A Computationally Grounded Logic of Awareness

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Abstract

We introduce a multi-agent logic of explicit, implicit belief and awareness with a semantics using belief bases. The novelty of our approach is that an agent's awareness is not a primitive but is directly computed from the agent's belief base. We prove soundness and completeness of the logic relative to the belief base semantics. Furthermore, we provide a polynomial embedding of the logic of propositional awareness into it, and a polynomial embedding of our logic into the logic of general awareness. Thanks to the latter embedding, we show that the satisfiability checking problem for our logic is PSPACE-complete. Finally, we extend it by the notions of public announcement and private belief expansion.

1 Introduction

The notion of awareness was introduced in the area of epistemic logic by Fagin & Halpern (F&H) [7] to cope with the problem of *logical omniscience* [16]. Their approach is *syntactic* to the extent that they associate a subset of formulas to each agent at each state, indicating the formulas the agent is aware of. Following the idea suggested by Levesque [19], F&H make the distinction between *explicit belief* and *implicit belief*, where explicit belief is defined to be implicit belief *plus* awareness.

There is another tradition in the formalization of awareness, initiated by Modica & Rustichini [27, 28] and Heifetz et al. [13, 14]. They support a *semantic* approach by letting possible worlds be associated with a subset of all propositional variables being defined. Hence, an agent is aware of a formula if and only if, every atomic proposition occurring in the formula is defined at every epistemically accessible state for the agent. Such a notion of awareness is often called *propositional awareness* in opposition to the notion of *general awareness*, according to which an agent can be "primitively" aware not only of atomic propositions but also of complex formulas. Halpern et al. [9, 11] prove an equivalence result between the syntactic approach and the semantic approach for propositional awareness. van Ditmarsch et al. [38] give a novel notion called *speculative knowledge*, which is also built on propositional awareness.

The concept of explicit belief, which is central in the logic of awareness, is closely related to the concept of *belief base* [29, 26, 12, 31]. The latter plays an important role in the AGM approach to belief revision [2] and, more generally, in the area of knowledge representation and reasoning (KR). Recently, in [21, 23] we defined a formal semantics for multi-agent epistemic logic exploiting belief bases which clearly distinguishes explicit from implicit belief. Specifically, according to this semantics, an agent explicitly believes that a certain fact α is true if α is a piece of information included in the agent's belief base. On the contrary, the agent implicitly believes that α, if α is derivable from the agent's belief base. A logic of explicit and implicit belief, called Logic of Doxastic Attitudes (LDA), was defined on the top of this semantics. The logic was further enriched in a series of papers with the aim of capturing higher-order epistemic reasoning in robots [22], the notions of "only believing" [24] and graded belief [25], and of elucidating the connection between distributed belief and belief merging [15].

In this paper, we extend the semantics introduced in [23] and the corresponding logic LDA with propositional awareness. We call LDAA the resulting logic. The novelty of our approach lies in the fact that the notion of awareness is not primitive but is computed from, and therefore grounded on, the notion of belief base. In particular, for an agent to be aware of a proposition p, p has to be included in the agent's vocabulary, that is to say, there should exist a formula in the agent's belief base which contains p. From this perspective, we offer a minimalistic logic approach to explicit, implicit belief and awareness in which only the former concept is primitive, while the other two concepts are defined from it.

The paper is organized as follows. In Section 2, we present the language of our logic of explicit, implicit belief and awareness. In Section 3, we first present the belief base semantics with respect to which the language is interpreted. Then, we introduce two alternative semantics which are closer in spirit to the standard semantics for epistemic logic based on multi-relational Kripke structures. We show that the three semantics are all equivalent with respect to the language under consideration. Section 4 presents axiomatic results for our logic, while in Section 5 we explore the connection between our logic and the logic of propositional awareness (LPA) [9], by providing a satisfiability preserving translation of the latter into the former. In Section 6, we establish the connection between our logic and the logic of general awareness (LGA) [7], by providing a satisfiability preserving translation of the former into the latter. Thanks to the latter embedding, we show that that the satisfiability checking problem for our logic is PSPACE-complete. Section 7 clarifies the ontological foundation of our logic LDAA in connection with the logic of propositional awareness (LPA) and the logic of general awareness (LGA). Section 8 presents two dynamic extensions of the static setting, the first by public announcement and the second by the notion of private belief base expansion. In Section 9 we conclude.

2 Language

This section presents the language of the Logic of Doxastic Attitudes with Awareness (LDAA) to represent explicit beliefs, implicit beliefs, and awareness. It extends the language in [23] with the awareness modality. Let $Atm = \{p, q, \ldots\}$ be a countably infinite set of atomic propositions and let $Agt = \{1, \ldots, n\}$ be a finite set of agents. The language $\mathcal{L}_0(Atm, Agt)$ is defined as follows:

$$\alpha \quad ::= \quad p \mid \neg\alpha \mid \alpha_1 \wedge \alpha_2 \mid \triangle_i\alpha \mid \bigcirc_i\alpha,$$

where p ranges over Atm and i ranges over Agt.

The language $\mathcal{L}_{\mathsf{LDAA}}(Atm, Agt)$ adds to $\mathcal{L}_0(Atm, Agt)$ a new level with implicit belief operators and is defined as follows:

$$\varphi \quad ::= \quad \alpha \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Box_i\varphi \mid \bigcirc_i\varphi,$$

where α ranges over $\mathcal{L}_0(Atm, Agt)$ and i ranges over Agt.

When it is unambiguous from the context, we write $\mathcal{L}_0$ instead of $\mathcal{L}_0(Atm, Agt)$ and $\mathcal{L}_{\mathsf{LDAA}}$ instead of $\mathcal{L}_{\mathsf{LDAA}}(Atm, Agt)$. The other Boolean connectives $\vee$, $\rightarrow$, $\leftrightarrow$, $\top$ and $\bot$ are defined from $\neg$ and $\wedge$ in the standard way. The formula $\triangle_i\alpha$ is read "agent i explicitly believes that α is true". The formula $\bigcirc_i\varphi$ is read "agent i is aware of φ". The $\triangle_i$-operator can be iterated, which means that the language contains expressions for higher-order explicit beliefs, such as $\triangle_i\triangle_j\alpha$, which is read "agent i explicitly believes that agent j explicitly believes that α is true". The iteration is possibly a mix of explicit belief and awareness, such as $\triangle_i \bigcirc_j \alpha$, which is read "agent i explicitly believes that agent j is aware of α".

The formula $\Box_i\varphi$ is read "agent i implicitly believes that φ is true". The dual operator $\Diamond_i$ is defined as follows:

$$\Diamond_i\varphi \overset{\mathtt{def}}{=} \neg\Box_i\neg\varphi,$$

where $\Diamond_i\varphi$ is read "φ is consistent with agent i's explicit beliefs"[1]. Note that the modality $\bigcirc_i$ appears at both levels of the language, but the modality $\triangle_i$ only appears at the first level. As a result, we can have awareness operators in the scope of explicit belief operators, but not implicit belief operators in the scope of explicit belief operators. Moreover, both the explicit belief and implicit belief operator are allowed inside the awareness operator. It is because the concept of *propositional awareness* allows awareness of any formula that is constituted by atomic propositions of which the agent is aware.

Since we represent a propositional notion of awareness, i.e., being aware of a formula is equivalent to being aware of every atomic proposition occurring in it, we need the following inductive definition to represent the set of atomic

[1] Because $\Box_i\varphi$ bears the meaning that φ can be inferred from i's explicit beliefs, $\Diamond_i\varphi$ should be read "$\neg\varphi$ cannot be inferred from agent i's explicit beliefs", which is equivalent to "φ is consistent with agent i's explicit beliefs".

propositions occurring in a formula φ, noted $Atm(\varphi)$:

$$\begin{aligned}
Atm(p) &= \{p\}, \\
Atm(\neg\varphi) &= Atm(\varphi), \\
Atm(\varphi_1 \wedge \varphi_2) &= Atm(\varphi_1) \cup Atm(\varphi_2), \\
Atm(\triangle_i \alpha) &= Atm(\alpha) \\
Atm(Y_i \varphi) &= Atm(\varphi), \text{ for } Y \in \{\bigcirc, \Box\}.
\end{aligned}$$

Let $\Gamma \subseteq \mathcal{L}_{\mathsf{LDAA}}$ be finite, we define $Atm(\Gamma) = \bigcup_{\varphi \in \Gamma} Atm(\varphi)$.

3 Semantics

In this section, we present three families of formal semantics for $\mathcal{L}_{\mathsf{LDAA}}$. The first semantics exploits belief bases. An agent's set of doxastic alternatives and awareness set are not primitive but computed from them. The second semantics is a Kripke-style semantics, in which we require each agent's set of doxastic alternatives to be equal to the set of worlds in which his explicit beliefs are true, and the agent's awareness set to be equal to the set of of atomic propositions occurring in his explicit beliefs. The third semantics relaxes these requirements, so that an agent's set of doxastic alternatives is included in the set of worlds in which the agent's explicit beliefs are true, and the set of atomic propositions occurring in an agent's explicit beliefs is a subset of the agent's awareness set.

3.1 Multi-Agent Belief Base Semantics

The basic constituent of our semantics is the following notion of state.

Definition 3.1 *A state is a tuple $S = (B_1, \dots, B_n, A_1, \dots, A_n, V)$ where,*

- $B_i \subseteq \mathcal{L}_0$ *is agent i's belief base for any $i \in Agt$,*
- $A_i = Atm(B_i)$ *is agent i's awareness set for any $i \in Agt$,*
- $V \subseteq Atm$ *is the actual environment.*

The set of all states is denoted by $\mathbf{S}$.

With the definition of state, we have the following interpretations for the formulas in $\mathcal{L}_0$.

Definition 3.2 *For any $S = (B_1, \dots, B_n, A_1, \dots, A_n, V) \in \mathbf{S}$:*

$$\begin{aligned}
S \models p &\iff p \in V, \\
S \models \neg\alpha &\iff S \not\models \alpha, \\
S \models \alpha_1 \wedge \alpha_2 &\iff S \models \alpha_1 \text{ and } S \models \alpha_2, \\
S \models \triangle_i \alpha &\iff \alpha \in B_i, \\
S \models \bigcirc_i \alpha &\iff Atm(\alpha) \subseteq A_i.
\end{aligned}$$

Note that the awareness component of Definition 3.1 is unnecessary, as we could interpret the operator $\bigcirc_i$ equivalenty by postulating "$S \models \bigcirc_i \alpha$ iff $Atm(\alpha) \subseteq Atm(B_i)$". The reason we keep it is that it has counterparts in the notional model semantics and quasi-notional model semantics we will define in Sections 3.2 and 3.2. In the quasi-notional model semantics, an agent's awareness set is supposed to be a superset of the set of atomic propositions occurring in the agent's belief set.

Note that the notion of awareness represented by the $\bigcirc_i$-operator is propositional, i.e., being aware of a formula is equivalent to being aware of every atomic proposition occurring in the formula. Such a notion of awareness is different from the notion of general awareness according to which an agent can be aware of $p \wedge q$ without being aware of $p \vee q$.

The following definition introduces the concept of multi-agent belief-awareness model.

Definition 3.3 *A multi-agent belief-awareness model* (MABA) *is a pair* (S, Cxt)*, where* $S \in \mathbf{S}$ *and* $Cxt \subseteq \mathbf{S}$.

Cxt is the agents' context or common ground [33]. It corresponds to the body of information that the agents share and that they use to make inferences from their explicit beliefs. Following [23], in the following definition we compute the agents' epistemic accessibility relations from their belief bases.

Definition 3.4 *For any* $i \in Agt$*,* $\mathcal{R}_i$ *is the binary relation on* $\mathbf{S}$ *such that for any* $S = (B_1, \dots, B_n, A_1, \dots, A_n, V), S' = (B'_1, \dots, B'_n, A'_1, \dots, A'_n, V') \in \mathbf{S}$,

$$(S, S') \in \mathcal{R}_i \textit{ if and only if } \forall \alpha \in B_i,\ S' \models \alpha.$$

According to the previous definition, a state is considered possible by an agent if it satisfies all his explicit beliefs.

With the accessibility relation defined, we have the following semantic interpretation for formulas in $\mathcal{L}_{\mathsf{LDAA}}$. The boolean case is defined in the usual way and omitted.

Definition 3.5 *Let* (S, Cxt) *be a* MABA *with* $S = (B_1, \dots, B_n, A_1, \dots, A_n, V)$*. Then,*

$$\begin{aligned}
(S, Cxt) \models \alpha &\iff S \models \alpha, \\
(S, Cxt) \models \Box_i \varphi &\iff \forall S' \in Cxt,\ \textit{if } (S, S') \in \mathcal{R}_i \textit{ then } (S', Cxt) \models \varphi, \\
(S, Cxt) \models \bigcirc_i \varphi &\iff Atm(\varphi) \subseteq A_i.
\end{aligned}$$

The following two definitions specify two interesting properties of MABAs.

Definition 3.6 *The* MABA (S, Cxt) *satisfies global consistency* (GC) *if and only if, for every* $i \in Agt$ *and for every* $S' \in (\{S\} \cup Cxt)$*, there exists* $S'' \in Cxt$ *such that* $(S', S'') \in \mathcal{R}_i$.

Definition 3.7 *The* MABA (S, Cxt) *satisfies belief correctness* (BC) *if and only if* $S \in Cxt$ *and, for every* $i \in Agt$ *and for every* $S' \in Cxt$*,* $(S', S') \in \mathcal{R}_i$.

It is worth noting that (S, Cxt) satisfies belief correctness if and only if $S \in Cxt$ and for all $S' = (B'_1, \dots, B'_n, A'_1, \dots, A'_n, V') \in Cxt$, $i \in Agt$ and $\alpha \in B'_i$ we have $S' \models \alpha$.

For $\mathrm{X} \subseteq \{\mathrm{GC}, \mathrm{BC}\}$, $\mathbf{MABA}_{\mathrm{X}}$ is the class of MABAs satisfying all the conditions in X. $\mathbf{MABA}_{\emptyset}$ is the class of all MABAs, and we write $\mathbf{MABA}$ instead of $\mathbf{MABA}_{\emptyset}$. It is easy to see that $\mathbf{MABA}_{\{\mathrm{GC},\mathrm{BC}\}} = \mathbf{MABA}_{\{\mathrm{BC}\}}$.

Let $\varphi \in \mathcal{L}_{\mathsf{LDAA}}$, we say that φ is valid for the class $\mathbf{MABA}_{\mathrm{X}}$ if and only if, for every $(S, Cxt) \in \mathbf{MABA}_{\mathrm{X}}$ we have $(S, Cxt) \models \varphi$. We say that φ is satisfiable of the class $\mathbf{MABA}_{\mathrm{X}}$ if and only if $\neg\varphi$ is not valid for the class $\mathbf{MABA}_{\mathrm{X}}$.

3.2 Notional Model Semantics

In this section we introduce an alternative Kripke-style semantics for the language $\mathcal{L}_{\mathsf{LDAA}}$ based on notional doxastic-awareness models. The latter extend notional doxastic models defined in [21, 23] by awareness functions.

Definition 3.8 *A notional doxastic-awareness model* (NDAM) *is a tuple* $M = (W, \mathcal{D}, \mathcal{A}, \mathcal{N}, \mathcal{V})$ *where,*

- W *is a non-empty set of worlds,*
- $\mathcal{D}$*:* $Agt \times W \longrightarrow 2^{\mathcal{L}_0}$ *is a doxastic function,*
- $\mathcal{A}$*:* $Agt \times W \longrightarrow 2^{Atm}$ *is an awareness function,*
- $\mathcal{N}$*:* $Agt \times W \longrightarrow 2^{W}$ *is a notional function,*
- $\mathcal{V}$*:* $Atm \longrightarrow 2^{W}$ *is a valuation function.*

and such that, given the following inductive definition of the semantic interpretation of formulas in $\mathcal{L}_{\mathsf{LDAA}}$*:*

$$\begin{aligned}
(M, w) \models p &\iff w \in \mathcal{V}(p),\\
(M, w) \models \neg\varphi &\iff (M, w) \not\models \varphi,\\
(M, w) \models \varphi \wedge \psi &\iff (M, w) \models \varphi \text{ and } (M, w) \models \psi,\\
(M, w) \models \triangle_i \alpha &\iff \alpha \in \mathcal{D}(i, w),\\
(M, w) \models \Box_i \varphi &\iff \forall u \in \mathcal{N}(i, w),\ (M, u) \models \varphi,\\
(M, w) \models \bigcirc_i \varphi &\iff Atm(\varphi) \subseteq \mathcal{A}(i, w),
\end{aligned}$$

it satisfies the following conditions (C1) *and* (C2)*, for all* $i \in Agt$ *and for all* $w \in W$*:*

(C1) $\mathcal{A}(i, w) = Atm\big(\mathcal{D}(i, w)\big)$,

(C2) $\mathcal{N}(i, w) = \bigcap_{\alpha \in \mathcal{D}(i, w)} ||\alpha||_M$*, where* $||\alpha||_M = \{u \in W : (M, u) \models \alpha\}$.

