Topological Dynamics and Decidability of Infinite Constraint Satisfaction

B. Klin, E. Kopczyński, J. Ochremiak, S. Toruńczyk

University of Warsaw

FICS/GI meeting, 11/09/2015
Puzzle 1

\[ a, b, c, d, \ldots \in \mathbb{A} \]
Puzzle 1

\[ a, b, c, d, \ldots \in \mathbb{A} \]
Puzzle 1

\[ a, b, c, d, \ldots \in \mathbb{A} \]

A graph:

- nodes: \( ab \) \quad a \neq b
- edges: \( ab \rightarrow bc \) \quad a \neq c
A graph:

- nodes: \( ab \)
- edges: \( ab \rightarrow bc \)
Puzzle 1

A graph:

- nodes: \( ab \)
- edges: \( ab \rightarrow bc \)

\( a, b, c, d, \ldots \in A \)

\( a \neq b \)
\( a \neq c \)
Puzzle 1

A graph:

- nodes: \( ab \) \hspace{1cm} a \neq b
- edges: \( ab \rightarrow bc \) \hspace{1cm} a \neq c

Is it 3-colorable?
Puzzle 1

A graph:
- nodes: \( ab \)
- edges: \( ab \rightarrow bc \)

Is it 3-colorable?
Puzzle 1

A graph:

- nodes: \( ab \) \( a \neq b \)
- edges: \( ab \rightarrow bc \) \( a \neq c \)

Is it 3-colorable?

No.
Puzzle 1

A graph:
- nodes: \(ab\) \(a \neq b\)
- edges: \(ab\) \(bc\) \(a \neq c\)

Is it 3-colorable? No.

Is 3-colorability decidable?
Doubly periodic graphs
Doubly periodic graphs

$\nu_{ij_1} \rightarrow \nu_{ij_4}$

$\nu_{ij_1} \rightarrow \nu_{i(j+1)j_5}$

$\nu_{ij_4} \rightarrow \nu_{(i+1)j_5}$
Doubly periodic graphs
Doubly periodic graphs

Thm: [Freedman’98]

3-colorability of doubly periodic graphs is undecidable.
Doubly periodic graphs

Thm: [Freedman’98]

3-colorability of doubly periodic graphs is undecidable.

What if we only use $=$?
Puzzle II

A system of equations over $\mathbb{Z}_2$:
- variables: $ab$, $a \neq b$
- equations: $ab + bc + ca = 0$
Puzzle II

A system of equations over $\mathbb{Z}_2$:
- variables: $ab$ $a \neq b$
- equations: $ab + bc + ca = 0$

Does it have a solution?
Puzzle II

A system of equations over $\mathbb{Z}_2$:
- variables: $ab$ \quad $a \neq b$
- equations: $ab + bc + ca = 0$
  $ab + ba = 1$

Does it have a solution?
Puzzle II

A system of equations over $\mathbb{Z}_2$:
- variables: $ab$ $a \neq b$
- equations: $ab + bc + ca = 0$
  $ab + ba = 1$

Does it have a solution?

\[
\begin{align*}
ab + ba &= 1 \\
ab + bc + ca &= 0 \\
ba + ac + cb &= 0 \\
bc + cd + db &= 0 \\
ca + ae + ec &= 0 \\
ac + cd &= 0 \\
bc + be + ec &= 0 \\
db + be + ed &= 0 \\
ae + ed + da &= 0
\end{align*}
\]  

No.
Puzzle II

A system of equations over $\mathbb{Z}_2$:
- variables: $ab \quad a \neq b$
- equations: $ab + bc + ca = 0$
  $ab + ba = 1$

Does it have a solution?

$$
\begin{align*}
ab + ba & = 1 \\
ab + bc + ca & = 0 \\
ba + ac + cb & = 0 \\
bc + cd + db & = 0 \\
ca + ae + ec & = 0 \\
ac + cd + da & = 0 \\
cb + be + ec & = 0 \\
db + be + ed & = 0 \\
ae + ed + da & = 0
\end{align*}
$$

No.

