THE UNREASONABLE EFFECTIVENESS OF NONSTANDARD ANALYSIS

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Abstract

As suggested by the title, the topic of my talk is the *vast computational content* of *classical* Nonstandard Analysis. In particular, I will present a template $\mathfrak{CI}$ which allows one to obtain *effective* theorems from theorems in 'pure' Nonstandard Analysis, i.e. only involving the *nonstandard* definitions (of continuity, compactness, Riemann integration, convergence, et cetera). This template $\mathfrak{CI}$ has been applied successfully to the Big Five systems of Reverse Mathematics ([7]), the Reverse Mathematics zoo ([3, 6, 8]), and computability theory ([5]). The template $\mathfrak{CI}$ often produces theorems of Bishop's *Constructive Analysis* ([2]).

The framework for the template $\mathfrak{CI}$ is Nelson's syntactic approach to Nonstandard Analysis, called *internal set theory* ([4]), and its fragments based on Gödel's T as introduced in [1]. Notable results are that applying the template $\mathfrak{CI}$ to theorems involving the nonstandard definitions of respectively continuity, compactness, and open set, the latter definitions are converted into the associated definitions from constructive or computable analysis (resp. continuity with a modulus, totally boundedness, and effectively open set).

Finally, we establish that a theorem of Nonstandard Analysis has the *same computational content* as its 'highly constructive' *Herbrandisation*. Thus, we establish an 'algorithmic two-way street' between so-called hard and soft analysis, i.e. between the worlds of numerical and qualitative results. However, the study of the Herbrandisations of nonstandard theorems also leads to a new class of functionals (not involving Nonstandard Analysis) with rather strange properties. Chief among these new functionals is the special fan functional (See [5, §3]) which can be computed easily in intuitionistic mathematics, but cannot be computed by the Turing jump functional ($\exists^2$) or even much stronger comprehension axioms. Similar functionals exist for most theorems from the Reverse Mathematics zoo.

References

[1] Benno van den Berg, Eyvind Briseid, and Pavol Safarik, *A functional interpretation for nonstandard arithmetic*, Ann. Pure Appl. Logic **163** (2012), no. 12, 1962–1994.

[2] Errett Bishop and Douglas S. Bridges, *Constructive analysis*, Grundlehren der Mathematischen Wissenschaften, vol. 279, Springer-Verlag, Berlin, 1985.

[3] Damir D. Dzhafarov, *Reverse Mathematics Zoo*. `http://rmzoo.uconn.edu/`.

[4] Edward Nelson, *Internal set theory: a new approach to nonstandard analysis*, Bull. Amer. Math. Soc. **83** (1977), no. 6, 1165–1198.

[5] Sam Sanders, *The Gandy-Hyland functional and a hitherto unknown computational aspect of Nonstandard Analysis*, Submitted, Available from: `http://arxiv.org/abs/1502.03622` (2015).

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[6] ______, *The taming of the Reverse Mathematics zoo*, Submitted, `http://arxiv.org/abs/1412.2022` (2015).

[7] ______, *The unreasonable effectiveness of Nonstandard Analysis*, Submitted to APAL special issue of LFCS, `http://arxiv.org/abs/1508.07434` (2016).

[8] ______, *The refining of the taming of the Reverse Mathematics zoo*, To appear in Notre Dame Journal for Formal Logic, `http://arxiv.org/abs/1602.02270` (2016).