We recall that the term 'notional' is borrowed from the philosopher D. Dennett [5, 6] (see, also, [18]): an agent's notional world is a world at which all the agent's explicit beliefs are true.

This idea is clearly expressed by Condition (C2): a world is in $\mathcal{N}(i, w)$ (i.e., agent i's set of notional worlds at w) if and only if it satisfies all explicit beliefs that agent i has at w.

The following definitions specify global consistency (GC) and belief correctness (BC) for notional models.

Definition 3.9 *The* NDAM $M = (W, \mathcal{D}, \mathcal{A}, \mathcal{N}, \mathcal{V})$ *satisfies global consistency if and only if, for any $i \in Agt$ and for any $w \in W$, $\mathcal{N}(i, w) \neq \emptyset$.*

Definition 3.10 *The* NDAM $M = (W, \mathcal{D}, \mathcal{A}, \mathcal{N}, \mathcal{V})$ *satisfies belief correctness if and only if, for any $i \in Agt$ and for any $w \in W$, $w \in \mathcal{N}(i, w)$.*

For any $\mathrm{X} \subseteq \{\mathrm{GC}, \mathrm{BC}\}$, $\mathbf{NDAM}_{\mathrm{X}}$ is the class of NDAMs satisfying the conditions in X. $\mathbf{NDAM}_{\emptyset}$ is the class of all NDAMs, and we write **NDAM** instead of $\mathbf{NDAM}_{\emptyset}$. Analogously to MABAs, we have $\mathbf{NDAM}_{\{\mathrm{GC},\mathrm{BC}\}} = \mathbf{NDAM}_{\{\mathrm{BC}\}}$. A NDAM $M = (W, \mathcal{D}, \mathcal{A}, \mathcal{N}, \mathcal{V})$ is finite if and only if W, $\mathcal{D}(i, w)$, and $\mathcal{V}^{\leftarrow}(w)$ are finite sets for every $i \in Agt$ and every $w \in W$, where $\mathcal{V}^{\leftarrow}(w) = \{p \in Atm : w \in \mathcal{V}(p)\}$. As $\mathcal{A}(i, w) = Atm\big(\mathcal{D}(i, w)\big)$, it follows that, if M is finite, $\mathcal{A}(i, w)$ is also a finite set for any $i \in Agt$ and any $w \in W$. We use finite-$\mathbf{NDAM}_{\mathrm{X}}$ to denote the class of finite NDAMs satisfying the conditions in X.

Let $\varphi \in \mathcal{L}_{\mathsf{LDAA}}$, we say that φ is valid for the class $\mathbf{NDAM}_{\mathrm{X}}$ if and only if, for every $M = (W, \mathcal{D}, \mathcal{A}, \mathcal{N}, \mathcal{V}) \in \mathbf{NDAM}_{\mathrm{X}}$ and for every $w \in W$, we have $(M, w) \models \varphi$. We say that φ is satisfiable for the class $\mathbf{NDAM}_{\mathrm{X}}$ if and only if $\neg\varphi$ is not valid for the class $\mathbf{NDAM}_{\mathrm{X}}$.

3.3 Quasi-Model Semantics

This section provides an alternative semantics for the language $\mathcal{L}_{\mathsf{LDAA}}$ based on a more general class of models, called quasi-notional doxastic-awareness models (quasi-NDAMs) in which the restrictions on the notional and awareness function are weakened.

Definition 3.11 *A quasi-notional doxastic-awareness model (*quasi-NDAM*) is a tuple $M = (W, \mathcal{D}, \mathcal{A}, \mathcal{N}, \mathcal{V})$ where $W, \mathcal{D}, \mathcal{A}, \mathcal{N}, \mathcal{V}$ together with the semantic interpretation of formulas in $\mathcal{L}_{\mathsf{LDAA}}$ are as in Definition 3.8, except that Condition* (C1) *and* (C2) *are replaced by the following weaker conditions, for all $i \in Agt$ and for all $w \in W$:*

(C1*) $Atm\big(\mathcal{D}(i, w)\big) \subseteq \mathcal{A}(i, w)$,

(C2*) $\mathcal{N}(i, w) \subseteq \bigcap_{\alpha \in \mathcal{D}(i, w)} ||\alpha||_M$.

As for NDAMs, for any $\mathrm{X} \subseteq \{\mathrm{GC}, \mathrm{BC}\}$, $\mathbf{QNDAM}_{\mathrm{X}}$ is the class of quasi-NDAMs satisfying the conditions in X. $\mathbf{QNDAM}_{\emptyset}$ is the class of all quasi-NDAMs, and we write **QNDAM** instead of $\mathbf{QNDAM}_{\emptyset}$. As for MABAs and NDAMs, we have $\mathbf{QNDAM}_{\{\mathrm{GC},\mathrm{BC}\}} = \mathbf{QNDAM}_{\{\mathrm{BC}\}}$. A quasi-NDAM $M = (W, \mathcal{D}, \mathcal{A}, \mathcal{N}, \mathcal{V})$ is finite if W, $\mathcal{D}(i, w)$, $\mathcal{A}(i, w)$ and $\mathcal{V}^{\leftarrow}(w)$ are finite sets for every $i \in Agt$ and every $w \in W$. We use finite-$\mathbf{QNDAM}_{\mathrm{X}}$ to denote the class of finite quasi-NDAMs satisfying the conditions in X. Validity and satisfiability of formulas for a class $\mathbf{QNDAM}_{\mathrm{X}}$ are defined in the usual way.

3.4 Equivalence Results

In this section, we present equivalence results between the five different semantics for $\mathcal{L}_{\mathsf{LDAA}}$ we presented above (i.e., **MABA**, **NDAM**, finite-**NDAM**, **QNDAM** and finite-**QNDAM**).

3.4.1 Equivalence between quasi-NDAMs and finite quasi-NDAMs

First of all, we consider the relationship between **QNDAM** and finite-**QNDAM**. Let us define a filtrated model for the proof.

Let $M = (W, \mathcal{D}, \mathcal{A}, \mathcal{N}, \mathcal{V})$ be a (possibly infinite) quasi-NDAM and let $\Sigma \subseteq \mathcal{L}_{\mathsf{LDAA}}$ be an arbitrary finite set of formulas which is closed under subformulas. The equivalence relation $\equiv_\Sigma$ on W is defined as follows:

$$\equiv_\Sigma = \{(w, v) \in W \times W : \forall \varphi \in \Sigma, (M, w) \models \varphi \text{ iff } (M, v) \models \varphi\}.$$

Let $[w]_\Sigma$ be the equivalence class of the world w generated by the relation $\equiv_\Sigma$. The model $M_\Sigma = (W_\Sigma, \mathcal{D}_\Sigma, \mathcal{A}_\Sigma, \mathcal{N}_\Sigma, \mathcal{V}_\Sigma)$ is the filtration of M under Σ where:

- $W_\Sigma = \{[w]_\Sigma : w \in W\}$,
- for any $i \in Agt$ and for any $[w]_\Sigma \in W_\Sigma$, $\mathcal{D}_\Sigma(i, [w]_\Sigma) = (\bigcap_{u \in [w]_\Sigma} \mathcal{D}(i, u)) \cap \Sigma$,
- for any $i \in Agt$ and for any $[w]_\Sigma \in W_\Sigma$, $\mathcal{A}_\Sigma(i, [w]_\Sigma) = \big(\bigcap_{u \in [w]_\Sigma} \mathcal{A}(i, u)\big) \cap \Sigma$,
- for any $i \in Agt$ and for any $[w]_\Sigma \in W_\Sigma$, $\mathcal{N}_\Sigma(i, [w]_\Sigma) = \{[u]_\Sigma \in W_\Sigma : \exists v \in [w]_\Sigma, \exists v' \in [u]_\Sigma$ such that $v' \in \mathcal{N}(i, v)\}$,
- for any $p \in Atm$, $\mathcal{V}_\Sigma(p) = \{[w]_\Sigma : (M, w) \models p\}$ if $p \in Atm(\Sigma)$, $\mathcal{V}_\Sigma(p) = \emptyset$ otherwise.

We have the following filtration lemma showing that the filtrated model is semantically equivalent with the original model with respect to Σ.

Lemma 3.1 *Let $\varphi \in \Sigma$ and let $w \in W$. Then, $(M, w) \models \varphi$ if and only if $(M_\Sigma, [w]_\Sigma) \models \varphi$.*

Proof The proof is by induction on the structure of φ. For the cases other than $\varphi = \bigcirc_i \psi$, the proof is identical with that of Lemma 4 in the appendix of [23]. So we only need to prove the case $\varphi = \bigcirc_i \psi$.

($\Rightarrow$) Suppose $(M, w) \models \bigcirc_i \psi$ with $\bigcirc_i \psi \in \Sigma$. Thus, $Atm(\psi) \subseteq \mathcal{A}(i, w)$. Hence, by the definition of $\mathcal{A}_\Sigma(i, [w]_\Sigma)$ and the fact that Σ is closed under subformulas, we have $Atm(\psi) \subseteq \mathcal{A}_\Sigma(i, [w]_\Sigma)$. It follows that $(M_\Sigma, [w]_\Sigma) \models \bigcirc_i \psi$.

($\Leftarrow$) For the other direction, suppose $(M_\Sigma, [w]_\Sigma) \models \bigcirc_i \psi$ with $\bigcirc_i \psi \in \Sigma$. Thus, $Atm(\psi) \subseteq \mathcal{A}_\Sigma(i, [w]_\Sigma)$. Hence, by the definition of $\mathcal{A}_\Sigma(i, [w]_\Sigma)$, $Atm(\psi) \subseteq \mathcal{A}(i, w)$. It follows that $(M, w) \models \bigcirc_i \psi$. $\square$

The following proposition highlights that M_Σ is finite and preserves the properties of M.

Proposition 3.1 *Let $\Sigma \subseteq \mathcal{L}_{\mathsf{LDAA}}$ be an arbitrary finite set of formulas which is closed under subformulas and let $M_\Sigma = (W_\Sigma, \mathcal{D}_\Sigma, \mathcal{A}_\Sigma, \mathcal{N}_\Sigma, \mathcal{V}_\Sigma)$ be a filtration of $M = (W, \mathcal{D}, \mathcal{A}, \mathcal{N}, \mathcal{V})$ under Σ. Then, M_Σ is a finite* quasi-NDAM. *Moreover, for any $x \in \{\mathrm{GC}, \mathrm{BC}\}$, if M satisfies x, then M_Σ also satisfies it.*

Proof By the proof of Proposition 12 in the appendix of [23], we have that, M_Σ is finite and satisfies Condition (C2*) in Definition 3.11, and that, for any $x \in \{\mathrm{GC}, \mathrm{BC}\}$, if M satisfies x, then M_Σ also satisfies it. Here, we only need to prove that M satisfies Condition (C1*) in Definition 3.11. Suppose $\varphi \in \mathcal{D}_\Sigma(i, [w]_\Sigma)$, we need to prove that $Atm(\varphi) \subseteq \mathcal{A}_\Sigma(i, [w]_\Sigma)$. By the definition of $\mathcal{D}_\Sigma(i, [w]_\Sigma)$, we have $\varphi \in \mathcal{D}(i, w)$. By Condition (C1*), it follows that, $Atm(\varphi) \subseteq \mathcal{A}(i, w)$. By the definition of $\mathcal{A}_\Sigma(i, [w]_\Sigma)$ and the fact that Σ is closed under subformulas, we have $Atm(\varphi) \subseteq \mathcal{A}_\Sigma(i, [w]_\Sigma)$. As a result, $\mathcal{A}_\Sigma(i, [w]_\Sigma) \supseteq Atm(\mathcal{D}_\Sigma(i, [w]_\Sigma))$. $\square$

The following lemma is a straighforward consequence of Lemma 3.1 and Proposition 3.1.

Lemma 3.2 *Let $\mathrm{X} \subseteq \{\mathrm{GC}, \mathrm{BC}\}$ and $\varphi \in \mathcal{L}_{\mathsf{LDAA}}$. If φ is satisfiable for the class* $\mathbf{QNDAM}_{\mathrm{X}}$ *then φ is satisfiable for the class finite-*$\mathbf{QNDAM}_{\mathrm{X}}$.

3.4.2 Equivalence between finite NDAMs and finite quasi-NDAMs

Our next result concerns the equivalence between finite-**NDAM** and finite-**QNDAM**.

Lemma 3.3 *Let $\mathrm{X} \subseteq \{\mathrm{GC}, \mathrm{BC}\}$ and $\varphi \in \mathcal{L}_{\mathsf{LDAA}}$. If φ is satisfiable for the class finite-*$\mathbf{QNDAM}_{\mathrm{X}}$*, then φ is satisfiable for the class finite-*$\mathbf{NDAM}_{\mathrm{X}}$.

Proof We are going to build a finite NDAM from a finite quasi-NDAM without changing the satisfiability of φ. To accomplish this goal, two things are essential in the construction. Firstly, we enlarge each agent's belief base with an identifier proposition to make his set of doxastic alternatives smaller and coincide with his set of notional worlds. Secondly, we combine the identifier with some tautologies by conjunctions, so that the set of atomic propositions occurring in his belief base is equal to his awareness set.

Since the set of identifiers and $Atm(\varphi)$ are disjoint, the satisfiability of φ is not affected if φ is of the form p, $\triangle_i \psi$ or $\bigcirc_i \psi$. If φ is of the form $\Box_i \psi$, its satisfiability is not affected since agent i's set of epistemically accessible states does not change as a consequence of the transformation.

Let $M = (W, \mathcal{D}, \mathcal{A}, \mathcal{N}, \mathcal{V})$ be a finite quasi-NDAM that satisfies φ, i.e., there exists $w \in W$ such that $(M, w) \models \varphi$. We define the set of all atomic propositions occurring in some awareness set of some agent at some world in M as follows:

$$\mathcal{T}(M) = \bigcup_{w \in W, i \in Agt} \mathcal{A}(i, w).$$

Since M is finite, $\mathcal{T}(M)$ is also finite.

We have the following injective function which assigns an identifier to each agent at each world in W.

$$f : Agt \times W \longrightarrow Atm \setminus \big(\mathcal{T}(M) \cup Atm(\varphi)\big).$$

As Atm is infinite while W, $\mathcal{T}(M)$, Agt and $Atm(\varphi)$ are finite, such an injection exists.

We define a new model $M' = (W', \mathcal{D}', \mathcal{A}', \mathcal{N}', \mathcal{V}')$ with $W' = W$, $\mathcal{N}' = \mathcal{N}$ and where $\mathcal{D}'$, $\mathcal{V}'$ and $\mathcal{A}'$ are defined as follows. For every $i \in Agt$ and for every $w \in W$:

$$\begin{aligned}
&\mathcal{A}'(i, w) = \mathcal{A}(i, w) \cup \{f(i, w)\},\\
&\mathcal{D}'(i, w) = \mathcal{D}(i, w) \cup \left\{ f(i, w) \wedge \left(\bigwedge_{p \in \mathcal{A}(i,w) \setminus Atm\left(\mathcal{D}(i,w)\right)} (p \vee \neg p) \right) \right\}.
\end{aligned}$$

Moreover, for every $p \in Atm$:

$$\begin{aligned}
\mathcal{V}'(p) &= \mathcal{V}(p) && \text{if } p \in \mathcal{T}(M) \cup Atm(\varphi),\\
\mathcal{V}'(p) &= \mathcal{N}(i, w) && \text{if } p = f(i, w),\\
\mathcal{V}'(p) &= \emptyset && \text{otherwise .}
\end{aligned}$$

It is easy to verify that M' satisfies Condition (C1) and (C2) in Definition 3.8. Thus, M' is a finite NDAM.

