Reachability Problems for Continuous Linear Dynamical Systems

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(Joint work with Ventsislav Chonev and Joël Ouaknine)

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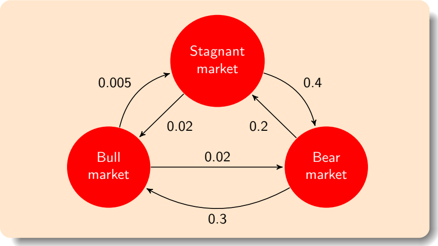
September 12th, 2015

Reachability for Continuous-Time Markov Chains



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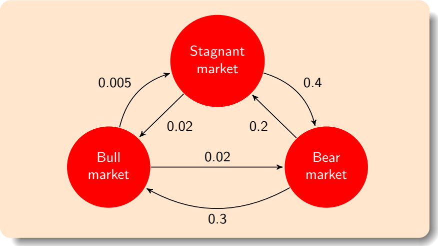

Distribution $P(t)$ at time t satisfies $P'(t) = P(t)Q$, where

$$Q = \begin{pmatrix} -0.025 & 0.02 & 0.005 \\ 0.3 & -0.5 & 0.2 \\ 0.02 & 0.4 & -0.42 \end{pmatrix}$$

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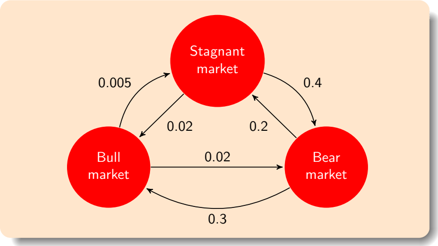

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- Semantic shift — consider whether the closure of the orbit $\{P(t) : t \geq 0\}$ meets the closure of the target set.

Cyber-Physical Systems

"To analyze a cyber-physical system, such as a pacemaker, we need to consider the **discrete software controller** *interacting with the physical world, which is typically modelled by* **differential equations***"*

Rajeev Alur (CACM, 2013)

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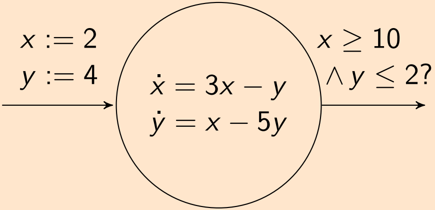

Reachability for Continuous Linear Dynamical Systems

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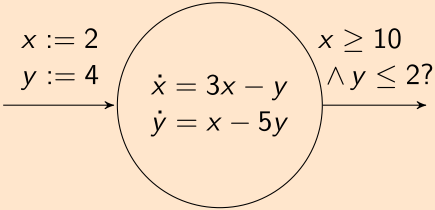

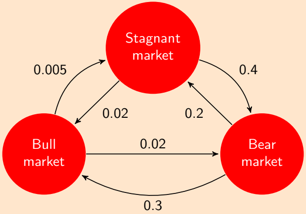

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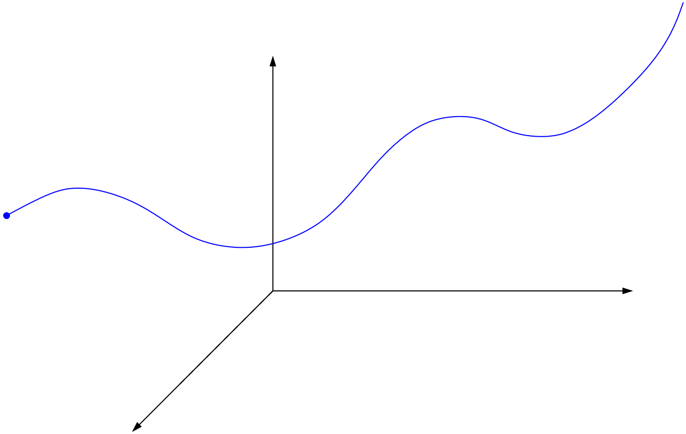

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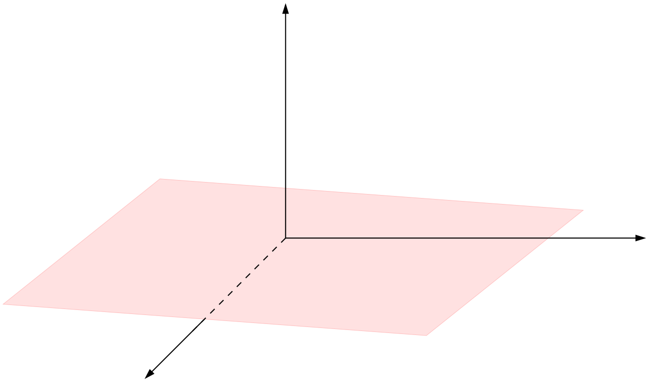

