

Reachability Problems for Continuous Linear Dynamical Systems

James Worrell

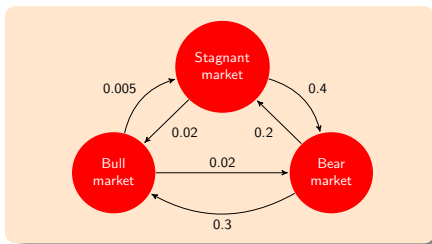
Department of Computer Science, Oxford University

(Joint work with Ventsislav Chonev and Joël Ouaknine)

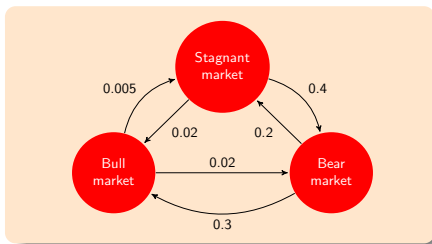
FICS 2015

September 12th, 2015

Reachability for Continuous-Time Markov Chains



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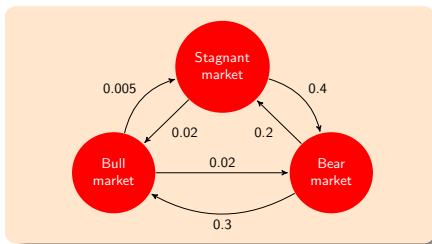


Distribution $P(t)$ at time t satisfies $P'(t) = P(t)Q$, where

$$Q = \begin{pmatrix} -0.025 & 0.02 & 0.005 \\ 0.3 & -0.5 & 0.2 \\ 0.02 & 0.4 & -0.42 \end{pmatrix}$$

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$$\exists t (P(t)_{\text{Bear}} \geq P(t)_{\text{Bull}})$$

Reachability - Some Basic Insights

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- Reduce to the **time-bounded case** by computing the stationary distribution:

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- Reduction does not work if π is on boundary of the target set.
- Semantic shift — consider whether the closure of the orbit $\{P(t) : t \geq 0\}$ meets the closure of the target set.

*“To analyze a cyber-physical system, such as a pacemaker, we need to consider the **discrete software controller** interacting with the physical world, which is typically modelled by **differential equations**”*

Rajeev Alur (CACM, 2013)



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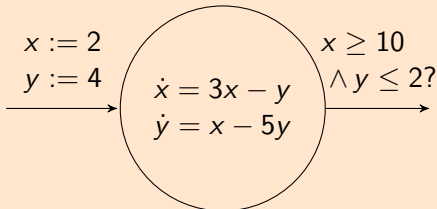
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- o-minimal flows + strong resets \Rightarrow reachability decidable

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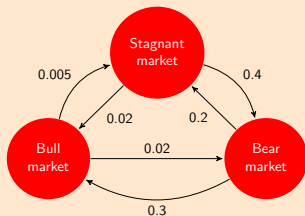
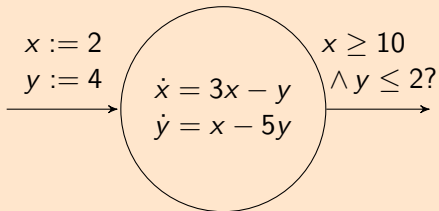
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Is this location a trap?



Reachability for Continuous Linear Dynamical Systems

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Is ever more likely to be a Bear market than a Bull market:

$$\exists t (P(t)_{\text{Bear}} \geq P(t)_{\text{Bull}}) ?$$

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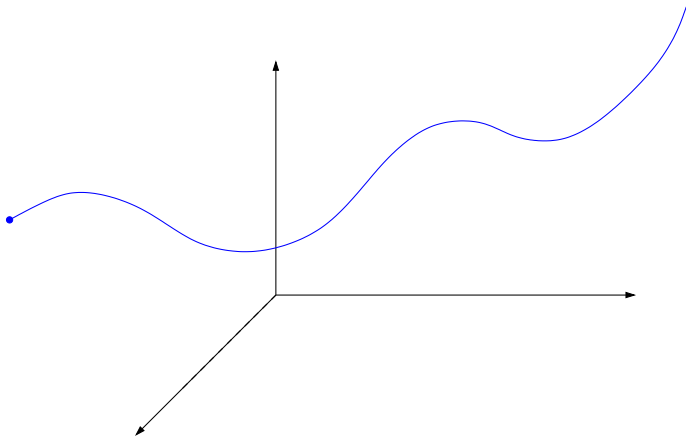
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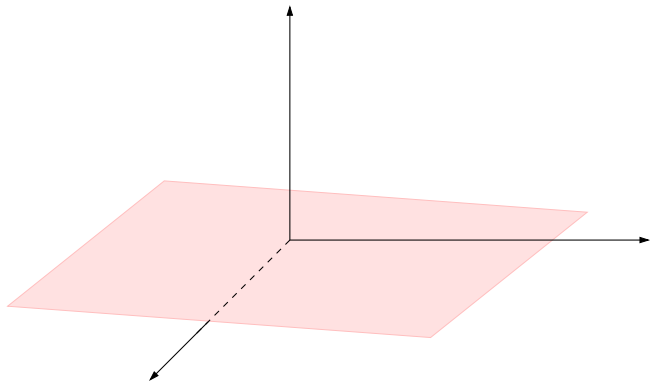


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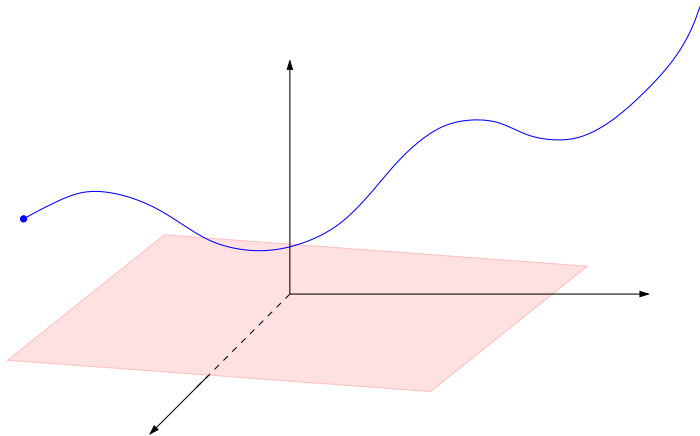


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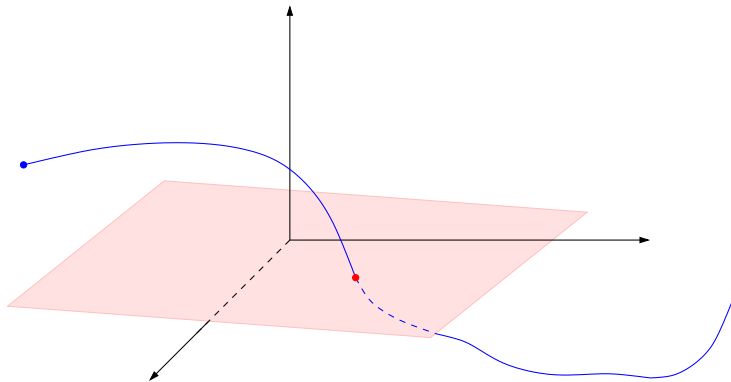


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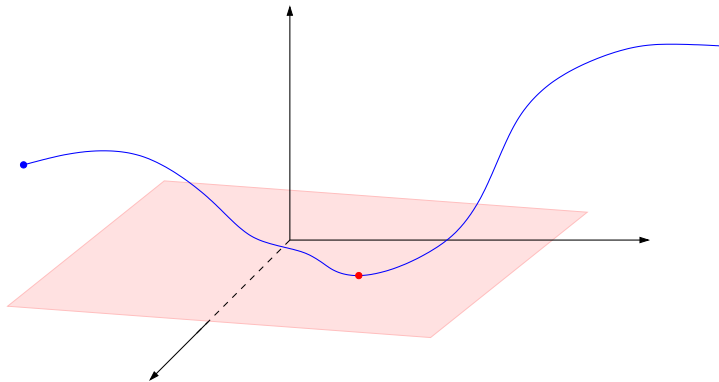