Is solvability decidable?
Topological dynamics

For decidability, we will use

Thm: [Pestov’98]

Every continuous action of $\text{Aut}(\mathbb{Q}, <)$ on a non-empty compact space has a fixpoint.
Topological dynamics

For decidability, we will use

Thm: [Pestov’98]

Every continuous action of $\text{Aut}(\mathbb{Q}, <)$ on a non-empty compact space has a fixpoint.
Topological dynamics

For decidability, we will use

Thm: [Pestov’98]

Every continuous action of $\text{Aut}(\mathbb{Q}, \prec)$ on a non-empty compact space has a fixpoint.

(proof: by Ramsey Theorem)
CSPs

An instance:
- a set $V$ of variables
- a set $T$ of values
- a set of constraints: $\left( x_1, \ldots, x_k; R \right)$

\[
\bigcap_{V} \bigcap_{T^k}
\]
CSPs

An instance:
- a set $V$ of variables
- a set $T$ of values
- a set of constraints: $(x_1, \ldots, x_k; R)$

A solution: $f : V \rightarrow T$

such that $(f(x_1), \ldots, f(x_k)) \in R$

for each constraint.
CSPs

An instance:
- a set $V$ of variables
- a set $T$ of values
- a set of constraints: $\left( x_1, \ldots, x_k; R \right)$

A solution: $f : V \rightarrow T$

such that $(f(x_1), \ldots, f(x_k)) \in R$

for each constraint.

CSP: does a given instance have a solution?
CSPs

An instance:
- a set $V$ of variables
- a set $T$ of values
- a set of constraints: $(x_1, \ldots, x_k; R)$

A solution: $f : V \to T$

such that $(f(x_1), \ldots, f(x_k)) \in R$

for each constraint.

CSP: does a given instance have a solution?
CSPs

An instance:
- a set $V$ of variables
- a set $T$ of values
- a set of constraints: $\left( x_1, \ldots, x_k; R \right)$

A solution: $f : V \rightarrow T$
such that $(f(x_1), \ldots, f(x_k)) \in R$

for each constraint.

CSP: does a given instance have a solution?
Decidable, NP-complete
CSP examples

Graph 3-colorability:

variables - nodes
values - \(\bullet\), \(\bullet\), \(\bullet\)
constraints - \((x, y; \neq)\), f.e. edge \(x \rightarrow y\)
CSP examples

Graph 3-colorability:

variables - nodes
values - ⊙, □, ●
constraints - \((x, y; \neq)\), f.e. edge \(x \longrightarrow y\)

Solving equations mod 2:

variables - variables
values - 0, 1
constraints - \((x, y, z; R)\), f.e. eqn. \(x + y + z = 0\)

\[ R \subseteq \{0, 1\}^3 \]

\[ R = \{(0, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0)\} \]
CSPs and homomorphisms

A CSP instance \((V, T, (\overline{x}_i, R_i)_{i \in I})\) defines:

- a **signature** \(\{R_i \mid i \in I\}\)
CSPs and homomorphisms

A CSP instance \((V, T, (\overline{x}_i, R_i)_{i \in I})\) defines:

- a signature \(\{R_i \mid i \in I\}\)

- a structure \(\mathcal{A}\):
  -- universe \(V\)
  -- \(R_i\) interpreted as \(\{\overline{x}_j \mid j \in I, R_j = R_i\}\)
CSPs and homomorphisms

A CSP instance \((V, T, (x_i, R_i)_{i \in I})\) defines:

- a **signature** \(\{R_i \mid i \in I\}\)
- a structure \(\mathcal{A}:\)
  -- universe \(V\)
  -- \(R_i\) interpreted as \(\{x_j \mid j \in I, \ R_j = R_i\}\)
- a structure \(\mathcal{B}:(\text{template})\)
  -- universe \(T\)
  -- \(R_i\) interpreted as \(R_i\)
CSPs and homomorphisms

A CSP instance \((V, T, (\overline{x}_i, R_i)_{i \in I})\) defines:

- a **signature** \(\{R_i \mid i \in I\}\)