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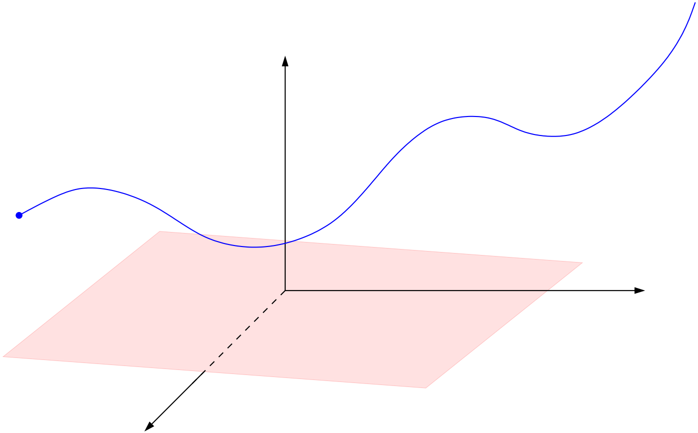

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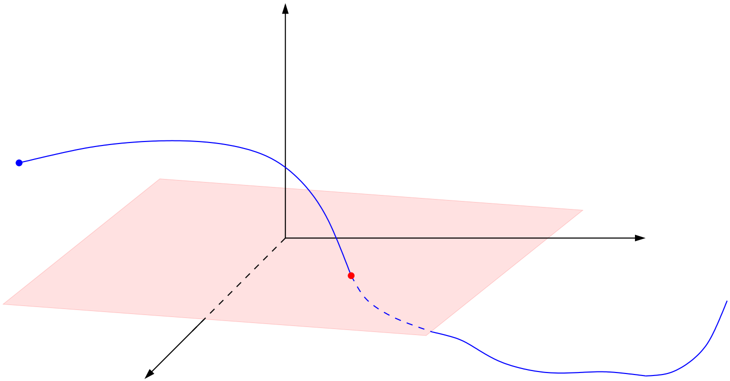

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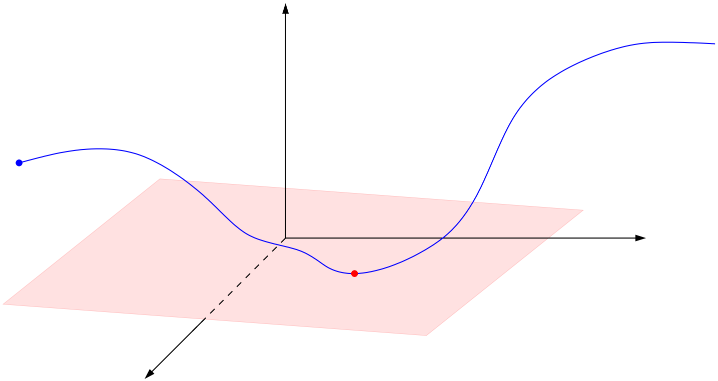

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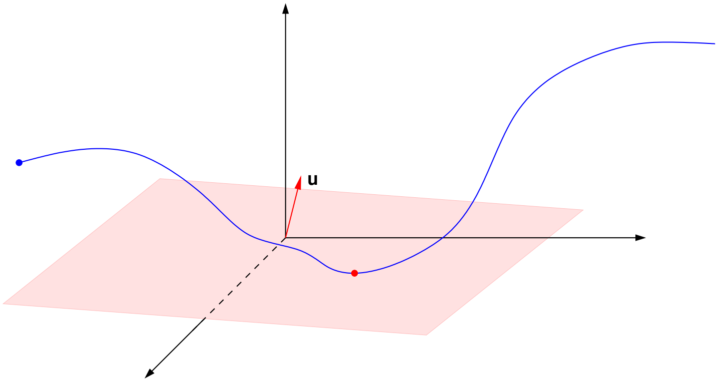

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Note – the λ_j are complex in general.

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Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be given as above, with all coefficients algebraic.

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Instance: f and bounded interval $[a, b]$

Question: Is there $t \in [a, b]$ such that $f(t) = 0$?

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- **Decidability open!** [Bell, Delvenne, Jungers, Blondel 2010]

Related Work

A lot of work since 1920s on the zeros of exponential polynomials

$$f(z) = \sum_{j=1}^{m} P_j(z)e^{\lambda_j z}$$

(Polya, Ritt, Tamarkin, Kac, Voorhoeve, van der Poorten, ...)
but mostly on distribution of *complex* zeros.