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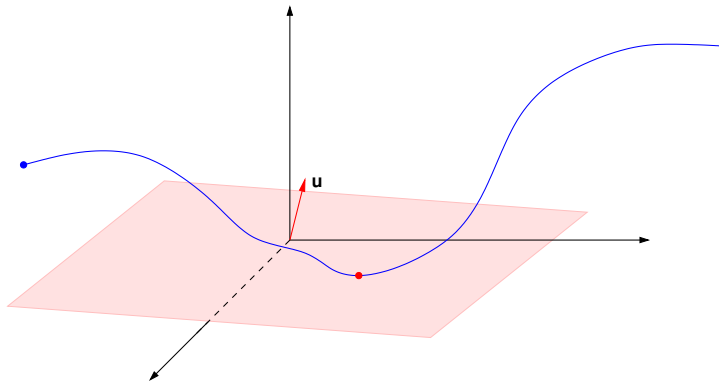


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Note – the λ_j are complex in general.

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Let $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ be given as above, with all coefficients algebraic.

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BOUNDED-ZERO Problem

Instance: f and bounded interval $[a, b]$

Question: Is there $t \in [a, b]$ such that $f(t) = 0$?

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- **Decidability open!** [Bell, Delvenne, Jungers, Blondel 2010]

A lot of work since 1920s on the zeros of exponential polynomials

$$f(z) = \sum_{j=1}^m P_j(z) e^{\lambda_j z}$$

(Polya, Ritt, Tamarkin, Kac, Voorhoeve, van der Poorten, . . .)
but mostly on distribution of *complex* zeros.

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CONTINUOUS-ORBIT Problem

The problem of whether the trajectory $\mathbf{x}(t) = e^{\mathbf{A}t} \mathbf{x}(0)$ reaches a given **target point** was shown to be decidable by Hainry (2008) and in PTIME by Chen, Han and Yu (2015).

Reachability for Continuous Linear Dynamical Systems

Theorem (Bell, Delvenne, Jungers, Blondel 2010)

In dimension 2, BOUNDED-ZERO and ZERO are decidable.

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Theorem (Chonev, Ouaknine, W. 2015)

Assuming Schanuel's Conjecture, BOUNDED-ZERO is decidable in all dimensions.

It turns out that this result (in fact, a powerful generalisation of it) had already been discovered (but never published) in the early 1990s by Macintyre and Wilkie!

[Angus Macintyre, personal communication, July 2015]

Theorem (Chonev, Ouaknine, W. 2015)

In dimension 8 or less, ZERO reduces to BOUNDED-ZERO.

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Theorem (Chonev, Ouaknine, W. 2015)

In dimension 9 (and above), decidability of ZERO would entail major breakthroughs in Diophantine approximation—the Diophantine approximation type of α would be computable to within arbitrary precision.

Schanuel's Conjecture

Theorem (Lindemann-Weierstrass)

If a_1, \dots, a_n are algebraic numbers linearly independent over \mathbb{Q} , then e^{a_1}, \dots, e^{a_n} are algebraically independent.

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Schanuel's Conjecture

If z_1, \dots, z_n are complex numbers linearly independent over \mathbb{Q} then some n -element subset of $\{z_1, \dots, z_n, e^{z_1}, \dots, e^{z_n}\}$ is algebraically independent.

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Example

By Schanuel's conjecture some two-element subset of $\{1, \pi i, e^1, e^{\pi i}\}$ is algebraically independent.