The rest of the proof consists in checking that, for every $x \in \{\mathrm{GC}, \mathrm{BC}\}$, if M satisfies x then M' also satisfies x, which is straightforward, and that, $(M, w) \models \varphi$ iff $(M, w) \models \varphi$. We prove the latter by induction on the structure of φ.

The case $\varphi = p$ is immediate from the definition of $\mathcal{V}'$. The boolean cases are straightforward. Let us prove the case $\varphi = \triangle_i \alpha$.

($\Rightarrow$) Suppose $(M, w) \models \triangle_i \alpha$. Then, we have $\alpha \in \mathcal{D}(i, w)$. Hence, by the definition of $\mathcal{D}'$, $\alpha \in \mathcal{D}'(i, w)$. Thus, $(M', w) \models \triangle_i \alpha$.

($\Leftarrow$) Suppose $(M', w) \models \triangle_i \alpha$. Then, we have $\alpha \in \mathcal{D}'(i, w)$. Since $f(i, w) \notin Atm(\triangle_i \alpha)$, by the definition of $\mathcal{D}'$, we have that,

$$\alpha \neq f(i, w) \wedge \left(\bigwedge_{p \in \mathcal{A}(i,w) \setminus Atm(\mathcal{D}(i,w))} (p \vee \neg p) \right).$$

Thus, $\alpha \in \mathcal{D}(i, w)$ and, consequently, $(M, w) \models \triangle_i \alpha$.

Then, let us prove the case $\varphi_i = \bigcirc_i \psi$.

($\Rightarrow$) Suppose $(M, w) \models \bigcirc_i \psi$. Then, we have $Atm(\psi) \subseteq \mathcal{A}(i, w)$. Hence, by the definition of $\mathcal{A}'$, $Atm(\psi) \subseteq \mathcal{A}'(i, w)$. Thus, $(M', w) \models \bigcirc_i \psi$.

($\Leftarrow$) Suppose $(M', w) \models \bigcirc_i \psi$. Then, we have $Atm(\psi) \subseteq \mathcal{A}'(i, w)$. The definition of $\mathcal{A}'$ ensures that $f(i, w) \notin Atm(\psi)$. Thus, $Atm(\psi) \subseteq \mathcal{A}(i, w)$ and, consequently, $(M, w) \models \bigcirc_i \psi$.

At last, let us prove the case $\varphi = \Box_i \psi$. $(M, w) \models \Box_i \psi$ means that $(M, u) \models \psi$ for all $u \in \mathcal{N}(i, w)$, which is equivalent to $(M', u) \models \psi$ for all $u \in \mathcal{N}'(i, w)$ by the induction hypothesis and the the fact that $\mathcal{N}'(i, w) = \mathcal{N}(i, w)$. The latter means that $(M', w) \models \Box_i \psi$.

Now we have that $(M, w) \models \varphi$ iff $(M, w) \models \varphi$. Then, if M satisfies φ, M' satisfies φ as well. $\square$

3.4.3 Equivalence between MABAs and NDAMs

The following lemma concerns the equivalence between **MABA** and **NDAM**.

Lemma 3.4 *Let $\varphi \in \mathcal{L}_{\mathsf{LDAA}}$ and $\mathrm{X} \subseteq \{\mathrm{GC}, \mathrm{BC}\}$. Then, φ is satisfiable for the class* $\mathbf{MABA}_{\mathrm{X}}$ *if and only if φ is satisfiable for the class* $\mathbf{NDAM}_{\mathrm{X}}$.

Proof The proof is almost identical to that of Lemma 7 in the appendix of [23]. Here we only give a quick sketch. For the left-to-right direction, we prove the following weaker result: if φ is satisfiable for the class $\mathbf{MABA}_{\mathrm{X}}$, then φ is satisfiable for the class $\mathbf{QNDAM}_{\mathrm{X}}$. Then, by Lemma 3.2 and Lemma 3.3, we have that if φ is satisfiable for the class $\mathbf{MABA}_{\mathrm{X}}$, then φ is satisfiable for the class $\mathbf{NDAM}_{\mathrm{X}}$. For the right-to-left direction, we first transform the NDAM $M = (W, \mathcal{D}, \mathcal{A}, \mathcal{N}, \mathcal{V})$ into a non-redundant NDAM, where there are no identical worlds in W. Then we build a MABA from the latter. It is easy to prove that such transformations preserve satisfiability of φ. $\square$

The following theorem is the main result of this section.

Theorem 3.1 *Let $\varphi \in \mathcal{L}_{\mathsf{LDAA}}$ and $\mathrm{X} \subseteq \{\mathrm{GC}, \mathrm{BC}\}$. Then, the following five statements are equivalent:*

- *φ is satisfiable for the class* $\mathbf{MABA}_{\mathrm{X}}$,
- *φ is satisfiable for the class* $\mathbf{NDAM}_{\mathrm{X}}$,
- *φ is satisfiable for the class* $\mathbf{QNDAM}_{\mathrm{X}}$,

$$\begin{tikzcd}
\mathbf{MABA}_{\mathrm{X}} \arrow[r, leftrightarrow] & \mathbf{NDAM}_{\mathrm{X}} \arrow[r, dotted] & \mathbf{QNDAM}_{\mathrm{X}} \arrow[d, shift left] \\
& finite - \mathbf{NDAM}_{\mathrm{X}} \arrow[u, dotted] \arrow[r, dotted, shift left] & finite - \mathbf{QNDAM}_{\mathrm{X}} \arrow[u, dotted, shift left] \arrow[l, shift left]
\end{tikzcd}$$

Figure 1: Relations between semantics for the language $\mathcal{L}_{\mathsf{LDAA}}$. An arrow means that satisfiability relative to the first class of structures implies satisfiability relative to the second class of structures. Full arrows correspond to the results stated in Lemmas 3.2, 3.3 and 3.4. Dotted arrows denote relations that follow straightforwardly given the inclusion between classes of structures.

- *φ is satisfiable for the class finite-$\mathbf{QNDAM}_{\mathrm{X}}$,*
- *φ is satisfiable for the class finite-$\mathbf{NDAM}_{\mathrm{X}}$.*

Proof The theorem is a direct consequence of Lemmas 3.2, 3.3 and 3.4. $\square$

4 Axiomatics

In this section, we define some variants of the LDAA logics and prove their soundness and completeness for their corresponding model classes.

We define the base logic LDAA to be the extension of classical propositional logic given by the following axioms and rule of inference:

$$\big(\Box_i\varphi \wedge \Box_i(\varphi \to \psi)\big) \to \Box_i\psi \qquad (\mathbf{K}_{\Box_i})$$

$$\triangle_i\alpha \to \Box_i\alpha \qquad (\mathbf{Int}_{\triangle_i,\Box_i})$$

$$\triangle_i\alpha \to \bigcirc_i\alpha \qquad (\mathbf{Int}_{\triangle_i,\bigcirc_i})$$

$$\bigcirc_i\,\varphi \leftrightarrow \bigwedge_{p\in Atm(\varphi)} \bigcirc_i p \qquad (\mathbf{AGPP})$$

$$\frac{\varphi}{\Box_i\varphi} \qquad (\mathbf{Nec}_{\Box_i})$$

For $\mathrm{X} \subseteq \{\mathbf{D}_{\Box_i}, \mathbf{T}_{\Box_i}\}$, let $\mathsf{LDAA}_{\mathrm{X}}$ be the extension of logic LDAA by every axiom in X, where,

$$\neg(\Box_i\varphi \wedge \Box_i\neg\varphi) \qquad (\mathbf{D}_{\Box_i})$$

$$\Box_i\varphi \to \varphi \qquad (\mathbf{T}_{\Box_i})$$

As usual, for every logic $\mathsf{LDAA}_{\mathrm{X}}$ with $\mathrm{X} \subseteq \{\mathbf{D}_{\Box_i}, \mathbf{T}_{\Box_i}\}$ and for every $\varphi \in \mathcal{L}_{\mathsf{LDAA}}$, we write $\vdash_{\mathsf{LDAA}_{\mathrm{X}}} \varphi$ to mean that φ is deducible in $\mathsf{LDAA}_{\mathrm{X}}$, which is defined as usual. We say that the set of formulas Γ from $\mathcal{L}_{\mathsf{LDAA}}$ is LDAA_X-consistent if there are no formulas $\varphi_1, \ldots, \varphi_m \in \Gamma$ such that $\vdash_{\mathsf{LDAA}_X} (\varphi_1 \wedge \ldots \wedge \varphi_m) \to \bot$. In particular, φ is LDAA_X-consistent if $\{\varphi\}$ is LDAA_X-consistent.

Clearly, the logics $\mathsf{LDAA}_{\{\mathbf{D}_{\Box_i}, \mathbf{T}_{\Box_i}\}}$ and $\mathsf{LDAA}_{\{\mathbf{T}_{\Box_i}\}}$ are identical since Axiom $\mathbf{D}_{\Box_i}$ is deducible in $\mathsf{LDAA}_{\{\mathbf{T}_{\Box_i}\}}$.

The rest of this section is devoted to prove completeness for the logics in the LDAA family.

We first prove completeness of each logic $\mathsf{LDAA}_{\mathrm{X}}$. To this aim, let us define the following correspondence function between axioms and semantic properties:

- $cf(\mathbf{D}_{\Box_i}) = \mathrm{GC}$,
- $cf(\mathbf{T}_{\Box_i}) = \mathrm{BC}$.

As usual, we have the following property for maximally consistent sets (MCSs).

Proposition 4.1 *Let Γ be a $\mathsf{LDAA}_{\mathrm{X}}$-MCS with $\mathrm{X} \subseteq \{\mathbf{D}_{\Box_i}, \mathbf{T}_{\Box_i}\}$. Then,*

- *if $\varphi, \varphi \to \psi \in \Gamma$ then $\psi \in \Gamma$,*
- *$\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$,*
- *$\varphi \vee \psi \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$.*

The following is the Lindenbaum's lemma for our logics. Its proof is standard (cf. Lemma 4.17 in [4]) and we omit it.

Lemma 4.1 *Let Γ be a set of formulas being LDAA_{X}-consistent with $\mathrm{X} \subseteq \{\mathbf{D}_{\Box_i}, \mathbf{T}_{\Box_i}\}$, then there exists a $\mathsf{LDAA}_{\mathrm{X}}$-MCS Γ' such that $\Gamma \subseteq \Gamma'$.*

To prove completeness with respect to the class $\mathbf{QNDAM}_{X}$, we construct a canonical model as follows.

Definition 4.1 *Let $\mathrm{X} \subseteq \{\mathbf{D}_{\Box_i}, \mathbf{T}_{\Box_i}\}$. Then, the $\mathsf{LDAA}_{\mathrm{X}}$-canonical model is a tuple $M^{\mathsf{LDAA}_{\mathrm{X}}} = (W^{\mathsf{LDAA}_{\mathrm{X}}}, \mathcal{D}^{\mathsf{LDAA}_{\mathrm{X}}}, \mathcal{A}^{\mathsf{LDAA}_{\mathrm{X}}}, \mathcal{N}^{\mathsf{LDAA}_{\mathrm{X}}}, \mathcal{V}^{\mathsf{LDAA}_{\mathrm{X}}})$ such that:*

- *$W^{\mathsf{LDAA}_{\mathrm{X}}}$ is the set of maximally consistent sets (MCSs) for $\mathsf{LDAA}_{\mathrm{X}}$,*
- *$\forall w \in W^{\mathsf{LDAA}_{\mathrm{X}}}$, $\forall i \in Agt$, and $\forall \alpha \in \mathcal{L}_0$, $\alpha \in \mathcal{D}^{\mathsf{LDAA}_{\mathrm{X}}}(w,i)$ iff $\triangle_i \alpha \in w$,*
- *$\forall w \in W^{\mathsf{LDAA}_{\mathrm{X}}}$, $\forall i \in Agt$, and $\forall p \in Atm$, $p \in \mathcal{A}^{\mathsf{LDAA}_{\mathrm{X}}}(w,i)$ iff $\bigcirc_i p \in w$,*
- *$\forall w, u \in W^{\mathsf{LDAA}_{\mathrm{X}}}$ and $\forall i \in Agt$, $u \in \mathcal{N}^{\mathsf{LDAA}_{\mathrm{X}}}(i,w)$ iff $\forall \varphi \in \mathcal{L}_{\mathsf{LDAA}}$, if $\Box_i \varphi \in w$ then $\varphi \in u$,*
- *$\forall w \in W^{\mathsf{LDAA}_{\mathrm{X}}}$ and $\forall p \in Atm$, $w \in \mathcal{V}^{\mathsf{LDAA}_{\mathrm{X}}}(p)$ iff $p \in w$.*

The following existence lemma is necessary for the proof of completeness. The proof is again standard (cf. Lemma 4.20 in [4]) and we omit it.

Lemma 4.2 *Let $\varphi \in \mathcal{L}_{\mathsf{LDAA}}$ and let $w \in W^{\mathsf{LDAA}_{\mathrm{X}}}$ with $\mathrm{X} \subseteq \{\mathbf{D}_{\Box_i}, \mathbf{T}_{\Box_i}\}$. Then, if $\Diamond_i \varphi \in w$ then there exists $u \in \mathcal{N}^{\mathsf{LDAA}_{\mathrm{X}}}(i,w)$ such that $\varphi \in u$.*

The following is the truth lemma for our logics.

Lemma 4.3 *Let $\varphi \in \mathcal{L}_{\mathsf{LDAA}}$ and let $w \in W^{\mathsf{LDAA}_{\mathrm{X}}}$ with $\mathrm{X} \subseteq \{\mathbf{D}_{\Box_i}, \mathbf{T}_{\Box_i}\}$. Then, $(M^{\mathsf{LDAA}_{\mathrm{X}}}, w) \models \varphi$ iff $\varphi \in w$.*

Proof The proof is by induction on the structure of the formula φ. For the cases φ atomic, Boolean, or of the form $\Box_i\psi$, the proof is standard by means of Proposition 4.1 and Lemma 4.2. The proof for the case $\varphi = \triangle_i\alpha$ goes as follows: $\triangle_i\alpha \in w$ iff $\alpha \in \mathcal{D}^{\mathsf{LDAA_X}}(i, w)$ iff $(M^{\mathsf{LDAA_X}}, w) \models \triangle_i\alpha$.

For the case $\varphi = \bigcirc_i\psi$, by the axiom **AGPP**, $\bigcirc_i\psi \in w$ iff $\forall p \in Atm(\psi)$, $\bigcirc_i p \in w$. By the definition of the canonical model, the latter is equivalent to the fact that, $\forall p \in Atm(\psi)$, $p \in \mathcal{A}^{\mathsf{LDAA_X}}(i, w)$. The latter is equivalent to $Atm(\psi) \subseteq \mathcal{A}^{\mathsf{LDAA_X}}(i, w)$, which means $(M^{\mathsf{LDAA_X}}, w) \models \bigcirc_i\psi$ by our semantics. $\Box$

The last step consists in proving that the $\mathsf{LDAA_X}$-canonical model belongs to the appropriate model class for the logic $\mathsf{LDAA_X}$.

Proposition 4.2 *Let* $\mathrm{X} \subseteq \{\mathbf{D}_{\Box_i}, \mathbf{T}_{\Box_i}\}$. *Then,* $M^{\mathsf{LDAA_X}} \in \mathbf{QNDAM}_{\{cf(x):x\in\mathrm{X}\}}$.