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CONTINUOUS-ORBIT Problem

The problem of whether the trajectory $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$ reaches a given **target point** was shown to be decidable by Hainry (2008) and in PTIME by Chen, Han and Yu (2015).

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Theorem (Bell, Delvenne, Jungers, Blondel 2010)

In dimension 2, BOUNDED-ZERO and ZERO are decidable.

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Theorem (Chonev, Ouaknine, W. 2015)

Assuming Schanuel's Conjecture, BOUNDED-ZERO is decidable in all dimensions.

It turns out that this result (in fact, a powerful generalisation of it) had already been discovered (but never published) in the early 1990s by Macintyre and Wilkie!

[Angus Macintyre, personal communication, July 2015]

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Theorem (Chonev, Ouaknine, W. 2015)

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In dimension 9 (and above), decidability of ZERO would entail major breakthroughs in Diophantine approximation—the Diophantine approximation type of α would be computable to within arbitrary precision.

Schanuel's Conjecture

Theorem (Lindemann-Weierstrass)

If $a_1, \ldots, a_n$ are algebraic numbers linearly independent over $\mathbb{Q}$, then $e^{a_1}, \ldots, e^{a_n}$ are algebraically independent.

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If $z_1, \ldots, z_n$ are complex numbers linearly independent over $\mathbb{Q}$ then some n-element subset of $\{z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}\}$ is algebraically independent.

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Example

By Schanuel's conjecture some two-element subset of $\{1, \pi i, e^1, e^{\pi i}\}$ is algebraically independent.

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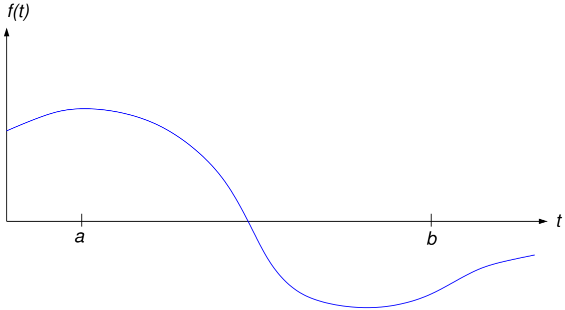

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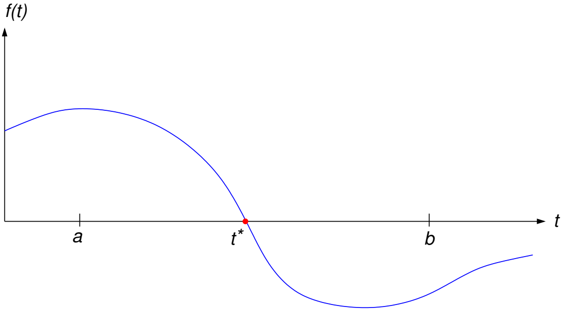

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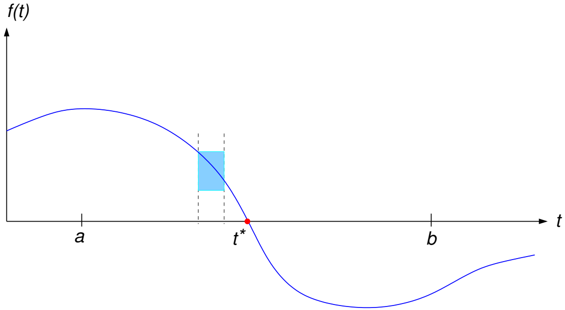

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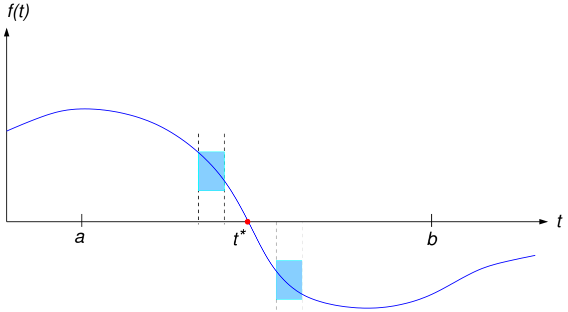

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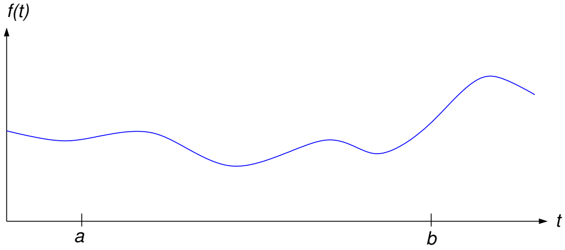