The BOUNDED-ZERO Problem

Real-valued exponential polynomial $f(t) = \sum_{j=1}^m P_j(t)e^{\lambda_j t}$

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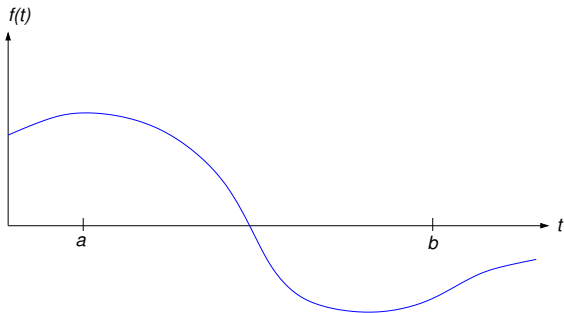
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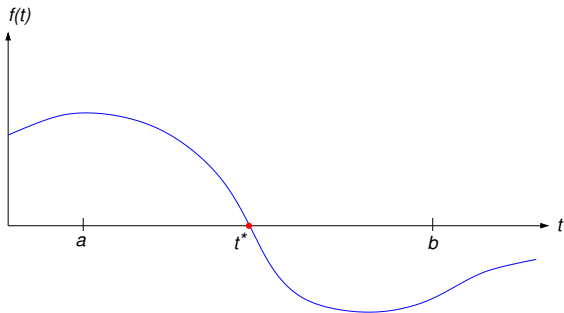
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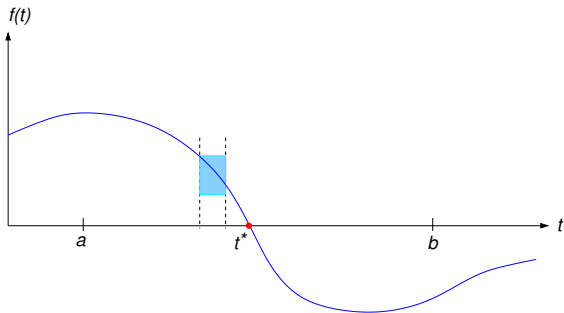
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'non-trivial' zero $\Rightarrow t^*$ transcendental

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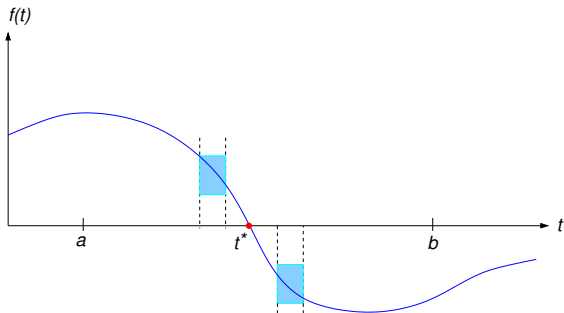
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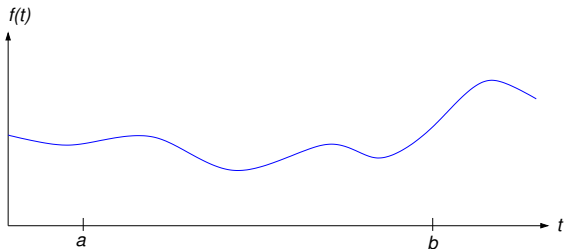
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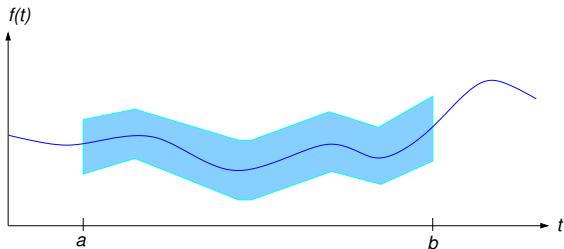
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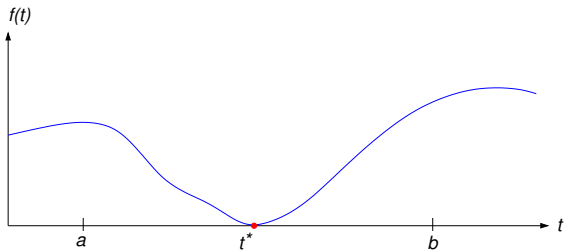
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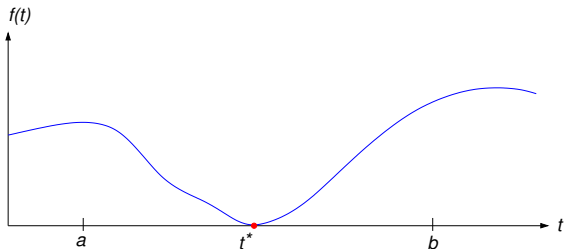
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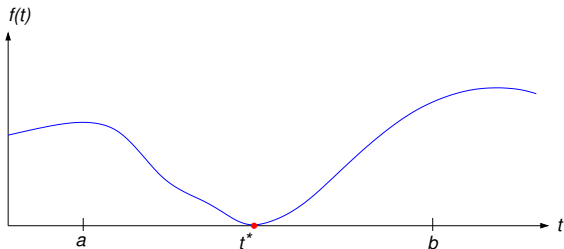
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Can this situation arise?

