,y) : \prod_{e,d:x \simeq_A y} \mathcal{U}$ is an equivalence relation on $x \simeq_A y$, for every $x, y : A$ such that appropriate properties that generalize the properties of equality paths hold.

If $\mathcal{A}, \mathcal{B}$ are typoids, $f : A \to B$ is a *typoid function*, if there are $\Phi_f : \prod_{x,y:A} \prod_{e:x \simeq_A y} f(x) \simeq_B f(y)$, and $\Phi_f^2 : \prod_{x,y:A} \prod_{e,d:x \simeq_A y} \prod_{i:e \cong_A d} \Phi_f(x,y,e) \cong_B \Phi_f(x,y,d)$, which we call an 1-*associate* of f and a 2-*associate* of f with respect to Φ_f, respectively, which preserve the typoid structure.

The notion of a univalent type, a type which satisfies an abstract version of Voevodsky's univalence axiom (see [3]), can be described in this setting. In this work in progress we study the category of typoids and we prove some fundamental properties of univalent typoids.

References

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