Proof Firstly, we need to prove that $M^{\mathsf{LDAA_X}}$ satisfies Condition (C1*) and (C2*) in Definition 3.11. For Condition (C1*), we have to prove that if $\alpha \in \mathcal{D}^{\mathsf{LDAA_X}}(i, w)$ then $Atm(\alpha) \subseteq \mathcal{A}^{\mathsf{LDAA_X}}(i, w)$. Suppose $\alpha \in \mathcal{D}^{\mathsf{LDAA_X}}(i, w)$. Thus, $\triangle_i\alpha \in w$. Hence, by the axiom $\mathbf{Int}_{\triangle_i,\bigcirc_i}$, $\bigcirc_i\alpha \in w$. By the axiom **AGPP**, it follows that, $\forall p \in Atm(\alpha)$, $\bigcirc_i p \in w$. Then, by the definition of $M^{\mathsf{LDAA_X}}$, $\forall p \in Atm(\alpha)$, $p \in \mathcal{A}^{\mathsf{LDAA_X}}(i, w)$, which means $Atm(\alpha) \subseteq \mathcal{A}^{\mathsf{LDAA_X}}(i, w)$. For Condition (C2*), we have to prove that if $\alpha \in \mathcal{D}^{\mathsf{LDAA_X}}(i, w)$ then $\mathcal{N}^{\mathsf{LDAA_X}}(i, w) \subseteq ||\alpha||_{M^{\mathsf{LDAA_X}}}$. Suppose $\alpha \in \mathcal{D}^{\mathsf{LDAA_X}}(i, w)$. Thus, $\triangle_i\alpha \in w$. Hence, by the axiom $\mathbf{Int}_{\triangle_i,\Box_i}$, $\Box_i\alpha \in w$. By the definition of $M^{\mathsf{LDAA_X}}$, if follows that, $\forall u \in \mathcal{N}^{\mathsf{LDAA_X}}(i, w)$, $\alpha \in u$. Thus, by Lemma 4.3, we have that, $\forall u \in \mathcal{N}^{\mathsf{LDAA_X}}(i, w)$, $(M^{\mathsf{LDAA_X}}, u) \models \alpha$. The latter means that $\mathcal{N}^{\mathsf{LDAA_X}}(i, w) \subseteq ||\alpha||_{M^{\mathsf{LDAA_X}}}$.

It is easy to verify that $M^{\mathsf{LDAA_X}}$ satisfies the corresponding properties in $\{cf(x) : x \in \mathrm{X}\}$ using the standard proof. $\Box$

By Lemma 4.3 and Proposition 4.2, we are able to prove the following soundness and completeness theorem. Proving soundness is just a routine exercise.

Theorem 4.1 *Let* $\mathrm{X} \subseteq \{\mathbf{D}_{\Box_i}, \mathbf{T}_{\Box_i}\}$. *Then, the logic* $\mathsf{LDAA_X}$ *is sound and complete for the class* $\mathbf{QNDAM}_{\{cf(x):x\in\mathrm{X}\}}$.

The following is a corollary of Theorem 3.1 and Theorem 4.1.

Corollary 4.1 *Let* $\mathrm{X} \subseteq \{\mathbf{D}_{\Box_i}, \mathbf{T}_{\Box_i}\}$. *Then,*

- $\mathsf{LDAA_X}$ *is sound and complete for the class* $\mathbf{NDAM}_{\{cf(x):x\in\mathrm{X}\}}$,
- $\mathsf{LDAA_X}$ *is sound and complete for the class* $\mathbf{MABA}_{\{cf(x):x\in\mathrm{X}\}}$.

5 Relationship with Propositional Awareness

In this section, we build a connection between LDAA and the logic of propositional awareness (LPA), where the latter, first introduced in [9], is a special case of the logic of general awareness (LGA) by Fagin & Halpern [7]. Specifically, we provide a polynomial, satisfiability preserving translation from the language of LPA to the language of LDAA. The language of LPA, denoted by $\mathcal{L}_{\mathsf{LPA}}(Atm, Agt)$, is defined by the following grammar:

$$\varphi \quad ::= \quad p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \mathsf{B}_i\varphi \mid \mathsf{A}_i\varphi \mid \mathsf{X}_i\varphi,$$

where p ranges over Atm and i ranges over Agt. When it is unambiguous from the context, we write $\mathcal{L}_{\mathsf{LPA}}$ instead of $\mathcal{L}_{\mathsf{LPA}}(Atm, Agt)$. At the semantics level, the logic of propositional awareness exploits awareness structures in which the awareness function is assumed to be propositional.

Definition 5.1 *A propositional awareness model* (PAM) *is a tuple* $M = (\Omega, \Rightarrow, \rho, \pi)$ *where,*

- Ω *is a non-empty set of states,*
- $\Rightarrow: Agt \times \Omega \longrightarrow 2^{\Omega}$ *is a doxastic accessibility function,*
- $\rho : Agt \times \Omega \longrightarrow 2^{Atm}$ *is a propositional awareness function,*
- $\pi : Atm \longrightarrow 2^{\Omega}$ *is a valuation function.*

The class of propositional awareness models is denoted by **PAM**.

For a PAM $M = (\Omega, \Rightarrow, \rho, \pi)$ and $s \in \Omega$, a pair (M, s) is called a pointed PAM. We write $s \Rightarrow_i t$ for $t \in \Rightarrow(i, s)$. We say that a PAM $M = (\Omega, \Rightarrow, \rho, \pi)$ satisfies global consistency (GC) if, for every $s \in \Omega$ and $i \in Agt$, there is $t \in \Omega$ such that $s \Rightarrow_i t$. We say that it satisfies belief correctness (BC) if, for every $s \in \Omega$ and $i \in Agt$, $s \Rightarrow_i s$. For every $\mathrm{X} \subseteq \{\mathrm{GC}, \mathrm{BC}\}$, we note $\mathbf{PAM}_{\mathrm{X}}$ the class of propositional awareness models satisfying every property in X. The semantic interpretation of formulas in $\mathcal{L}_{\mathsf{LPA}}$ relative to a pointed PAM is defined as follows.

Definition 5.2 *Let* $M = (\Omega, \Rightarrow, \rho, \pi)$ *be a* PAM *and* $s \in \Omega$. *Then,*

$$\begin{aligned}
(M, s) \models p &\iff s \in \pi(p),\\
(M, s) \models \neg\varphi &\iff (M, s) \not\models \varphi,\\
(M, s) \models \varphi \wedge \psi &\iff (M, s) \models \varphi \text{ and } (M, s) \models \psi,\\
(M, s) \models \mathsf{B}_i\varphi &\iff \forall t \in \Omega,\ \text{if } s \Rightarrow_i t \text{ then } (M, t) \models \varphi,\\
(M, s) \models \mathsf{A}_i\varphi &\iff Atm(\varphi) \subseteq \rho(i, s),\\
(M, s) \models \mathsf{X}_i\varphi &\iff (M, s) \models \mathsf{B}_i\varphi \text{ and } (M, s) \models \mathsf{A}_i\varphi.
\end{aligned}$$

We translate formulas of $\mathcal{L}_{\mathsf{LPA}}(Atm, Agt)$ into formulas of $\mathcal{L}_{\mathsf{LDAA}}(Atm, Agt)$

via the following translation function $tr_1 : \mathcal{L}_{\mathsf{LPA}}(Atm, Agt) \longrightarrow \mathcal{L}_{\mathsf{LDAA}}(Atm, Agt)$:

$$\begin{aligned}
tr_1(p) &= p \text{ for } p \in Atm,\\
tr_1(\neg\varphi) &= \neg tr_1(\varphi),\\
tr_1(\varphi_1 \wedge \varphi_2) &= tr_1(\varphi_1) \wedge tr_1(\varphi_2),\\
tr_1(\mathsf{A}_i\varphi) &= \bigcirc_i tr_1(\varphi),\\
tr_1(\mathsf{B}_i\varphi) &= \Box_i tr_1(\varphi),\\
tr_1(\mathsf{X}_i\varphi) &= \bigwedge_{p \in Atm(\varphi)} \bigcirc_i p \wedge \Box_i tr_1(\varphi).
\end{aligned}$$

The interesting aspect of the previous translation is that the LPA notion of explicit belief is mapped into the combination of implicit belief *plus* awareness in our logic LDAA, and not directly into the LDAA notion of explicit belief. This highlights that the two notions of explicit belief do not capture the same type of epistemic attitude. While the LDAA notion represents an agent's actual belief which is active in his working memory and instantly accessible to him (we assume an agent's belief base to be a rough approximation of his working memory), the LPA notion is aimed at capturing the agent's relevant beliefs that are built from his actual vocabulary.[2]

As the following theorem highlights, the translation is satisfiability preserving.

Theorem 5.1 *Let $\varphi \in \mathcal{L}_{\mathsf{LPA}}$ and $\mathrm{X} \subseteq \{\mathrm{GC}, \mathrm{BC}\}$. Then, φ is satisfiable for the class $\mathbf{PAM}_{\mathrm{X}}$ if and only if $tr_1(\varphi)$ is satisfiable for the class $\mathbf{NDAM}_{\mathrm{X}}$.*

Proof We first prove a weaker result of the left-to-right direction, i.e., if φ is satisfiable for the class $\mathbf{PAM}_{\mathrm{X}}$, then $tr_1(\varphi)$ is satisfiable for the class $\mathbf{QNDAM}_{\mathrm{X}}$. Let $M = (\Omega, \Rightarrow, \rho, \pi)$ be a PAM and let $s \in \Omega$ such that $(M, s) \models \varphi$. We build the corresponding $M' = (W, \mathcal{D}, \mathcal{A}, \mathcal{N}, \mathcal{V})$ as follows:

- $W = \Omega$,
- $\forall i \in Agt$ and $\forall s \in \Omega$, $\mathcal{D}(i, s) = \{p \vee \neg p : (M, s) \models \mathsf{A}_i p\}$,
- $\forall i \in Agt$ and $\forall s \in \Omega$, $\mathcal{A}(i, s) = \rho(i, s)$,
- $\forall i \in Agt$ and $\forall s \in \Omega$, $\mathcal{N}(i, s) = \Rightarrow(i, s)$,
- $\forall p \in Atm$, $\mathcal{V}(p) = \pi(p)$.

We prove that M' is a quasi-NDAM by showing that it satisfies Condition (C1*) and (C2*) in Definition 3.11.

[2]Note that if we defined the translation sending explicit beliefs of LPA into explicit beliefs of LDAA, satisfiability would be preserved only in the direction from LDAA to LPA. For the other direction, a formula of the form $\mathsf{X}_i\mathsf{B}_i\varphi$ in $\mathcal{L}_{\mathsf{LPA}}$ cannot be translated into $\mathcal{L}_{\mathsf{LDAA}}$ with this alternative translation.

For Condition (C1*), by the semantics of PAM and the definitions of $\mathcal{D}(i,s)$ and $\mathcal{A}(i,s)$, it is easy to show that, $Atm\big(\mathcal{D}(i,s)\big) = \mathcal{A}(i,s)$ for every $i \in Agt$ and every $s \in W$, which implies $Atm\big(\mathcal{D}(i,s)\big) \subseteq \mathcal{A}(i,s)$.

For Condition (C2*), by the definition of $\mathcal{D}(i,s)$, there are only tautologies in it. So we have that $\bigcap_{tr_1(\varphi)\in\mathcal{D}(i,w)} ||tr_1(\varphi)||_M = W$. Then, clearly, Condition (C2*) is satisfied.

It is easy to verify that, for every $x \in \{\mathrm{GC}, \mathrm{BC}\}$, if M satisfies x then M' satisfies it as well.

By induction on the structure of φ, we prove that, for all $s \in \Omega$, $(M,s) \models \varphi$ iff $(M',s) \models tr_1(\varphi)$.

For the case $\varphi = p$ and the boolean cases $\varphi = \neg\psi$ and $\varphi = \psi_1 \wedge \psi_2$, it is straightforward.

Now we consider the case $\varphi = \mathsf{A}_i\psi$. Suppose $(M,s) \models \mathsf{A}_i\psi$. By the semantics of PAMs, it is equivalent to $Atm(\psi) \subseteq \rho(i,s)$. By the definition of $\mathcal{A}(i,s)$ and the function tr_1, the latter is equivalent to $Atm\big(tr_1(\psi)\big) \subseteq \mathcal{A}(i,s)$. And in turn the latter means $(M',s) \models \bigcirc_i tr_1(\psi)$. Then, by the function tr_1, the latter is equivalent to $(M',s) \models tr_1(\mathsf{A}_i\psi)$.

Let us consider the case $\varphi = \mathsf{B}_i\psi$. Suppose $(M,s) \models \mathsf{B}_i\psi$. By the induction hypothesis, we have $||\psi||_M = ||tr_1(\psi)||_{M'}$. $(M,s) \models \mathsf{B}_i\psi$ means that $\Rightarrow(i,s) \subseteq ||\psi||_M$. By the definition of $\mathcal{N}(i,s)$ and the fact that $||\psi||_M = ||tr_1(\psi)||_{M'}$, the latter it equivalent to $\mathcal{N}(i,s) \subseteq ||tr_1(\psi)||_{M'}$, which is equivalent to $(M',s) \models \Box_i tr_1(\psi)$. The latter means $(M',s) \models tr_1(\mathsf{B}_i\psi)$ by the definition of the function tr_1.

Finally, let us consider the case $\varphi = \mathsf{X}_i\psi$. Suppose $(M,s) \models \mathsf{X}_i\psi$. Given the fact that $\mathsf{X}_i\psi$ is equivalent to $\mathsf{A}_i\psi \wedge \mathsf{B}_i\psi$ and $\mathsf{A}_i\psi$ is equivalent to $\bigwedge_{p\in Atm(\psi)} \mathsf{A}_i p$, by the previous cases, it means that, $(M',s) \models \bigwedge_{p\in Atm(\psi)} \mathsf{A}_i p \wedge \Box_i tr_1(\psi)$. By the function tr_1, the latter is equivalent to $(M',s) \models tr_1(\mathsf{X}_i\psi)$.

Thus, we conclude that $(M,s) \models \varphi$ iff $(M',s) \models tr_1(\varphi)$ for all $s \in \Omega$. Then we have that, if φ is satisfiable for the class $\mathbf{PAM}_{\mathrm{X}}$, then $tr_1(\varphi)$ is satisfiable for the class $\mathbf{QNDAM}_{\mathrm{X}}$. By Theorem 3.1, it follows that, if φ is satisfiable for the class $\mathbf{PAM}_{\mathrm{X}}$, then $tr_1(\varphi)$ is satisfiable for the class $\mathbf{NDAM}_{\mathrm{X}}$.

Then we prove the right-to-left direction. Let $M = (W, \mathcal{D}, \mathcal{A}, \mathcal{N}, \mathcal{V})$ be a NDAM. We build the model $M' = (\Omega, \Rightarrow, \rho, \pi)$ as follows:

- $\Omega = W$,
- $\forall i \in Agt$ and $\forall w \in W$, $\Rightarrow(i,w) = \mathcal{N}(i,w)$,
- $\forall i \in Agt$ and $\forall w \in W$, $\rho(i,w) = \mathcal{A}(i,w)$,
- $\forall p \in Atm$, $\pi(p) = \mathcal{V}(p)$.

Obviously, M' is a PAM. And it is easy to verify that, for every $x \in \{\mathrm{GC}, \mathrm{BC}\}$, if M satisfies x then M' satisfies it as well.

The next step is to prove that for all $w \in W$, $(M,w) \models tr_1(\varphi)$ iff $(M',w) \models \varphi$.

The case $\varphi = p$ and the boolean cases are straightforward.

Let us consider the case $\varphi = \mathsf{A}_i\psi$. Suppose $(M, w) \models tr_1(\mathsf{A}_i\psi)$. By the semantics of NDAMs and the function tr_1, it is equivalent to $Atm(\psi) \subseteq \mathcal{A}(i, w)$. By the definition of $\rho(i, w)$, the latter is equivalent to $Atm(\psi) \subseteq \rho(i, w)$. Then by the semantics of PAMs, the latter is equivalent to $(M', w) \models \mathsf{A}_i\psi$.