- a structure \(A\):
  -- universe \(V\)
  -- \(R_i\) interpreted as \(\{\overline{x}_j \mid j \in I, R_j = R_i\}\)

- a structure \(B\) (template)
  -- universe \(T\)
  -- \(R_i\) interpreted as \(R_i\)

Instance solutions = homomorphisms \(A \rightarrow B\)
CSPs for fixed templates

\( \mathcal{B} \) - a relational structure

\( \text{CSP}(\mathcal{B}) \) - instances with template \( \mathcal{B} \)

Example:

3-colorability = \( \text{CSP}(\Delta) \)
CSPs for fixed templates

\( \mathcal{B} \) - a relational structure

\( \text{CSP}(\mathcal{B}) \) - instances with template \( \mathcal{B} \)

Example:

3-colorability = \( \text{CSP}(\Delta) \)

Conjecture: [Feder-Vardi’98]

For any \( \mathcal{B} \), the problem \( \text{CSP}(\mathcal{B}) \)
is either NP-complete or in PTIME
Infinite CSPs

- infinite templates ($\mathcal{B}$)
  -- example: instance = set of triples $(x, y, z) \in V^3$
  solution = $f : V \rightarrow \mathbb{Q}$ such that
  $x < y < z$ or $z < y < x$
Infinite CSPs

- infinite templates ($\mathcal{B}$)
  -- example: instance = set of triples $(x, y, z) \in V^3$
  solution = $f : V \rightarrow \mathbb{Q}$ such that
  $x < y < z$ or $z < y < x$

- infinite instances ($\mathcal{A}$)
  -- examples: Puzzles I and II
Infinite CSPs

- infinite templates \((\mathcal{B})\)
  -- example: instance = set of triples \((x, y, z) \in V^3\)
  solution = \(f : V \rightarrow \mathbb{Q}\) such that
  \(x < y < z\) or \(z < y < x\)

- infinite instances \((\mathcal{A})\)
  -- examples: Puzzles I and II

**Problem**: input needs to be finitely presented
Infinite CSPs

- infinite templates \((\mathcal{B})\)
  -- example: instance = set of triples \((x, y, z) \in V^3\)
  solution = \(f : V \rightarrow \mathbb{Q}\) such that
  \[x < y < z \text{ or } z < y < x\]

- infinite instances \((\mathcal{A})\)
  -- examples: Puzzles I and II

\textbf{Problem}: input needs to be finitely presented

- infinite instances and templates
Sets with atoms

Also known as nominal sets [GP’99], FM-sets, ...
Sets with atoms

Also known as nominal sets [GP’99], FM-sets, ...

Atoms: $\mathbb{A} \ni a, b, c, d, \ldots$
Sets with atoms

Also known as nominal sets [GP’99], FM-sets, ...

Atoms: $\mathbb{A} \ni a, b, c, d, \ldots$

A set with atoms:
- a set $X$
- a group action $\cdot_\_ : \text{Aut}(\mathbb{A}) \times X \rightarrow X$
Sets with atoms

Also known as nominal sets [GP’99], FM-sets, ...

Atoms: \( A \ni a, b, c, d, \ldots \)

A set with atoms:
- a set \( X \)
- a group action \( \_ \_ : \text{Aut}(A) \times X \to X \)

\( S \subseteq A \) supports \( x \in X \) if
\[ \forall a \in S. \pi(a) = a \quad \implies \quad \pi \cdot x = x \]
Sets with atoms

Also known as nominal sets [GP’99], FM-sets, ...

Atoms: $\mathbb{A} \ni a, b, c, d, \ldots$

A set with atoms:
- a set $X$
- a group action $\cdot: \text{Aut}(\mathbb{A}) \times X \to X$

$S \subseteq \mathbb{A}$ supports $x \in X$ if
$\forall a \in S. \pi(a) = a \implies \pi \cdot x = x$
Sets with atoms

Also known as nominal sets [GP'99], FM-sets, ...