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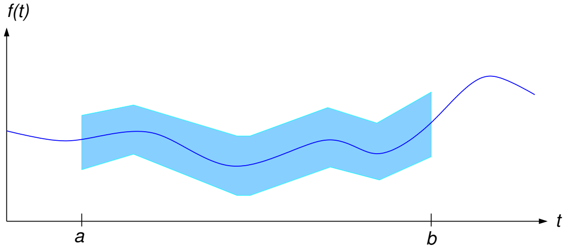

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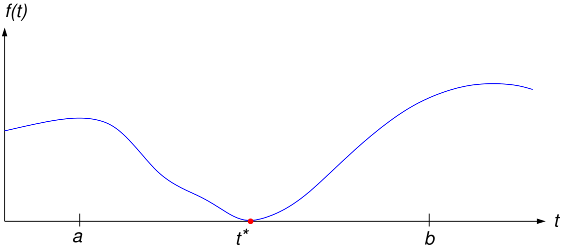

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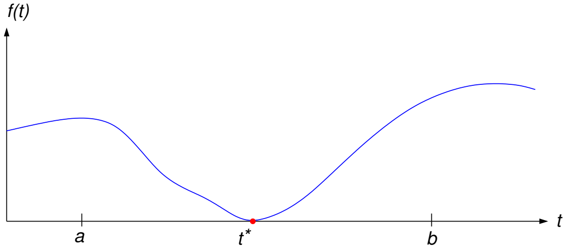

Can this situation arise?

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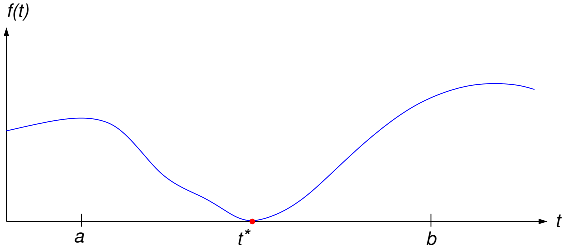

Easily! For example, $f(t) = 2 + e^{it} + e^{-it}$.

Laurent Polynomials and Factorisation

Example

- Write $f(t) = 2 + e^{it} + e^{-it}$ in the form $f(t) = P(e^{it})$ for the **Laurent polynomial**

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Idea: factorise f. Noting that factors may be complex-valued!

The Real Case

Any exponential polynomial $f(t)$ can be written

$$f(t) = P(t, e^{a_1 t}, \ldots, e^{a_m t})$$

with

$$P \in \mathbb{C}[x, x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$$

and $\{a_1, \ldots, a_m\}$ a set of real and imaginary algebraic numbers that is linearly independent over $\mathbb{Q}$.

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$$f(t) = P(t, e^{a_1 t}, \ldots, e^{a_m t})$$

with

$$P \in \mathbb{C}[x, x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$$

and $\{a_1, \ldots, a_m\}$ a set of real and imaginary algebraic numbers that is linearly independent over $\mathbb{Q}$.

Lemma

Assuming Schanuel's conjecture, if f is real valued and P is irreducible then f has no tangential zeros.

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Complex case requires some new ideas . . .

The Unbounded Case

"there are known unknowns; that is to say we know there are some things we do not know."

Continued Fractions

Finite continued fractions:

$$[3, 7, 15, 1, 292] = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292}}}}$$

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Infinite continued fractions:

$$[a_0, a_1, a_2, a_3, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

Real Algebraic Numbers

Theorem

The continued fraction expansion of a real quadratic irrational number is periodic.

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What about numbers of degree ≥ 3?

$$\sqrt[3]{2} = [1, 3, 1, 5, 1, 1, 4, 1, 1, 8, 1, 14, 1, 10, 2, 1, 4, 12, 2, 3, 2, 1$$
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Lang and Trotter: "*no significant departure from random behaviour*"

An Open Problem

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"Is there an algebraic number of degree higher than two whose simple continued fraction has unbounded partial quotients? Does every such number have unbounded partial quotients?"

R. K. Guy, 2004

A Mathematical Obstacle at Dimension 9

Given $x = [a_0, a_1, a_2, \ldots]$, define $S(x) = \sup_{n \in \mathbb{N}} a_n$.

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Remark

Perhaps this set is recursive—it may even be $\emptyset$ or $\mathbb{R} \cap \mathbb{A}$. However proving recursive enumerability would be a significant achievement.

Diophantine Approximation

How well can one approximate a real number x with rationals?

$$\left| x - \frac{m}{n} \right|$$

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- Relate this to the existence of zeros of order-9 exponential polynomial $f(t)$ with terms e^{ixt} and e^{it}.