Let us consider the case $\varphi = \mathsf{B}_i\psi$. Suppose $(M, w) \models tr_1(\mathsf{B}_i\psi)$. By the induction hypothesis, we have $||\psi||_{M'} = ||tr_1(\psi)||_M$. By the function tr_1, $(M, w) \models tr_1(\mathsf{B}_i\psi)$ means $(M, w) \models \Box_i tr_1(\psi)$. By the semantics of NDAM, the latter is equivalent to $\mathcal{N}(i, w) \subseteq ||tr_1(\psi)||_M$. By the definition of $\Rightarrow(i, w)$ and the fact $||\psi||_{M'} = ||tr_1(\psi)||_M$, the latter is equivalent to $\Rightarrow(i, w) \subseteq ||\psi||_{M'}$, which is equivalent to $(M', w) \models \mathsf{B}_i\psi$.

Finally, let us consider the case $\varphi = \mathsf{X}_i\psi$. Suppose $(M, w) \models tr_1(\mathsf{X}_i\psi)$. Given the fact that $\mathsf{X}_i\psi$ is equivalent to $\mathsf{B}_i\psi \wedge \mathsf{A}_i\psi$ and $\mathsf{A}_i\psi$ is equivalent to $\bigwedge_{p \in Atm(\psi)} \mathsf{A}_i p$, by the previous cases, it is equivalent to $(M', w) \models \mathsf{B}_i\psi \wedge \mathsf{A}_i\psi$, which in turn is equivalent to $(M', w) \models \mathsf{X}_i\psi$.

Thus, we conclude that $(M, w) \models tr_1(\varphi)$ iff $(M', w) \models \varphi$ for all $w \in W$. Therefore, if $tr_1(\varphi)$ is satisfiable for the class $\mathbf{NDAM}_{\mathrm{X}}$, then φ is satisfiable for the class $\mathbf{PAM}_{\mathrm{X}}$. $\square$

Theorem 5.1 shows that the translation of a LPA-formula is satisfiable if and only if the LPA-formula is satisfiable too. This highlights that $\mathcal{L}_{\mathsf{LDAA}}$ is at least as expressive as $\mathcal{L}_{\mathsf{LPA}}$. We do not know whether they have the same expressivity. What we can affirm is that the formula $\neg\triangle_i(p \wedge p) \wedge \Box_i p \wedge \bigcirc_i p$ is satisfiable in the class **NDAM**, but it cannot be satisfied in the class **PAM**, if we translate $\triangle_i$, $\Box_i$, and $\bigcirc_i$ into X_i, B_i, and A_i, respectively. Again this shows that the LPA notion of explicit belief and the LDAA notion of explicit belief capture epistemic attitudes of different nature.

6 Relationship with General Awareness

In the previous section, we have studied the relationship between our logic LDAA and the logic of propositional awareness. We have provided a polynomial embedding of the latter into the former.

As we have emphasized in the introduction, the notion of propositional awareness is generally distinguished from general awareness introduced for the first time in [7]. Unlike propositional awareness whereby an agent can only be "primitively" aware about atomic propositions and awareness about complex formulas is generated from it, in general awareness an agent can also be "primitively" aware about complex formulas. On the conceptual level, the crucial difference between propositional and general awareness is that the former captures a 'relevance-based' form of awareness, while the latter is closer to an 'accessibility-based' interpretation of the notion of awareness. On the one side, propositional awareness corresponds to all propositions that are relevant for the agent in the actual situation and which constitute the agent's actual vocabulary. From a cognitive point of view, being propositionally aware of a certain fact or proposition means imagining it, thinking about it, holding a mental representation of it. On the other side, general awareness corresponds to all facts whose truth is instantly accessible to the agent. Being generally aware of a certain fact means being able to *instantly* assess whether this fact is true, given the information at our disposal. This second conception is made explicit by Fagin & Halpern, according to whom the notion of awareness is "...open to a number of interpretations. One of them is that an agent is aware of a formula if he can compute whether or not it is true in a given situation within a certain time or space bound" [7, p. 41].[3]

In this section, we explore the connection between the logic LDAA and Fagin & Halpern's logic of general awareness. We will provide a polynomial embedding of the former into the latter and, thanks to this embedding, we will obtain a complexity result for the LDAA-satisfiability checking problem.

The logic of general awareness (LGA) has the same language as the logic of propositional awareness we introduced in Section 5, that is, $\mathcal{L}_{\mathsf{LGA}} = \mathcal{L}_{\mathsf{LPA}}$. Nonetheless, it differs at the semantic level, where formulas are interpreted with respect to the following notion of general awareness model.

Definition 6.1 *A general awareness model* (GAM) *is a tuple* $M = (\Omega, \Rightarrow, \rho, \pi)$ *where,*

- Ω *is a non-empty set of states,*
- $\Rightarrow: Agt \times \Omega \longrightarrow 2^{\Omega}$ *is a doxastic accessibility function,*
- $\rho : Agt \times \Omega \longrightarrow 2^{\mathcal{L}_{\mathsf{LGA}}}$ *is a general awareness function,*
- $\pi : Atm \longrightarrow 2^{\Omega}$ *is a valuation function.*

The class of general awareness models is denoted by $\mathbf{GAM}$.

For every $\mathrm{X} \subseteq \{\mathrm{GC}, \mathrm{BC}\}$, we note $\mathbf{GAM}_{\mathrm{X}}$ the class of general awareness models satisfying every property in X, where GC and BC for GAMs are defined in the same way as for PAMs.

The only difference between propositional awareness models and general awareness models is that in the latter the awareness function is about generic formulas while in the former it is only about atomic propositions. The interpretation of formulas in $\mathcal{L}_{\mathsf{LGA}}$ relative to a pointed GAM is exactly as in Definition 5.2 except for the awareness operators which are interpreted by the following rule:

$$(M, s) \models \mathsf{A}_i\varphi \Longleftrightarrow \varphi \in \rho(i, s).$$

In order to relate our logic LDAA to LGA, let us extend the set of atomic formulas as follows:

$$Atm^{+} = Atm \cup \{aw(i,p) : i \in Agt \text{ and } p \in Atm\},$$

[3]See [32], for more details on the different interpretations of the notion of awareness.

where $aw(i,p)$ is a special atom denoting the fact that agent i is propositionally aware of p. Such special atoms are crucial for the embedding since they allow us to "simulate" the notion of propositional awareness in the context of the general awareness framework. Note that we have made a conceptual shift when moving from LPA to LGA, since in LPA the operator A_i represents propositional awareness, while in LGA the same operator represents general awareness and the special atoms $aw(i,p)$ are used to represent propositional awareness.

We translate formulas of LDAA into formulas of LGA via the following translation function $tr_2 : \mathcal{L}_{\mathsf{LDAA}}(Atm, Agt) \longrightarrow \mathcal{L}_{\mathsf{LGA}}(Atm^+, Agt)$:

$$\begin{aligned}
tr_2(p) &= p \text{ for } p \in Atm,\\
tr_2(\neg\varphi) &= \neg tr_2(\varphi),\\
tr_2(\varphi_1 \wedge \varphi_2) &= tr_2(\varphi_1) \wedge tr_2(\varphi_2),\\
tr_2(\triangle_i \alpha) &= \mathsf{X}_i tr_2(\alpha) \wedge \bigwedge_{p \in Atm(\alpha)} aw(i,p),\\
tr_2(\Box_i \varphi) &= \mathsf{B}_i tr_2(\varphi),\\
tr_2(\bigcirc_i \varphi) &= \bigwedge_{p \in Atm(\varphi)} aw(i,p).
\end{aligned}$$

For every set of formulas $\Gamma \subseteq \mathcal{L}_{\mathsf{LDAA}}(Atm, Agt)$, we slightly abuse notation and define $tr_2(\Gamma) = \bigcup_{\varphi\in\Gamma} \big\{tr_2(\varphi)\big\}$. As the previous translation indicates, the LDAA notions of implicit belief and propositional awareness are mapped into the corresponding LGA notions. The LDAA notion of explicit belief is mapped into LGA explicit belief *plus* propositional awareness, represented by the special atoms $aw(i,p)$. In this sense, the LDAA notion of explicit belief is stronger than the LGA notion. This is due to the fact that in LDAA propositional awareness and explicit belief are logically related since the former is computed from the latter, whereas in LGA explicit belief is logically related to general awareness but is not logically related to propositional awareness. Indeed, in LGA there is no connection between the explicit belief operator X_i and the propositional awareness atoms $aw(i,p)$. A more thoughtful discussion on this conceptual difference will be given in Section 7. We have the following embedding theorem.

Theorem 6.1 *Let $\varphi \in \mathcal{L}_{\mathsf{LDAA}}(Atm, Agt)$ and $\mathrm{X} \subseteq \{\mathrm{GC}, \mathrm{BC}\}$. Then, φ is satisfiable for the class $\mathbf{NDAM}_{\mathrm{X}}$ if and only if $tr_2(\varphi)$ is satisfiable for the class $\mathbf{GAM}_{\mathrm{X}}$.*

Proof We first prove the left-to-right direction.

Suppose $M = (W, \mathcal{D}, \mathcal{A}, \mathcal{N}, \mathcal{V})$ is a NDAM and $w \in W$ such that $(M, w) \models \varphi$. We build the GAM $M' = (\Omega, \Rightarrow, \rho, \pi)$ as follows:

- $\Omega = W$,
- for every $w \in \Omega$, $\Rightarrow(i, w) = \{v \in \Omega : v \in \mathcal{N}(i, w)\}$,
- for every $w \in \Omega$, $\rho(i, w) = tr_2\big(\mathcal{D}(i, w)\big)$,

- $\pi(p) = \mathcal{V}(p)$ for each $p \in Atm$,
- $\pi\big(aw(i,p)\big) = \{w \in W : p \in \mathcal{A}(i,w)\}$ for each $i \in Agt$ and $w \in W$.

It is easy to verify that M' is a general awareness structure and that, for every $x \in \{\mathrm{GC}, \mathrm{BC}\}$, if M satisfies x then M' satisfies it as well.

By induction on the structure of φ, we are going to show that $(M,w) \models \varphi$ if and only if $(M',w) \models tr_2(\varphi)$. The atomic case, boolean cases and case $\varphi = \bigcirc_i\psi$ are straightforward and we do not need to prove them. Let us prove the case $\varphi = \triangle_i\alpha$.

($\Rightarrow$) $(M,w) \models \triangle_i\alpha$ means $\alpha \in \mathcal{D}(i,w)$ and $\forall p \in Atm(\alpha)$, $p \in \mathcal{A}(i,w)$. By definition of M', the latter implies $tr_2(\alpha) \in \rho(i,w)$ and $w \in \pi\big(aw(i,p)\big)$ for all $p \in Atm(\alpha)$. The latter is equivalent to (i) $(M',w) \models \mathsf{A}_i tr_2(\alpha)$ and $(M',w) \models aw(i,p)$ for all $p \in Atm(\alpha)$. Moreover, $(M,w) \models \triangle_i\alpha$ implies $\mathcal{N}(i,w) \subseteq ||\alpha||_M$. By induction hypothesis, we have $||\alpha||_M = ||tr_2(\alpha)||_{M'}$. Thus, by the definition of $\Rightarrow$, it follows that $\Rightarrow(i,w) \subseteq ||tr_2(\alpha)||_{M'}$. The latter means that (ii) $(M',w) \models \mathsf{B}_i tr_2(\alpha)$. The previous items (i) and (ii) together imply that $(M',w) \models \mathsf{X}_i tr_2(\alpha) \wedge \bigwedge_{p \in Atm(\alpha)} aw(i,p)$.

($\Leftarrow$) Suppose $(M',w) \models \mathsf{X}_i tr_2(\alpha) \wedge \bigwedge_{p \in Atm(\alpha)} aw(i,p)$. The latter implies that $(M',w) \models \mathsf{A}_i tr_2(\alpha)$. Hence, by definition of M', $\alpha \in \mathcal{D}(i,w)$ which is equivalent to $(M,w) \models \triangle_i\alpha$.

Finally, let us prove the case $\varphi = \Box_i\psi$. By induction hypothesis, we have $||\psi||_M = ||tr_2(\psi)||_{M'}$. $(M,w) \models \Box_i\psi$ means that $\mathcal{N}(i,w) \subseteq ||\psi||_M$. By definition of $\Rightarrow$ and $||\psi||_M = ||tr_2(\psi)||_{M'}$, the latter is equivalent to $\Rightarrow(i,w) \subseteq ||tr_2(\psi)||_{M'}$ which in turn is equivalent to $(M',w) \models \mathsf{B}_i tr_2(\psi)$.

Thus, $(M',w) \models tr_2(\varphi)$, since we supposed $(M,w) \models \varphi$.

As for the right-to-left direction, we are going to prove a weaker result: if $tr_2(\varphi)$ is satisfiable for the class $\mathbf{GAM}_{\mathrm{X}}$ then φ is satisfiable for the class $\mathbf{QNDAM}_{\mathrm{X}}$. Let $M = (\Omega, \Rightarrow, \rho, \pi)$ be a GAM and $s \in \Omega$ such that $(M,s) \models tr_2(\varphi)$.

We build the structure $M' = (W, \mathcal{D}, \mathcal{A}, \mathcal{N}, \mathcal{V})$ as follows:

- $W = \Omega$,
- for every $s \in \Omega$ and $i \in Agt$, $\mathcal{D}(i,s) = \{\alpha \in \mathcal{L}_{\mathsf{LDAA}} : \Rightarrow(i,s) \subseteq ||tr_2(\alpha)||_M$, $tr_2(\alpha) \in \rho(i,s)$ and $\forall p \in Atm, (M,s) \models aw(i,p)\}$,
- for every $s \in \Omega$ and $i \in Agt$, $\mathcal{N}(i,s) = \Rightarrow(i,s)$,
- for every $s \in \Omega$ and $i \in Agt$, $\mathcal{A}(i,s) = \big\{p \in Atm : s \in \pi\big(aw(i,p)\big)\big\}$,
- $\mathcal{V}(p) = \pi(p)$ for $p \in Atm$.

It is easy to verify that M' belongs to the class **QNDAM** and that, for every $x \in \{\mathrm{GC}, \mathrm{BC}\}$, if M satisfies x then M' satisfies it as well.

By induction on the structure of φ, we are going to show that $(M,s) \models tr_2(\varphi)$ if and only if $(M',s) \models \varphi$. The atomic case, boolean cases and case $\varphi = \bigcirc_i\psi$ are straightforward and we do not need to prove them.

Let us prove the case $\varphi = \triangle_i \alpha$. $(M, s) \models tr_2(\triangle_i \alpha)$ is equivalent to $(M, s) \models \mathsf{X}_i tr_2(\alpha) \wedge \bigwedge_{p \in Atm(\alpha)} aw(i,p)$. The latter is equivalent to $\Rightarrow(i, s) \subseteq ||tr_2(\alpha)||_M$, $tr_2(\alpha) \in \rho(i, s)$ and $\forall p \in Atm, s \in \pi\big(aw(i,p)\big)$. By definition of $\mathcal{D}$, the latter is equivalent to $\alpha \in \mathcal{D}(i, s)$ which in turn is equivalent to $(M', s) \models \triangle_i \alpha$.

Let us finally prove the case $\varphi = \Box_i \psi$. By induction hypothesis, we have $||tr_2(\psi)||_M = ||\psi||_{M'}$. $(M, s) \models tr_2(\Box_i \psi)$ means that $(M, s) \models \mathsf{B}_i tr_2(\psi)$ which in turn means that $\Rightarrow(i, s) \subseteq ||tr_2(\psi)||_M$. By definition of $\mathcal{N}(i, s)$ and $||\psi||_M = ||tr_2(\psi)||_{M'}$, the latter is equivalent to $\mathcal{N}(i, s) \subseteq ||\psi||_{M'}$ which in turn is equivalent to $(M', s) \models \Box_i \psi$.

Thus, $(M', s) \models \varphi$, since we supposed $(M, s) \models tr_2(\varphi)$.