Atoms: \( \mathcal{A} \ni a, b, c, d, \ldots \)

A set with atoms:
- a set \( X \)
- a group action \( \_ \cdot \_ : \text{Aut}(\mathcal{A}) \times X \rightarrow X \)
- such that every \( x \in X \) has a finite support.

\[ S \subseteq \mathcal{A} \text{ supports } x \in X \text{ if } \forall a \in S. \pi(a) = a \implies \pi \cdot x = x \]
Sets with atoms

Also known as nominal sets [GP’99], FM-sets, ...

Atoms: $\mathbb{A} \ni a, b, c, d, \ldots$

A set with atoms:
- a set $X$
- a group action $\cdot : \text{Aut}(\mathbb{A}) \times X \rightarrow X$
- such that every $x \in X$ has a finite support.

Equivariant functions: $f : X \rightarrow Y$ such that

$$f(\pi \cdot x) = \pi \cdot f(x)$$
Orbit-finite sets

**Orbit of** $x \in X : \quad \{ \pi \cdot x \mid \pi \in \text{Aut}(\mathbb{A}) \} \subseteq X$
Orbit-finite sets

Orbit of $x \in X : \{\pi \cdot x \mid \pi \in \text{Aut}(A)\} \subseteq X$

Define orbit-finite sets by set expressions:

- variables (ranging over $A$)
- set builders $\{e : v_1, \ldots, v_k \in A : \phi\}$
- unions, tuples
Orbit-finite sets

Orbit of $x \in X$ : $\{ \pi \cdot x \mid \pi \in \text{Aut}(A) \} \subseteq X$

Define orbit-finite sets by set expressions:
- variables (ranging over $A$)
- set builders $\{ e : v_1, \ldots, v_k \in A : \phi \}$
- unions, tuples

Examples:

$\{(a, b) : a, b \in A : a \neq b\}$

$\{\{(a, b, c), (b, c, a), (c, a, b)\} : a, b, c \in A : \top\}$
CSPs with atoms

An instance:
- a set $V$ of variables
- a set $T$ of values
- a set of constraints: $\left( x_1, \ldots, x_k; R \right)$

A solution: $f : V \to T$

such that $(f(x_1), \ldots, f(x_k)) \in R$

for each constraint.
CSPs with atoms

An instance:
- a set $V$ of variables
- a set $T$ of values
- a set of constraints: $(x_1, \ldots, x_k; R)$

A solution: $f : V \rightarrow T$
such that $(f(x_1), \ldots, f(x_k)) \in R$
for each constraint.
CSPs with atoms

An instance:
- a set $V$ of variables
- a set $T$ of values
- a set of constraints: $(x_1, \ldots, x_k; R)$

A solution: $f : V \rightarrow T$

such that $(f(x_1), \ldots, f(x_k)) \in R$

for each constraint.
CSPs with atoms

An instance:
- a set $V$ of variables
- a set $T$ of values
- a set of constraints: $\bigcap_{V} (x_1, \ldots, x_k; R) \subseteq T^k$

A solution: $f : V \rightarrow T$

such that $(f(x_1), \ldots, f(x_k)) \in R$

for each constraint.

CSP: does a given instance have a solution?
CSPs with atoms

An instance:
- a set $V$ of variables
- a set $T$ of values
- a set of constraints: $\{x_1, \ldots, x_k; R\}$

A solution: $f : V \rightarrow T$

such that $(f(x_1), \ldots, f(x_k)) \in R$

for each constraint.

CSP: does a given instance have a solution?
Examples

Graph from Puzzle I:

- nodes: \( ab \) \( a \neq b \)
- edges: \( ab \rightleftharpoons bc \) \( a \neq c \)
Examples

Graph from Puzzle I:

- nodes: \( ab \quad a \neq b \)
- edges: \( ab \rightarrow bc \quad a \neq c \)