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Instance: f

Question: Is there $t \in \mathbb{R}_{\geq 0}$ such that $f(t) = 0$?

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- Diophantine-approximation bounds play a key role in the proof.

Illustrative Example

Consider the exponential polynomial

$$f(t) = 1.9 + \cos(t + \varphi_1) + \cos(\sqrt{2}t + \varphi_2) - e^{-t}$$

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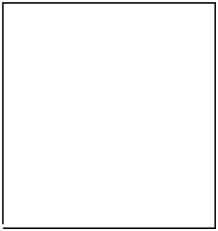

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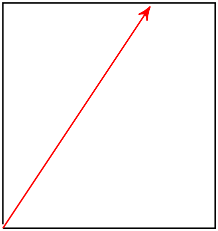

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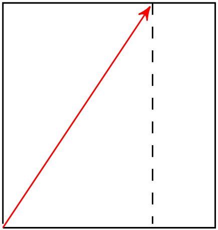

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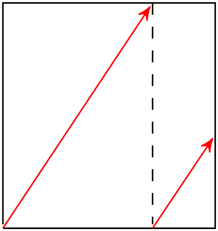

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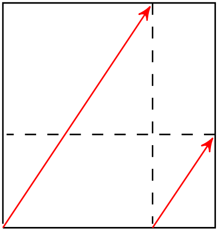

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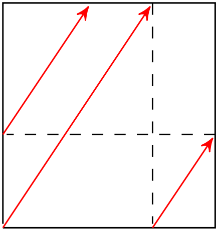

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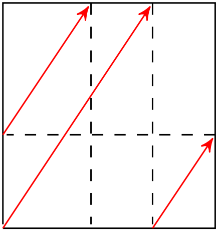

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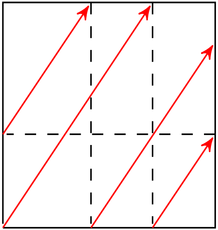

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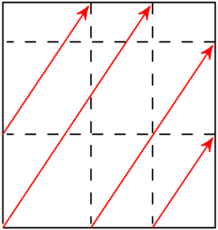

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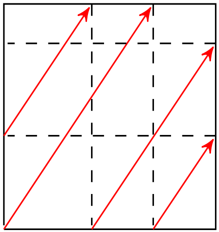

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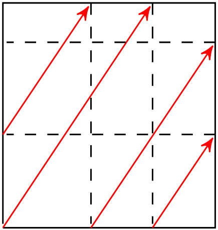

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Consider the exponential polynomial

$$f(t) = {\color{red}2} + \cos(t + \varphi_1) + \cos(\sqrt{2}t + \varphi_2) - e^{-t}$$

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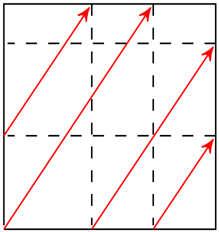

Baker's Theorem:

$$\left|\left|\left(t, \sqrt{2}t\right) - (\pi - \varphi_1, \pi - \varphi_2)\right|\right| \geq \frac{1}{\mathrm{poly}(t)}$$

Conclusion and Perspectives

The Discrete Case

A **linear recurrence sequence** is a sequence $\langle u_0, u_1, u_2, \ldots \rangle$ of integers such that there exist constants $a_1, \ldots, a_k$, such that

$$u_{n+k} = a_1 u_{n+k-1} + a_2 u_{n+k-2} + \ldots + a_k u_n$$

for all $n \geq 0$.

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Theorem (Skolem 1934; Mahler 1935, 1956; Lech 1953)

The set of zeros of a linear recurrence sequence is semi-linear:

$$\{n : u_n = 0\} = F \cup A_1 \cup \ldots \cup A_\ell$$

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Theorem (Berstel and Mignotte 1976)

In Skolem-Mahler-Lech, the infinite part (arithmetic progressions $A_1, \ldots, A_\ell$) is fully constructive.

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Does $\exists n$ such that $u_n = 0$?

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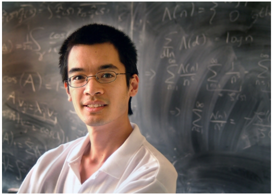
Terence Tao



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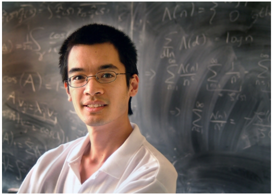

". . . a mathematical embarrassment . . ."

Richard Lipton

Wrapping Things Up

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- Even the bounded problem is hard (apparently).
- Formidable "mathematical obstacle" at dimension 9 in the unbounded case.
- The infinite-zeros problem is also hard.
- Diophantine-approximation techniques unavoidable.