We have proved that if $tr_2(\varphi)$ is satisfiable for the class $\mathbf{GAM}_{\mathrm{X}}$, then φ is satisfiable for the class $\mathbf{QNDAM}_{\mathrm{X}}$. From Theorem 3.1, it follows that if $tr(\varphi)$ is satisfiable for the class $\mathbf{GAM}_{\mathrm{X}}$, then φ is satisfiable for the class $\mathbf{NDAM}_{\mathrm{X}}$. $\square$

In the light of Theorems 3.1 and 6.1, we have that, for every formula $\varphi \in \mathcal{L}_{\mathsf{LDAA}}(Atm, Agt)$, φ is satisfiable for the class $\mathbf{MABA}_{\mathrm{X}}$ if and only if $tr_2(\varphi)$ is satisfiable for the class $\mathbf{GAM}_{\mathrm{X}}$.

In [1], it is proved that the satisfiability problem for the logic of general awareness is in PSPACE, even in the case in which every accessibility relation $\Rightarrow_i$ is assumed to be reflexive. The proof relies on a tableau-based PSPACE satisfiability checking procedure. Ågotnes & Alechina's tableau-based method can be easily adapted to show that the satisfiability problem of the logic LGA interpreted over general awareness structures whose accessibility relations $\Rightarrow_i$ are assumed to be serial is also in PSPACE. In the light of this observation, we get the following complexity result.

Theorem 6.2 *Let* $\mathrm{X} \subseteq \{\mathrm{GC}, \mathrm{BC}\}$. *Then, checking satisfiability of formulas in* $\mathcal{L}_{\mathsf{LDAA}}(Atm, Agt)$ *relative to the class* $\mathbf{NDAM}_{\mathrm{X}}$ *(resp.* $\mathbf{MABA}_{\mathrm{X}}$*) is a PSPACE-complete problem.*

Proof Theorem 6.1 guarantees that the translation tr_2 provides a polynomial-time reduction of the satisfiability problem for formulas in $\mathcal{L}_{\mathsf{LDAA}}$ relative to the class $\mathbf{NDAM}_{\mathrm{X}}$ (resp. $\mathbf{MABA}_{\mathrm{X}}$) to the satisfiability problem of formulas in $\mathcal{L}_{\mathsf{LGA}}$ relative to the class $\mathbf{GAM}_{\mathrm{X}}$. Since the latter problem is in PSPACE, it follows that the former problem is also in PSPACE. PSPACE-hardness follows from known PSPACE-hardness results for multimodal logic K^n, KD^n and KT^n [10]. $\square$

Figure 2: Embeddings

Figure 2 summarizes the connection between our logic LDAA and the awareness logics LPA and LGA we established in Theorems 5.1 and 6.1. It highlights that LDAA stays in between LPA and LGA. In the following section we discuss the relationship between the three frameworks and justify the two translations tr_1 and tr_2 from a conceptual perspective.

7 Discussion

In this section, we provide an intuitive explanation of the two embeddings of LPA into LDAA and of LDAA into LGA given in Sections 5 and 6. Moreover, we elucidate some important differences between the three frameworks in relating explicit belief to implicit belief and awareness.

Conceptual justification of the translations Table 1 outlines the properties of the concepts of awareness, explicit belief and implicit belief in the three frameworks. In LPA and LGA the concepts of awareness and implicit belief are primitive and the concept of explicit belief is defined from them. On the contrary, in LDAA the only primitive concept is explicit belief, while awareness and implicit belief are computed from it. Moreover, LPA and LDAA only consider propositional awareness, while LGA is conceptually richer as it can easily combine general awareness with propositional awareness, by means of the special atoms $aw(i,p)$. We recall that propositional awareness captures a 'relevance-based' form of awareness, whereas general awareness captures an 'accessibility-based' form. The former corresponds to an agent's mental vocabulary, the propositions that are relevant for him in a given situation and about which the agent is thinking. General awareness corresponds to the facts whose truth is accessible to the agent, for which the agent can instantly assess whether they are true. Being generally aware of a fact implies no cost of time and computation for inferring whether the fact is true.

		LPA	LGA	LDAA
Awareness	Primitive	yes	yes	no
	Types	relevance-based	accessibility-based (primarily), relevance-based (via special atoms $aw(i,p)$)	relevance-based
Explicit belief	Primitive	no	no	yes
	Types	relevance-based	accessibility-based	accessibility-based
	Deductively closed	locally	no	no
Implicit belief	Primitive	yes	yes	no
	Deductively closed	globally	globally	globally

Table 1: Concepts of awareness, explicit belief and implicit belief in LPA, LGA and LDAA

We have shown that (i) the LPA notions of propositional awareness and implicit belief are directly mapped into the LDAA notions of propositional awareness and implicit belief, and (ii) the LDAA notions of propositional awareness and implicit belief are directly mapped into the LGA notions of propositional awareness and implicit belief. As for explicit belief, there is no direct correspondence between LPA, LGA and LDAA as it has different meanings and properties in the three frameworks. In particular, the LPA notion of explicit belief is mapped into the combination of propositional awareness and implicit belief in LDAA, and the LDAA notion is mapped into the combination of explicit belief and propositional awareness in LGA.

LGA defines explicit belief from general awareness and implicit belief. Like LDAA, LGA views explicit belief as a belief available in the agent's mind and instantly accessible to him. In particular, according to the LGA and LDAA interpretation, having an explicit belief that α means being able to instantly ascertain that α is true. It presupposes that the agent to be modeled is resource-bounded and is not logical omniscient, since not all facts that the agent can deduce from his explicit (instantly accessible) beliefs are instantly accessible to him. Indeed, deduction and, more generally inference, takes time so that not all deducible facts are instantly accessible to the agent. Therefore, according to LGA, an agent's explicit beliefs are not necessarily closed under deduction.[4]

In LPA explicit belief is defined from propositional awareness together with implicit belief and is viewed as an agent's relevant belief, as a belief which is built from the agent's vocabulary. In LPA the agent's explicit beliefs are deductively closed within the scope of his propositional awareness (i.e., with respect to his actual vocabulary). Specifically, in LPA, all facts that are deducible from the agent's explicit beliefs and whose atoms are parts of the agent's vocabulary are explicitly believed by the agent. For instance, LPA satisfies closure under logical consequence within the scope of the agent's vocabulary: if ψ is a logical consequence of the facts $\varphi_1, \ldots, \varphi_k$, the agent explicitly believes each of these facts and the atoms that ψ does not share with these facts are all included in the agent's vocabulary, then the agent explicitly believes ψ as well. That is, the following rule of inference preserves validity in LPA (while it does not in LGA):

$$\frac{(\varphi_1 \wedge \ldots \wedge \varphi_k) \to \psi}{(\mathsf{X}_i\varphi_1 \wedge \ldots \wedge \mathsf{X}_i\varphi_k \wedge \bigwedge_{p \in Atm(\psi) \setminus Atm(\{\varphi_1, \ldots, \varphi_k\})} \mathsf{A}_i p) \to \mathsf{X}_i\psi} \tag{1}$$

This form of *local* deductive closure for explicit belief should be kept distinct from *global* deductive closure for implicit belief which is shared altogether by LPA, LGA and LDAA.

Given the conceptual similarity between LDAA-explicit belief and LGA-explicit belief, one would expect the former to directly map into the latter. But, in LDAA propositional awareness is computed from explicit belief. In other words, in LDAA relevance is grounded on accessibility: what is relevant for an agent (i.e., the agent's actual vocabulary) is computed from the beliefs that are instantly

[4]Explicit belief in the sense of LGA and LDAA can be conceived as a belief which is active in the agent's mind and is under the focus of his attention. The latter interpretation is in line with some existing cognitive theories of epistemic states [17, 20].

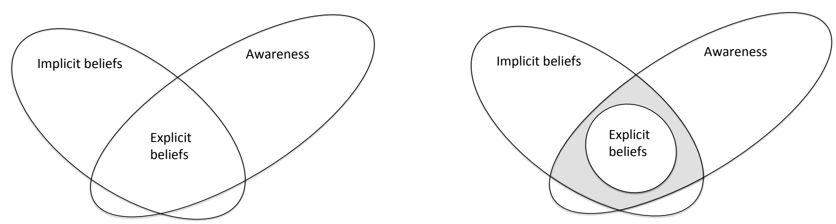
accessible to him. The latter is not the case in LGA in which explicit belief is defined from implicit belief and general awareness, but there is no connection between explicit belief and propositional awareness represented by the special atoms $aw(i,p)$. In LGA an agent can have propositions in his vocabulary that do not appear in any of his explicit beliefs. In formal terms, it might be the case that $(M,s) \models aw(i,p)$ for some pointed GAM (M,s) but $p \notin Atm(\varphi)$ for every φ such that $(M,s) \models \mathsf{X}_i\varphi$. In this sense, LDAA-explicit belief is stronger than LGA-explicit belief. This explains why in Section 6 LDAA-explicit belief is translated into LGA-explicit belief *plus* the fact that all propositions in the belief are part of the agent's propositional awareness.

The concept of explicit belief Let us explain in more detail how explicit belief relates to awareness and implicit belief in the three frameworks.

In LPA and LGA explicit belief is defined in terms of awareness and implicit belief. Therefore, the following formula is valid for the class **PAM** (resp. **GAM**):

$$\mathsf{X}_i\varphi \leftrightarrow (\mathsf{B}_i\varphi \wedge \mathsf{A}_i\varphi). \tag{2}$$

This means that explicit belief is coextensive with the intersection of implicit belief and awareness, as highlighted in Figure 3a.



(a) LPA and LGA (b) LDAA

Figure 3: Relationship between awareness, explicit and implicit belief

In LDAA only the left-to-right direction of the equivalence holds (i.e., explicit belief implies implicit belief and awareness). Specifically, the following formula is valid for the class **MABA**:

$$\triangle_i\alpha \rightarrow (\Box_i\alpha \wedge \bigcirc_i\alpha). \tag{3}$$

But the formula $(\Box_i\alpha \wedge \bigcirc_i\alpha) \rightarrow \triangle_i\alpha$ is not valid for the class **MABA**. To show the latter, in the light of Theorem 3.1, it suffices to exhibit a NDAM which invalidates the previous formula. Consider the pointed NDAM (M, w_1)

such that $M = (W, \mathcal{D}, \mathcal{A}, \mathcal{N}, \mathcal{V})$ with

$$\begin{aligned}
&W = \{w_1, w_2\},\\
&\mathcal{D}(i, w_1) = \mathcal{D}(i, w_2) = \{p_1, p_1 \to p_2\},\\
&\mathcal{N}(i, w_1) = \mathcal{N}(i, w_2) = \{w_2\},\\
&\mathcal{V}(p_1) = \{w_1, w_2\},\\
&\mathcal{V}(p_2) = \{w_2\},\\
&\mathcal{D}(j, w_k) = \emptyset \text{ for every } j \neq i \text{ and } k \in \{1, 2\},\\
&\mathcal{N}(j, w_k) = W \text{ for every } j \neq i \text{ and } k \in \{1, 2\},\\
&\mathcal{V}(q) = \emptyset \text{ for every } q \notin \{p_1, p_2\}.
\end{aligned}$$

Clearly, we have $(M, w_1) \models \Box_i p_2 \wedge \bigcirc_i p_2$ but $(M, w_1) \models \neg\triangle_i p_2$.

This means that in LDAA there are facts that are implicitly believed by an agent and of which the agent is propositionally aware, but that are not explicitly believed by him. Such facts are represented by the grey zone in Figure 3b.

This difference between LDAA and the competing frameworks LPA and LGA is due to their diverging conceptions of the relationship between explicit belief and awareness. As we have emphasized above, LPA is a theory of relevance-based explicit belief and awareness, whereas LGA is a theory of accessibility-based explicit belief and awareness. LDAA is a theory of accessibility-based explicit belief and relevance-based awareness, in which the notion of relevance for awareness is grounded on the notion of accessibility for explicit belief. Given this interpretation, in LDAA it makes perfect sense to assume that an agent can implicitly believe α and be propositionally aware of α without explictly believing α, simply because it has not yet inferred α from his explicit beliefs. For example, suppose an agent explicitly believes each premise in $\{p_1, p_2, (p_2 \wedge p_3) \to p_4, p_1 \to p_3\}$. Atom p_4 is part of the agent's vocabulary and the agent is thinking about it since it is included in one of his explicit beliefs thereby being relevant for him.[5] It is deducible from the agent's explicit beliefs through a sequence of two inference steps. For this reason the agent implicitly believes that p_4. But since the agent has not deduced it yet, he does not believe it explicitly.

Being aware requires having an explicit belief Before concluding, we would like to shed further light on the connection between awareness and explicit belief in the three frameworks. LDAA satisfies the following two properties:

(P1) if an agent is propositionally aware of an atom p,
then he has an explicit belief about some formula involving p,

(P2) if an agent has some explicit belief about the atom p,
then he is propositionally aware of p.

[5] See [3, 39] for an epistemic logic in which inference steps conceived as mental actions are explicitly modeled.

LPA satisfies them too, while LGA does not. In particular, for every MABA (S, Cxt) the following holds:

$$\begin{aligned} &\text{if } (S, Cxt) \models \bigcirc_i p \text{ then} \\ &\text{there exists } \alpha \text{ such that } p \in Atm(\alpha) \text{ and } (S, Cxt) \models \triangle_i \alpha, \end{aligned} \tag{4}$$

and for every pointed PAM (M, s), the following holds:

$$\begin{aligned} &\text{if } (M, s) \models \mathsf{A}_i p \text{ then} \\ &\text{there exists } \varphi \text{ such that } p \in Atm(\varphi) \text{ and } (M, s) \models \mathsf{X}_i \varphi. \end{aligned} \tag{5}$$

We have (5) since both $\mathsf{B}_i(p \vee \neg p)$ and $\mathsf{A}_i p \to \mathsf{A}_i(p \vee \neg p)$ are valid in the class **PAM**. Therefore, $(M, w) \models \mathsf{A}_i p$ implies $(M, s) \models \mathsf{X}_i(p \vee \neg p)$. We have (4) since in LDAA propositional awareness is computed from explicit belief. This excludes situations in which the agent thinks about a concept or a proposition without having available in his mind an explicit belief about it. More generally, in LDAA it is assumed that mere thinking and imagining are subordinate to the epistemic activity. Moreover, the following LDAA-formula is valid for the class **MABA**:

$$\triangle_i \alpha \to \bigcirc_i p \text{ if } p \in Atm(\alpha), \tag{6}$$

and the following LPA-formula is valid for the class **PAM**:

$$\mathsf{X}_i \varphi \to \mathsf{A}_i p \text{ if } p \in Atm(\varphi). \tag{7}$$

The validity (6) excludes situations in which an agent has some belief about p active in his mind and, at the same time, he does not think about p. Such situations are not possible since in LDAA having an explicit about a proposition p makes p relevant for the agent and induces the agent to think about it.

8 Dynamic Extensions

This section focuses on some extensions of the logic LDAA capturing different types of belief dynamics and awareness dynamics in a multi-agent setting.

The logics in the LDA family introduced in [23] allow us to model common ground change and private belief base change. These properties are inherited altogether by the LDAA logics naturally. For private belief base change, the LDAA setting retains the virtue of the LDA setting by offering a "parsimonious" account of private informative actions whereby duplication of the original epistemic model in the style of Gerbrandy & Groeneveld [8] is not needed. We only need to add a piece of information to an agent's belief base, while keeping the other agents' belief bases unchanged, and then recompute the agent's doxastic accessibility relation. The novelty of LDAA compared to LDA is the account of awareness dynamics. In LDAA, the latter depends on belief base dynamics, since an agent's awareness is grounded on the agent's belief base. This distiguishes the LDAA approach from existing approaches to awareness change such as [34] in which the notion of awareness is taken as a primitive and its dynamics are independent of belief dynamics.

In the rest of this section, we first investigate the extension of LDAA by public announcements. Then, we present the extension of LDAA by private belief base expansion. Complexity results for these extensions are provided.

8.1 Public Announcements

In order to represent how public announcements influence the agents' common ground, we extend the language $\mathcal{L}_{\mathsf{LDAA}}(Atm, Agt)$ by modal operators of the form $[\varphi!]$, thereby obtaining the following language $\mathcal{L}_{\mathsf{LDAA-PA}}(Atm, Agt)$:

$$\varphi \quad ::= \quad \alpha \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Box_i\varphi \mid \bigcirc_i\varphi \mid [\varphi!]\psi,$$

where i ranges over Agt, α ranges over $\mathcal{L}_0$, and $\mathsf{LDAA} - \mathsf{PA}$ stands for "Logic of Doxastic Attitudes with Awareness and Public Announcements".