- nodes: \( \{(a, b) : a, b \in \mathbb{A} : a \neq b\} \)
- edges: \( \{(a, b), (b, c)\} : a, b, c \in \mathbb{A} \)

\[ : a \neq b \land b \neq c \land a \neq c \]
Examples

Equation system from Puzzle II:

\[ ab + bc + ca = 0 \]
\[ ab + ba = 1 \]
Examples

Equation system from Puzzle II:

\[
ab + bc + ca = 0
\]
\[
ab + ba = 1
\]

\[
\{(a, b), (b, c), (c, a), 1) : a, b, c \in \mathbb{A} \\
: a \neq b \land b \neq c \land a \neq c\} \\
\cup \\
\{(a, b), (b, a), 0) : a, b \in \mathbb{A} : a \neq b\}
\]
Solving CSPs, equivariantly

Does a given instance have an equivariant solution?
Solving CSPs, equivariantly

Does a given instance have an equivariant solution? Decidable.
Solving CSPs, equivariantly

Does a given instance have an equivariant solution?

Algorithm:

1. Compute orbits of variables.
Solving CSPs, equivariantly

Does a given instance have an equivariant solution?

Algorithm:

1. Compute orbits of variables.

Decidable.
Solving CSPs, equivariantly

Does a given instance have an equivariant solution?

Algorithm:

1. Compute orbits of variables.
2. Assign values to variable orbits

Decidable.
Solving CSPs, equivariantly

Does a given instance have an equivariant solution?

Algorithm: 

1. Compute orbits of variables.
2. Assign values to variable orbits
3. Check every constraint

Decidable.
Solving CSPs, equivariantly

Does a given instance have an **equivariant** solution?

**Algorithm:**

1. Compute orbits of variables.
2. Assign values to variable orbits.
3. Check every constraint.
4. If failed, try 2. again.

Decidable.
Solving CSPs, equivariantly

Does a given instance have an equivariant solution?

Algorithm:

1. Compute orbits of variables.
2. Assign values to variable orbits.
3. Check every constraint.
4. If failed, try 2. again.

Complexity: exponential slowdown

- 3-colorability: NEXPTIME-complete
- Equation solving: EXPTIME
Examples

Puzzle I:
Examples

Puzzle I:
- one orbit of nodes
Examples

Puzzle I:
- one orbit of nodes
- edges exist
Examples

Puzzle I:

- one orbit of nodes
- edges exist
- so: no equivariant coloring
Examples

Puzzle I:
- one orbit of nodes
- edges exist
- so: no equivariant coloring

Puzzle II:
\[ ab + bc + ca = 0 \]
\[ ab + ba = 1 \]
Examples

Puzzle I:
- one orbit of nodes
- edges exist
- so: no equivariant coloring

Puzzle II:
- one orbit of variables

\[
\begin{align*}
ab + bc + ca &= 0 \\
ab + ba &= 1
\end{align*}
\]
Examples

Puzzle I:
- one orbit of nodes
- edges exist
- so: no equivariant coloring

Puzzle II:
- one orbit of variables
- all-1 assignment does not work

\[
\begin{align*}
ab + bc + ca &= 0 \\
ab + ba &= 1
\end{align*}
\]
Examples

Puzzle I:
- one orbit of nodes
- edges exist
- so: no equivariant coloring

Puzzle II:
- one orbit of variables
- all-1 assignment does not work
- all-0 assignment does not work

\[ ab + bc + ca = 0 \]
\[ ab + ba = 1 \]
Examples

Puzzle I:
- one orbit of nodes
- edges exist
- so: no equivariant coloring

Puzzle II:
- one orbit of variables
- all-1 assignment does not work
- all-0 assignment does not work
- so: no equivariant solution

\[ ab + bc + ca = 0 \]
\[ ab + ba = 1 \]
Solutions vs. equivariant solutions

Problem: the graph

- nodes: \( ab \quad a \neq b \)
- edges: \( ab \rightarrow ba \)
Solutions vs. equivariant solutions

Problem: the graph

- nodes: \( ab \) \( a \neq b \)
- edges: \( ab \rightarrow ba \)

has

- a 2-coloring,
- no equivariant coloring
Solutions vs. equivariant solutions

Problem: the graph

- nodes: \( ab \) \( a \neq b \)
- edges: \( ab \rightarrow ba \)

has

- a 2-coloring,
- no equivariant coloring

Solution: order atoms in some way, put

\( ab \) if \( a < b \), \( ab \) otherwise
Solutions vs. equivariant solutions

Problem: the graph

- nodes: \( ab \) \( a \neq b \)
- edges: \( ab \) — \( ba \)

has
- a 2-coloring,
- no equivariant coloring

Solution: order atoms in some way, put

\[
\begin{align*}
ab & \quad \text{if } a < b, \quad ab & \quad \text{otherwise}
\end{align*}
\]