The BOUNDED-ZERO Problem

Real-valued exponential polynomial $f(t) = \sum_{j=1}^m P_j(t)e^{\lambda_j t}$



Easily! For example, $f(t) = 2 + e^{it} + e^{-it}$.

Example

- Write $f(t) = 2 + e^{it} + e^{-it}$ in the form $f(t) = P(e^{it})$ for the **Laurent polynomial**

$$P(z) = 2 + z + z^{-1}.$$

Laurent Polynomials and Factorisation

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- Factorisation $P(z) = (1 + z)(1 + z^{-1})$ induces a factorisation

$$f(t) = \underbrace{(1 + e^{it})}_{f_1(t)} \underbrace{(1 - e^{it})}_{f_2(t)}$$

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Idea: factorise f . Noting that factors may be complex-valued!

The Real Case

Any exponential polynomial $f(t)$ can be written

$$f(t) = P(t, e^{a_1 t}, \dots, e^{a_m t})$$

with

$$P \in \mathbb{C}[x, x_1^{\pm 1}, \dots, x_m^{\pm 1}]$$

and $\{a_1, \dots, a_m\}$ a set of real and imaginary algebraic numbers that is linearly independent over \mathbb{Q} .

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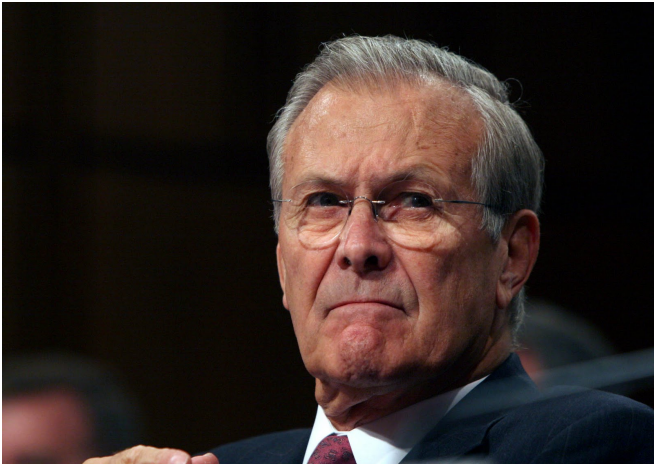
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Complex case requires some new ideas ...

The Unbounded Case

“there are known unknowns; that is to say we know there are some things we do not know.”



Continued Fractions

Finite continued fractions:

$$[3, 7, 15, 1, 292] = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292}}}}$$

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$$\begin{aligned} [3, 7, 15, 1, 292] &= 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292}}}} \\ &= 3.141592653\dots \end{aligned}$$

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Infinite continued fractions:

$$[a_0, a_1, a_2, a_3, \dots] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

Real Algebraic Numbers

Theorem

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Lang and Trotter: “*no significant departure from random behaviour*”

An Open Problem

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“Is there an algebraic number of degree higher than two whose simple continued fraction has unbounded partial quotients? Does every such number have unbounded partial quotients?”

R. K. Guy, 2004



A Mathematical Obstacle at Dimension 9

Given $x = [a_0, a_1, a_2, \dots]$, define $S(x) = \sup_{n \in \mathbb{N}} a_n$.

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Remark

Perhaps this set is recursive—it may even be \emptyset or $\mathbb{R} \cap \mathbb{A}$. However proving recursive enumerability would be a significant achievement.

Diophantine Approximation

How well can one approximate a real number x with rationals?

$$\left| x - \frac{m}{n} \right|$$

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- Relate this to the existence of zeros of order-9 exponential polynomial $f(t)$ with terms e^{ixt} and e^{it} .