For simplicity, we write $\mathcal{L}_{\mathsf{LDAA-PA}}$ instead of $\mathcal{L}_{\mathsf{LDAA-PA}}(Atm, Agt)$. The formula $[\varphi!]\psi$ is read "ψ holds after the public announcement of φ". Like in standard public announcement logic PAL [30], public announcements are assumed to be truthful, i.e., an announcement is executable if and only if the formula to be announced is true. As usual, $\langle\varphi!\rangle\psi$ is the abbreviation of $\neg[\varphi!]\neg\psi$, which means "the public announcement of φ is executable and ψ will hold after the public announcement". We generalize the function computing the atoms occurring in a formula in the expected way: $Atm([\varphi!]\psi) = Atm(\varphi) \cup Atm(\psi)$ Moreover, we add the following clause to the definition of the satisfaction relation between MABAs and formulas.

Definition 8.1 *Let (S, Cxt) be a* MABA*. Then,*

$$(S, Cxt) \models [\varphi!]\psi \quad \Longleftrightarrow \quad \textit{if } (S, Cxt) \models \varphi \textit{ then } (S, Cxt^{\varphi!}) \models \psi,$$

where $Cxt^{\varphi!} = \{S' \in Cxt : (S', Cxt) \models \varphi\}$.

The intuition is that ψ is the consequence of the public announcement of φ if and only if, if φ is true, then ψ is going to be true after restricting the agents' common ground to be the states satisfying φ.

The semantics so defined guarantees that the agents' implicit beliefs are changed, as a consequence of a public announcement, whilst their explicit beliefs and awareness remain intact. This corresponds to the concept of "implicit observation" in the sense of van Benthem & Velásquez-Quesada [34]. It is implicit because, although it removes worlds thereby affecting the agents' implicit beliefs, it does not modify their explicit beliefs in the preserved ones. This concept of implicit observation is useful for AI applications in which multiple artificial agents interact in a multi-agent system. Each agent is identified with its private belief base and share with the other agents a common body of information (the common ground). For example, two autonomous vehicles 1 and 2 can share some driving rules prescribing that a vehicle must stop whenever the traffic light is red (i.e., $red \rightarrow oughtStop$) and a vehicle is allowed to continue whenever the traffic light is green (i.e., $green \rightarrow allowContinue$). These two rules are parts of the agents' common ground and should be kept distinct from the agents' private information determined by their visibility conditions. For example, suppose agent 1 is in front of the traffic light so that he can see that it is red, whilst agent 2 is far from it and cannot see its colour. This means that agent 1 has the proposition *red* in its belief base, whilst agent 2 does not have it. Suppose a new rule is publicly announced by an external agent (e.g., the system designer) prescribing that a vehicle must slow down whenever the traffic light flashes orange (i.e., $flashOrange \rightarrow OughtSlowDown$). We can imagine the system is centralized so that agents 1 and 2 correctly perceive the external agent's message and have common knowledge of this. As a consequence, the new rule is added to the common ground thereby modifying their implicit beliefs. We can represent this scenario by the MABA (S, Cxt) with $S = (B_1, B_2, A_1, A_2, V)$ such that $B_1 = A_1 = V = \{red\}$, $B_2 = A_2 = \emptyset$ and $Cxt = \{S' \in \mathbf{S} : S' \models (red \rightarrow oughtStop) \wedge (green \rightarrow allowContinue)\}$. The result of the public announcement $(flashOrange \rightarrow OughtSlowDown)!$ is the new MABA $(S, Cxt^{(flashOrange \rightarrow OughtSlowDown)!})$ such that:

$$
\begin{aligned}
Cxt^{(flashOrange \rightarrow OughtSlowDown)!} = &\{S' \in \mathbf{S} : S' \models (red \rightarrow oughtStop) \wedge \\
&(green \rightarrow allowContinue) \wedge \\
&(flashOrange \rightarrow OughtSlowDown)\}.
\end{aligned}
$$

Note that the definition of public announcement is compatible with the class $\mathbf{MABA}_{\{\mathrm{BC}\}}$ given the following fact:

$$
\begin{aligned}
&\text{if } (S, Cxt) \models \varphi \text{ and } (S, Cxt) \in \mathbf{MABA}_{\{\mathrm{BC}\}}, \\
&\text{then } S \in Cxt^{\varphi!} \text{ and } (S', S') \in \mathcal{R}_i \text{ for every } S' \in Cxt^{\varphi!}.
\end{aligned}
$$

However, public announcements may not preserve the property of global consistency for the reason that, restricting the common ground to the φ-situations could empty an agent's set of doxastically accessible states. Consequently, the semantics for $\mathcal{L}_{\mathsf{LDAA-PA}}$ is compatible with the classes $\mathbf{MABA}$ and $\mathbf{MABA}_{\{\mathrm{BC}\}}$, but incompatible with the class $\mathbf{MABA}_{\{\mathrm{GC}\}}$. Like in PAL, we have reduction principles for the dynamic operators $[\varphi!]$.

Proposition 8.1 *The following formulas are valid relative to the class* $\mathbf{MABA}$*:*

$$
\begin{aligned}
&[\varphi!]p \leftrightarrow (\varphi \rightarrow p) \\
&[\varphi!]\neg\psi \leftrightarrow (\varphi \rightarrow \neg[\varphi!]\psi) \\
&[\varphi!](\psi_1 \wedge \psi_2) \leftrightarrow ([\varphi!]\psi_1 \wedge [\varphi!]\psi_2) \\
&[\varphi!]\Box_i\psi \leftrightarrow (\varphi \rightarrow \Box_i[\varphi!]\psi) \\
&[\varphi!]\triangle_i\alpha \leftrightarrow (\varphi \rightarrow \triangle_i\alpha) \\
&[\varphi!]\bigcirc_i\psi \leftrightarrow (\varphi \rightarrow \bigcirc_i\psi)
\end{aligned}
$$

Proof The first three validities are proved in the same way as in PAL. The proof of the fourth validity is analogous to the proof of the corresponding validity in the proof of [23, Proposition 6]. The fifth and sixth validities are clear since public announcements do not affect the agents' belief bases. □

The first four validities are standard reduction principles of PAL. The fifth and the sixth are reduction principles for explicit belief and awareness, respectively. They highlight that an agent's explicit beliefs and awareness are not affected by public announcements.

With the equivalences in Proposition 8.1, we are able to find for every formula of the language $\mathcal{L}_{\mathsf{LDAA-PA}}$ an equivalent formula of the language $\mathcal{L}_{\mathsf{LDAA}}$ by way of a mapping $red_{\mathsf{LDAA-PA}}$. The function $red_{\mathsf{LDAA-PA}}$ iteratively applies the equivalences of Proposition 8.1, so that every operator $[\varphi!]$ is moved inside the formula and finally eliminated by the first equivalence of Proposition 8.1.

Definition 8.2 *The mapping $red_{\mathsf{LDAA-PA}}$ is inductively defined as follows:*

$$
\begin{array}{lcl}
1. red_{\mathsf{LDAA-PA}}(p) & = & p \\
2. red_{\mathsf{LDAA-PA}}(\triangle_i \alpha) & = & \triangle_i red_{\mathsf{LDAA-PA}}(\alpha) \\
3. red_{\mathsf{LDAA-PA}}(\neg\varphi) & = & \neg red_{\mathsf{LDAA-PA}}(\varphi) \\
4. red_{\mathsf{LDAA-PA}}(\varphi \wedge \psi) & = & red_{\mathsf{LDAA-PA}}(\varphi) \wedge red_{\mathsf{LDAA-PA}}(\psi) \\
5. red_{\mathsf{LDAA-PA}}(\Box_i \varphi) & = & \Box_i red_{\mathsf{LDAA-PA}}(\varphi) \\
6. red_{\mathsf{LDAA-PA}}(\bigcirc_j \varphi) & = & \bigwedge_{p \in Atm(\varphi)} \bigcirc_j p \\
7. red_{\mathsf{LDAA-PA}}([\varphi!]p) & = & red_{\mathsf{LDAA-PA}}(\varphi \rightarrow p) \\
8. red_{\mathsf{LDAA-PA}}([\varphi!]\neg\psi) & = & red_{\mathsf{LDAA-PA}}(\varphi \rightarrow \neg [\varphi!]\psi) \\
9. red_{\mathsf{LDAA-PA}}\big([\varphi!](\psi_1 \wedge \psi_2)\big) & = & red_{\mathsf{LDAA-PA}}([\varphi!]\psi_1 \wedge [\varphi!]\psi_2) \\
10. red_{\mathsf{LDAA-PA}}([\varphi!]\Box_i \psi) & = & red_{\mathsf{LDAA-PA}}(\varphi \rightarrow \Box_i [\varphi!]\psi) \\
11. red_{\mathsf{LDAA-PA}}([\varphi!]\triangle_i \alpha) & = & red_{\mathsf{LDAA-PA}}(\varphi \rightarrow \triangle_i \alpha) \\
12. red_{\mathsf{LDAA-PA}}([\varphi!]\bigcirc_i \psi) & = & red_{\mathsf{LDAA-PA}}(\varphi \rightarrow \bigcirc_i \psi) \\
13. red_{\mathsf{LDAA-PA}}([\varphi!][\psi!]\chi) & = & red_{\mathsf{LDAA-PA}}\big([\varphi!] red_{\mathsf{LDAA-PA}}([\psi!]\chi)\big)
\end{array}
$$

Note that the last item in the definition of $red_{\mathsf{LDAA-PA}}$ is necessary for the case of iteration of public announcements.

Proposition 8.2 *Let $\varphi \in \mathcal{L}_{\mathsf{LDAA-PA}}$ and $\mathrm{X} \subseteq \{\mathrm{BC}\}$. Then, $\varphi \leftrightarrow red_{\mathsf{LDAA-PA}}(\varphi)$ is valid with respect to the class $\mathbf{MABA}_{\mathrm{X}}$.*

Proof The proposition is proved inductively on the structure of φ. The atomic case, the boolean cases and the cases for the operators $\triangle_i$, $\Box_i$ and $\bigcirc_i$ are evident. The case for the public announcement operator $[\varphi!]$ is provable via the valid equivalences in Proposition 8.1. □

The fact that checking satisfiability for formulas in $\mathcal{L}_{\mathsf{LDAA-PA}}$ is decidable follows from the decidability of satisfiability checking for formulas in $\mathcal{L}_{\mathsf{LDAA}}$

(Theorem 6.2) and the fact that $red_{\mathsf{LDAA-PA}}$ provides an effective procedure for reducing a formula φ in $\mathcal{L}_{\mathsf{LDAA-PA}}$ into an equivalent formula $red_{\mathsf{LDAA-PA}}(\varphi)$ in $\mathcal{L}_{\mathsf{LDAA}}$.

Theorem 8.1 *Let* $\mathrm{X} \subseteq \{\mathrm{BC}\}$. *Then, checking satisfiability of formulas in* $\mathcal{L}_{\mathsf{LDAA-PA}}$ *relative to the class* $\mathbf{MABA}_{\mathrm{X}}$ *is decidable.*

8.2 Private Belief Expansion

This section presents a second dynamic extension of the LDAA logics by private belief base expansion. Concretely, we extend the language $\mathcal{L}_{\mathsf{LDAA}}(Atm, Agt)$ by operators of the form $[+_i\alpha]$, thereby obtaining the language $\mathcal{L}_{\mathsf{LDAA-PBE}}(Atm, Agt)$ defined by the following grammar:

$$\varphi \quad ::= \quad \alpha \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Box_i\varphi \mid \bigcirc_i\varphi \mid [+_i\alpha]\,\varphi,$$

where i ranges over Agt, α ranges over $\mathcal{L}_0$, and $\mathsf{LDAA-PBE}$ stands for "Logic of Doxastic Attitudes with Awareness and Private Belief Expansion".

As in Section 2, we write $\mathcal{L}_{\mathsf{LDAA-PBE}}$ instead of $\mathcal{L}_{\mathsf{LDAA-PBE}}(Atm, Agt)$. The formula $[+_i\alpha]\,\varphi$ has to be read "φ holds after agent i has expanded his belief base with α".

Like in the extension by public announcement, we need to generalize the function computing the atoms occurring in a formula: $Atm([+_i\alpha]\varphi) = Atm(\alpha) \cup Atm(\varphi)$ Moreover, we need to provide the interpretation for the dynamic operator $[+_i\alpha]$.

Definition 8.3 *Let* (S, Cxt) *be a* MABA *with* $S = (B_1, \ldots, B_n, A_1, \ldots, A_n, V)$. *Then,*

$$(S, Cxt) \models [+_i\alpha]\varphi \iff (S^{+_i\alpha}, Cxt) \models \varphi,$$

with $S^{+_i\alpha} = (B_1^{+_i\alpha}, \ldots, B_n^{+_i\alpha}, A_1^{+_i\alpha}, \ldots, A_n^{+_i\alpha}, V^{+_i\alpha})$, *where*

$$V^{+_i\alpha} = V,$$

and for all $j \in Agt$*:*

$$\begin{aligned} B_j^{+_i\alpha} &= B_j \cup \{\alpha\} && \textit{if } i = j, \\ B_j^{+_i\alpha} &= B_j && \textit{otherwise}, \\ A_j^{+_i\alpha} &= Atm(B_j^{+_i\alpha}). \end{aligned}$$

The informative event $+_i\alpha$ just consists in agent i privately learning that α by adding α to his belief base, while the other agents' belief bases are not changed. Note that agent i's awareness set is indirectly affected by the informative event, since agent i's awareness is computed from his new belief base.

Unlike public announcement, private belief base expansion does not necessarily preserve belief correctness (BC) or global consistency (GC) of the original belief base. Therefore, in the rest of this section we focus on the most general model class **MABA**.

As for public announcement, we have reduction principles for the dynamic operators $[+_i\alpha]$. They are listed in the following proposition. (We omit the proof since it is entirely standard.)

Proposition 8.3 *The following formulas are valid relative to the class* **MABA***:*

$$
\begin{array}{ll}
[+_i\alpha]p \leftrightarrow p & \\
[+_i\alpha]\neg\varphi \leftrightarrow \neg[+_i\alpha]\varphi & \\
[+_i\alpha](\varphi_1 \wedge \varphi_2) \leftrightarrow ([+_i\alpha]\varphi_1 \wedge [+_i\alpha]\varphi_2) & \\
[+_i\alpha]\Box_j\varphi \leftrightarrow \Box_j\varphi & \textit{if } i \neq j \\
[+_i\alpha]\Box_i\varphi \leftrightarrow \Box_i(\alpha \rightarrow \varphi) & \\
[+_i\alpha]\triangle_j\beta \leftrightarrow \triangle_j\beta & \textit{if } i \neq j \textit{ or } \alpha \neq \beta \\
[+_i\alpha]\triangle_i\alpha \leftrightarrow \top & \\
[+_i\alpha]\bigcirc_j\varphi \leftrightarrow \bigcirc_j\varphi & \textit{if } i \neq j \\
[+_i\alpha]\bigcirc_i\varphi \leftrightarrow \bigwedge_{p \in Atm(\varphi)\setminus Atm(\alpha)} \bigcirc_i p &
\end{array}
$$

It is worth to pay attention to the last two reduction principles for awareness: by expanding his belief base with the input formula α, agent i simply adds to his awareness set the atomic propositions occurring in α. As indicated by the fourth reduction principle, agent i will be able to deduce that φ after expanding his belief base with α if, before the belief expansion, agent i is able to deduce that α implies φ. The sixth and seventh reduction principles just state that, by privately expanding his belief base with α, agent i adds α to his belief base, while all other agents keep their belief bases unchanged.

Analogously to the extension with public announcement, the valid equivalences in the previous proposition allow us to transform every formula of the language $\mathcal{L}_{\mathsf{LDAA-PBE}}$ into an equivalent formula of $\mathcal{L}_{\mathsf{LDAA}}$ with no occurrence of dynamic operators. The transformation is obtained via the following mapping $red_{\mathsf{LDAA-PBE}}$.