A monotone-equivariant solution
Ordered atoms

\( \mathbb{A} = \{a, b, c, d, \ldots\} \)

can be replaced by

\( \mathbb{Q} \)
Ordered atoms

\[ A = \{a, b, c, d, \ldots\} \quad \text{Aut}(A) \]

can be replaced by

\[ \mathbb{Q} \quad \text{Aut}(\mathbb{Q}, <) \]
Ordered atoms

\[ A = \{a, b, c, d, \ldots \} \quad \text{Aut}(A) \]

can be replaced by

\[ Q \quad \text{Aut}(Q, <) \]

(or by any countable homogenous structure)
Ordered atoms

\[ \mathbb{A} = \{a, b, c, d, \ldots\} \quad \text{Aut}(\mathbb{A}) \]

can be replaced by

\[ \mathbb{Q} \quad \text{Aut}(\mathbb{Q}, <) \]

(or by any countable homogenous structure)

New orbit-finite sets:

\[ \{(a, b) : a, b \in \mathbb{A} : a < b\} \]
Ordered atoms

\[ \mathbb{A} = \{ a, b, c, d, \ldots \} \quad \text{Aut}(\mathbb{A}) \]

can be replaced by

\[ \mathbb{Q} \quad \text{Aut}(\mathbb{Q}, <) \]

(or by any countable homogenous structure)

New orbit-finite sets:

\[ \{(a, b) : a, b \in \mathbb{A} : a < b\} \]

- definable sets remain definable
- equivariant functions are monotone-equivariant
Solving CSPs

Let $\mathbb{A} = \mathbb{Q}$, totally ordered.
Solving CSPs

Let \( \mathbb{A} = \mathbb{Q} \), totally ordered.

**Theorem:** if an instance has a solution, it has a monotone-equivariant one.
Solving CSPs

Let $\mathbb{A} = \mathbb{Q}$, totally ordered.

**Theorem:** if an instance has a solution, it has a monotone-equivariant one.

**Proof:**
- solutions form a compact space
Solving CSPs

Let $\mathbb{A} = \mathbb{Q}$, totally ordered.

**Theorem:** if an instance has a solution, it has a monotone-equivariant one.

**Proof:**
- solutions form a compact space
- the group $\text{Aut}(\mathbb{Q}, <)$ acts on it
Solving CSPs

Let $\mathbb{A} = \mathbb{Q}$, totally ordered.

**Theorem:** if an instance has a solution, it has a monotone-equivariant one.

**Proof:**
- solutions form a compact space
- the group $\text{Aut}(\mathbb{Q}, <)$ acts on it
- monotone-equiv. solutions are fixpoints of this action
Solving CSPs

Let $\mathbb{A} = \mathbb{Q}$, totally ordered.

**Theorem:** if an instance has a solution, it has a monotone-equivariant one.

Proof:
- solutions form a compact space
- the group $\text{Aut}(\mathbb{Q}, <)$ acts on it
- monotone-equiv. solutions are fixpoints of this action
- Pestov’s theorem: Every continuous action of $\text{Aut}(\mathbb{Q}, <)$ on a nonempty compact space has a fixpoint.
Solving CSPs

Let $A = \mathbb{Q}$, totally ordered.