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Instance: f

Question: Is there $t \in \mathbb{R}_{\geq 0}$ such that $f(t) = 0$?

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- Diophantine-approximation bounds play a key role in the proof.

Illustrative Example

Consider the exponential polynomial

$$f(t) = 1.9 + \cos(t + \varphi_1) + \cos(\sqrt{2}t + \varphi_2) - e^{-t}$$

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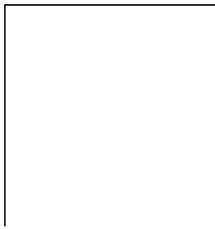
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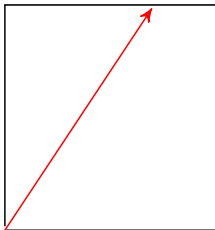


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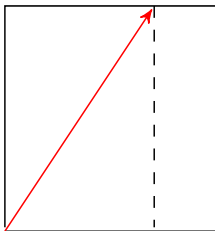


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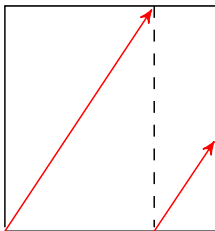


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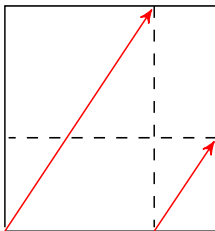


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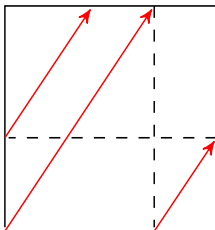


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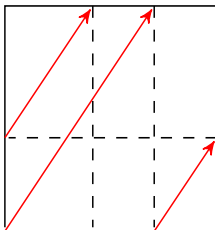


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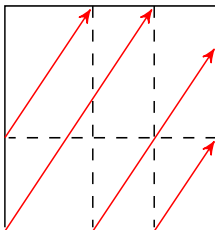


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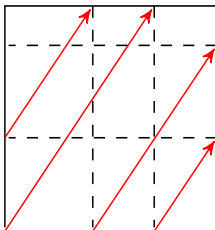


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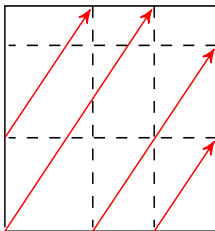


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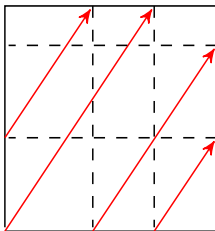


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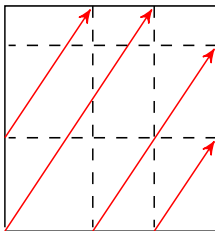


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Baker's Theorem:

$$\left\| (t, \sqrt{2}t) - (\pi - \varphi_1, \pi - \varphi_2) \right\| \geq \frac{1}{\text{poly}(t)}$$

Conclusion and Perspectives

The Discrete Case

A **linear recurrence sequence** is a sequence $\langle u_0, u_1, u_2, \dots \rangle$ of integers such that there exist constants a_1, \dots, a_k , such that

$$u_{n+k} = a_1 u_{n+k-1} + a_2 u_{n+k-2} + \dots + a_k u_n$$

for all $n \geq 0$.

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Theorem (Skolem 1934; Mahler 1935, 1956; Lech 1953)

The set of zeros of a linear recurrence sequence is semi-linear:

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Theorem (Berstel and Mignotte 1976)

In Skolem-Mahler-Lech, the infinite part (arithmetic progressions A_1, \dots, A_ℓ) is fully constructive.

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Does $\exists n$ such that $u_n = 0$?

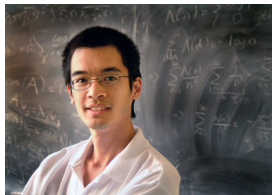
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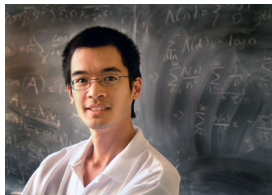
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"... a mathematical embarrassment ..."

Richard Lipton

Wrapping Things Up

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- The infinite-zeros problem is also hard.
- Diophantine-approximation techniques unavoidable.