Definition 8.4 *The mapping $red_{\mathsf{LDAA-PBE}}$ is inductively defined as follows:*

$$
\begin{array}{lcll}
1.red_{\mathsf{LDAA-PBE}}(p) & = & p & \\
2.red_{\mathsf{LDAA-PBE}}(\triangle_j \alpha) & = & \triangle_j red_{\mathsf{LDAA-PBE}}(\alpha) & \\
3.red_{\mathsf{LDAA-PBE}}(\neg\varphi) & = & \neg red_{\mathsf{LDAA-PBE}}(\varphi) & \\
4.red_{\mathsf{LDAA-PBE}}(\varphi \wedge \psi) & = & red_{\mathsf{LDAA-PBE}}(\varphi) \wedge red_{\mathsf{LDAA-PBE}}(\psi) & \\
5.red_{\mathsf{LDAA-PBE}}(\Box_j \varphi) & = & \Box_j red_{\mathsf{LDAA-PBE}}(\varphi) & \\
6.red_{\mathsf{LDAA-PBE}}(\bigcirc_j \varphi) & = & \bigwedge_{p \in Atm(\varphi)} \bigcirc_j p & \\
7.red_{\mathsf{LDAA-PBE}}([+_i\alpha]p) & = & red_{\mathsf{LDAA-PBE}}(p) & \\
8.red_{\mathsf{LDAA-PBE}}([+_i\alpha]\neg\psi) & = & red_{\mathsf{LDAA-PBE}}(\neg[+_i\alpha]\psi) & \\
9.red_{\mathsf{LDAA-PBE}}\big([+_i\alpha](\psi_1 \wedge \psi_2)\big) & = & red_{\mathsf{LDAA-PBE}}([+_i\alpha]\psi_1 \wedge [+_i\alpha]\psi_2) & \\
10.red_{\mathsf{LDAA-PBE}}([+_i\alpha]\Box_j\varphi) & = & red_{\mathsf{LDAA-PBE}}(\Box_j\varphi) & \textit{if } i \neq j \\
\quad red_{\mathsf{LDAA-PBE}}([+_i\alpha]\Box_i\varphi) & = & red_{\mathsf{LDAA-PBE}}\big(\Box_i(\alpha \to \varphi)\big) & \\
11.red_{\mathsf{LDAA-PBE}}([+_i\alpha]\triangle_j\beta) & = & red_{\mathsf{LDAA-PBE}}(\triangle_j\beta) & \textit{if } i \neq j \textit{ or } \alpha \neq \beta \\
\quad red_{\mathsf{LDAA-PBE}}([+_i\alpha]\triangle_i\alpha) & = & \top & \\
12.red_{\mathsf{LDAA-PBE}}([+_i\alpha]\bigcirc_j\varphi) & = & red_{\mathsf{LDAA-PBE}}(\bigcirc_j\varphi) & \textit{if } i \neq j \\
\quad red_{\mathsf{LDAA-PBE}}([+_i\alpha]\bigcirc_i\varphi) & = & red_{\mathsf{LDAA-PBE}}\left(\bigwedge_{p \in Atm(\varphi)\setminus Atm(\alpha)} \bigcirc_i p\right) & \\
 & & & \textit{otherwise} \\
13.red_{\mathsf{LDAA-PBE}}([+_i\alpha][+_j\beta]\varphi) & = & red_{\mathsf{LDAA-PBE}}\big([+_i\alpha]red_{\mathsf{LDAA-PBE}}([+_j\beta]\varphi)\big) &
\end{array}
$$

The following proposition is proved by induction on the structure of φ in a way similar to Proposition 8.2.

Proposition 8.4 *Let $\varphi \in \mathcal{L}_{\mathsf{LDAA-PBE}}$. Then, $\varphi \leftrightarrow red_{\mathsf{LDAA-PBE}}(\varphi)$ is valid with respect to the class* **MABA***.*

The fact that checking satisfiability for formulas in $\mathcal{L}_{\mathsf{LDAA-PBE}}$ is in PSPACE follows from (i) the fact that satisfiability checking for formulas in $\mathcal{L}_{\mathsf{LDAA}}$ is in PSPACE (Theorem 6.2) and (ii) the fact that $red_{\mathsf{LDAA-PBE}}$ provides an effective procedure for reducing a formula φ in $\mathcal{L}_{\mathsf{LDAA-PBE}}$ into an equivalent formula $red_{\mathsf{LDAA-PBE}}(\varphi)$ in $\mathcal{L}_{\mathsf{LDAA}}$ whose size is polynomial in the size of φ. PSPACE-hardness follows from the fact that the language $\mathcal{L}_{\mathsf{LDAA-PBE}}$ is a conservative extension of $\mathcal{L}_{\mathsf{LDAA}}$ (i.e., every valid formula of the latter is also a valid formula of the former).

Theorem 8.2 *Checking satisfiability of formulas in $\mathcal{L}_{\mathsf{LDAA-PBE}}$ relative to the class* **MABA** *is PSPACE-complete.*

8.3 Discussion

In this section, we are going to focus on some conceptual aspects of the logic $\mathsf{LDAA} - \mathsf{PBE}$, namely, how becoming aware of a proposition requires expanding the belief base with a formula about this proposition and how to represent "explicit" public announcement in opposition to "implicit" public announcement, as defined in Section 8.1.

Becoming aware requires acquiring an explicit belief $\mathsf{LDAA} - \mathsf{PBE}$ captures the subtle connection between awareness and explicit belief from a dynamic perspective. This is the main novelty compared to the extension of the logic LDA by private belief expansion presented in [23] which does not contemplate awareness.

The following two properties are the dynamic counterparts of the static properties P1 and P2 discussed in Section 7:

(P3) an agent cannot become propositionally aware of an atom p,
unless he expands his belief base with a formula involving p,

(P4) if an agent expands his belief base with some information about the atom p,
then he becomes propositionally aware of p.

$\mathsf{LDAA} - \mathsf{PBE}$ satisfies both properties. Indeed, the following two formulas are valid for the class **MABA**:

$$\neg\big(\neg\bigcirc_i p \wedge [+_i\alpha]\bigcirc_i p\big) \text{ if } p \notin Atm(\alpha), \tag{8}$$

$$[+_i\alpha]\bigcirc_i p \text{ if } p \in Atm(\alpha). \tag{9}$$

The previous two properties P3 and P4 are justified on the same grounds as properties P1 and P2. Since in $\mathsf{LDAA} - \mathsf{PBE}$, an agent's propositional awareness is computed from his belief base, there is no way to become propositionally aware of an atom p without adding to the belief base a new piece of information about p. Conversely, adding to the belief base a new piece of information about p makes the agent propositionally aware of p.

It is worth noting that such dynamic properties are not satisfied by LGA. Indeed, as highlighted in [34], in LGA awareness dynamics and explicit belief dynamics are not necessarily related: an agent can become aware of a formula without changing his explicit beliefs.[6]

k-level "explicit" public announcement As we emphasized in Section 8.1, public announcement is analogous to the notion of "implicit observation" studied

[6] van Benthem & Velásquez-Quesada [34] defines the "consider" operation which consists in extending an agent's awareness set with a new formula. This operation does not necessarily modify the agent's explicit beliefs, since in LGA explicit belief is defined from awareness and implicit belief. Therefore, explicit belief change may require that awareness change and implicit belief change are synchronously executed.

in [34]: an implicit observation only modifies the agents' common ground, and consequently their implicit beliefs, without affecting their explicit beliefs and awareness. In $\mathsf{LDAA}-\mathsf{PBE}$ we can represent a notion of k-level "explicit" public announcement that directly operates on the agents' belief bases and generates common belief up to a certain finite level k of both explicit and implicit type.

In order to define k-level "explicit" public announcement in $\mathsf{LDAA}-\mathsf{PBE}$, we first need to define explicit shared belief, denoted by the symbol ESB:

$$\mathsf{ESB}\ \alpha \stackrel{\mathtt{def}}{=} \bigwedge_{i \in Agt} \triangle_i \alpha.$$

Then, we need to define explicit mutual belief in an inductive way, for $k \geq 0$:

$$\mathsf{EMB}^0 \alpha \stackrel{\mathtt{def}}{=} \alpha,$$
$$\mathsf{EMB}^{k+1} \alpha \stackrel{\mathtt{def}}{=} \mathsf{ESB}\ \mathsf{EMB}^k \alpha.$$

From explicit mutual belief we can define k-level explicit common belief for $k \geq 0$, denoted by the symbol ECB^k, as the conjunction of all mutual beliefs between level 1 and level $k+1$:

$$\mathsf{ECB}^k \alpha \stackrel{\mathtt{def}}{=} \bigwedge_{1 \leq h \leq k+1} \mathsf{EMB}^h \alpha.$$

Notice that 0-level explicit common belief (i.e., $\mathsf{ECB}^0 \alpha$) is the same as explicit shared belief (i.e., $\mathsf{ESB}\ \alpha$).

We have everything we need to define the k-level "explicit" public announcement operator, denoted by the symbol $[!!^k \alpha]$:

$$[!!^k \alpha]\varphi \stackrel{\mathtt{def}}{=} [+_{\sigma(1)} \mathsf{EMB}^0 \alpha] \ldots [+_{\sigma(n)} \mathsf{EMB}^0 \alpha] \ldots [+_{\sigma(1)} \mathsf{EMB}^k \alpha] \ldots [+_{\sigma(n)} \mathsf{EMB}^k \alpha]\varphi,$$

where σ is any permutation of the set of agents $Agt = \{1, \ldots, n\}$, by virtue of the fact that in $\mathsf{LDAA}-\mathsf{PBE}$ the order of execution of some private belief expansion operations does not matter. The latter is a consequence of the fact that a single private belief expansion operation merely consists in adding a piece of information to an agent's belief base, while keeping the other agents' belief bases unchanged. Therefore, its position in a sequence of private belief expansion operations is irrelevant.[7] In other words, k-level explicit public announcement of α just consists in all agents expanding their belief bases with the information that α is mutual belief, for every level between 0 and k.

[7] Note that the order of execution would have been relevant, if we assumed that private belief expansion operations have executability preconditions. For example, suppose agent 1 can learn that p (i.e., $+_1 p$ is executable) if and only if agent 2 already believes that p (i.e., $\triangle_2 p$ is true), while agent 2 can learn that p (i.e., $+_2 p$ is executable) if and only if p is actually true. Therefore, if p is indeed true and none of them explicitly believes that p, it is possible for 1 to learn that p after 2 has learnt that p. On the contrary, it is not possible for 1 to learn that p before 2 learns that p.

It is easy to verify that k-level explicit public announcement of α generates k-level explicit common belief of α, k-level implicit common belief of α and $(k-1)$-level implicit common belief about shared awareness of α. Indeed, the following three formulas are valid in $\mathsf{LDAA}-\mathsf{PBE}$:

$$[!!^k\alpha]\mathsf{ECB}^k\alpha,$$
$$[!!^k\alpha]\mathsf{ICB}^k\alpha,$$
$$[!!^k\alpha]\mathsf{ICB}^{k-1}\bigwedge_{i\in Agt}\bigcirc_i\alpha \text{ if } k>0,$$

where the k-level implicit common belief operator ICB^k is defined as follows:

$$\mathsf{ICB}^k\alpha \stackrel{\mathtt{def}}{=} \bigwedge_{1\leq h\leq k+1}\mathsf{IMB}^h\alpha,$$

after having defined implicit mutual belief (i.e., IMB^k) from implicit shared belief (i.e., ISB) in the following way:

$$\mathsf{ISB}\ \alpha \stackrel{\mathtt{def}}{=} \bigwedge_{i\in Agt}\Box_i\alpha,$$
$$\mathsf{IMB}^0\alpha \stackrel{\mathtt{def}}{=} \alpha,$$
$$\mathsf{IMB}^{k+1}\alpha \stackrel{\mathtt{def}}{=} \mathsf{ISB}\ \mathsf{IMB}^k\alpha.$$

The $\mathsf{LDAA}-\mathsf{PBE}$ framework cannot handle unbounded explicit public announcement since its underlying language is finitary and defining it as an abbrevation would require an infinitary language. We leave for future investigation an extension of the $\mathsf{LDAA}-\mathsf{PBE}$ framework by a *primitive* notion of unbounded explicit public announcement whose effect is to expand the agents' belief bases with the information that α is mutual belief, for any level $k\geq 0$.

9 Conclusion and perspectives

We have provided a novel investigation of propositional awareness and of its relationship with explicit and implicit belief. In our approach, explicit belief is the only primitive concept, and awareness and implicit belief are grounded on it. Specifically, an agent's awareness set and set of doxastic alternatives are directly computed from the agent's belief base. The main results of the paper are an axiomatics for our logic of awareness, explicit and implicit belief, a polynomial embedding of the logic of propositional awareness into our logic, and a polynomial embedding of our logic into the logic of general awareness. By the latter embedding, we obtained a complexity result for our logic, namely, that its satisfiability checking problem is PSPACE-complete, the same complexity as standard multimodal logic K^n. We also investigated some dynamic aspects of our logical setting, by extending it with the notions of public announcement and private belief expansion. We have provided complexity results for these extensions. Directions of future research are manifold. We would like to briefly discuss some of them before concluding.

Introspection Future work will be devoted to explore more properties of awareness typically discussed in the literature, such as awareness/unawareness introspection. A simple way to obtain introspection over epistemic attitudes consists in strengthening the epistemic accessibility relation between two states $S = (B_1, \dots, B_n, A_1, \dots, A_n, V)$ and $S' = (B'_1, \dots, B'_n, A'_1, \dots, A'_n, V')$ as follows:

$$\begin{aligned}(S, S') \in \mathcal{R}_i \text{ if and only if } & (i) \forall \alpha \in B_i, S' \models \alpha, \text{ and} \\ & (ii) B_i = B'_i.\end{aligned}$$

Item (i) corresponds to the condition given in Definition 3.4, while item (ii) requires that a state S' is considered possible by an agent at a state S only if the agent has the same belief base in the two states. In other words, the two states are epistemically equivalent for agent i. This variant of the epistemic accessibility relation which is used for interpreting the implicit belief operator $\Box_i$ makes the following six formulas valid:

$$\triangle_i \alpha \to \Box_i \triangle_i \alpha, \tag{10}$$

$$\neg \triangle_i \alpha \to \Box_i \neg \triangle_i \alpha, \tag{11}$$

$$\bigcirc_i \varphi \to \Box_i \bigcirc_i \varphi, \tag{12}$$

$$\neg \bigcirc_i \varphi \to \Box_i \neg \bigcirc_i \varphi, \tag{13}$$

$$\Box_i \varphi \to \Box_i \Box_i \varphi, \tag{14}$$

$$\neg \Box_i \varphi \to \Box_i \neg \Box_i \varphi. \tag{15}$$

They capture, respectively, positive and negative introspection over explicit belief, awareness and implicit belief. We plan to study the axiomatic properties of this variant of the logic LDAA with introspection over epistemic attitudes.

Awareness of agents We also plan to extend our logical setting by the notion of *awareness of agents* and to compare it with the notion defined in [35, 36, 37]. Like propositional awareness, awareness of agents will be computed from belief bases. Formally, let $\bigcirc_{i,j}$ be a modal operator indicating that agent i is aware of agent j. We interpret it as follows with the help of our belief base semantics:

$$\begin{aligned}(S, Cxt) \models \bigcirc_{i,j} \iff & \text{there exists } \alpha \in \mathcal{L}_0 \text{ such that } j \text{ appears in } \alpha \\ & \text{and } (S, Cxt) \models \triangle_i \alpha.\end{aligned}$$

This means agent i is aware of agent j if agent i has a formula in his belief base mentioning agent j. We conjecture that the following principle together with the principles of the logic LDAA is sufficient to completely axiomatize this extension by the notion of agent awareness:

$$\triangle_i \alpha \to \bigcirc_{i,j} \text{ if } j \text{ appears in } \alpha.$$

Bisimulation Last but not least, we plan to define a notion of bisimulation for our belief base semantics and to compare it with the notion of awareness bisimulation for the logic of propositional awareness defined in [38].

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