**Theorem:** if an instance has a solution, it has a monotone-equivariant one.

Proof:
- solutions form a compact space
- the group $\text{Aut}(\mathbb{Q}, <)$ acts on it
- monotone-equiv. solutions are fixpoints of this action
- Pestov’s theorem: *Every continuous action of* $\text{Aut}(\mathbb{Q}, <)$ *on a nonempty compact space has a fixpoint.*

And existence of monotone-equivariant solutions is decidable!
Examples

Puzzle I:
Examples

Puzzle I:
- two orbits of nodes

\[
\{(a, b) : a, b \in \mathbb{A} : a < b\}
\]
\[
\{(a, b) : a, b \in \mathbb{A} : a > b\}
\]
Examples

Puzzle I:

- two orbits of nodes

\[ \{ (a, b) : a, b \in A : a < b \} \]
\[ \{ (a, b) : a, b \in A : a > b \} \]

- triangles exist
Examples

Puzzle I:

- two orbits of nodes

\[ \{(a, b) : a, b \in \mathbb{A} : a < b\} \]
\[ \{(a, b) : a, b \in \mathbb{A} : a > b\} \]

- triangles exist

- so: no monotone-equivariant coloring
Examples

Puzzle I:
- two orbits of nodes
  \[ \{(a, b) : a, b \in \mathbb{A} : a < b\} \]
  \[ \{(a, b) : a, b \in \mathbb{A} : a > b\} \]
- triangles exist
- so: no monotone-equivariant coloring

Puzzle II:
\[ ab + bc + ca = 0 \]
\[ ab + ba = 1 \]
Examples

Puzzle I:
- two orbits of nodes
  \[ \{(a, b) : a, b \in A : a < b\} \]
  \[ \{(a, b) : a, b \in A : a > b\} \]
- triangles exist
- so: no monotone-equivariant coloring

Puzzle II:
- two orbits of variables
  \[ ab + bc + ca = 0 \]
  \[ ab + ba = 1 \]
Examples

Puzzle I:
- two orbits of nodes
  \[\{(a, b) : a, b \in \mathbb{A} : a < b\}\]
  \[\{(a, b) : a, b \in \mathbb{A} : a > b\}\]
- triangles exist
- so: no monotone-equivariant coloring

Puzzle II:
- two orbits of variables
- four assignments, none works
  \[ab + bc + ca = 0\]
  \[ab + ba = 1\]
Examples

Puzzle I:
- two orbits of nodes
  \begin{align*}
  \{(a, b) : a, b \in \mathbb{A} : a < b\} \\
  \{(a, b) : a, b \in \mathbb{A} : a > b\}
  \end{align*}
- triangles exist
- so: no monotone-equivariant coloring

Puzzle II:
- two orbits of variables
  \begin{align*}
  ab + bc + ca &= 0 \\
  ab + ba &= 1
  \end{align*}
- four assignments, none works
- so: no monotone-equivariant solution
Complexity
Complexity

InfCSP(\(\mathcal{B}\)) - definable instances, template \(\mathcal{B}\)
Complexity

\text{InfCSP}(B) - definable instances, template \( B \)

\textbf{Thm:} \text{InfCSP}(B) is decidable for all finite \( B \)
Complexity

InfCSP(\(B\)) - definable instances, template \(B\)

**Thm:** InfCSP(\(B\)) is decidable for all finite \(B\)

**Thm:** if CSP(\(B\)) is \(C\)-complete, then
InfCSP(\(B\)) is \(\text{exp}(C)\)-complete
Complexity

InfCSP(ℬ) - definable instances, template ℬ

Thm: InfCSP(ℬ) is decidable for all finite ℬ

Thm: if CSP(ℬ) is C-complete, then InfCSP(ℬ) is exp(C)-complete

Examples: for definable instances,
- 3-colorability is NEXPTIME-complete
- Horn-SAT is EXPTIME-complete
- 2-colorability is PSPACE-complete
- etc.
Generalizations

Locally finite CSP:
- orbit-finite set of values,
- only finite relations allowed in constraints.
Generalizations

Locally finite CSP:
- orbit-finite set of values,
- only finite relations allowed in constraints.

Still decidable, the same proof works.
Generalizations

Locally finite CSP:
- orbit-finite set of values,
- only finite relations allowed in constraints.

Still decidable, the same proof works.

Orbit-finite CSPs: the following is undecidable:

Given orbit-finite relational structures $\mathcal{A}$ and $\mathcal{B}$, is there a homomorphism from $\mathcal{A}$ to $\mathcal{B}$?