Identities among relations for higher-dimensional rewriting systems

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Abstract – *We generalize the notion of identities among relations, well known for presentations of groups, to presentations of n-categories by polygraphs. To each polygraph, we associate a track n-category, generalizing the notion of crossed module for groups, in order to define the natural system of identities among relations. We relate the facts that this natural system is finitely generated and that the polygraph has finite derivation type.*
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INTRODUCTION

The notion of *identity among relations* originates in the work of Peiffer and Reidemeister, in combinatorial group theory [14, 17]. It is based on the notion of *crossed module*, introduced by Whitehead, in algebraic topology, for the classification of homotopy 2-types [20, 21]. Crossed modules have also been defined for other algebraic structures than groups, such as commutative algebras [16], Lie algebras [11] or categories [15]. Then Baues has introduced *track 2-categories*, which are categories enriched in groupoids, as a model of homotopy 2-type [2, 1], together with *linear track extensions*, as generalizations of crossed modules [4].

There exist several interpretations of identities among relations for presentations of groups: as homological 2-syzygies [5], as homotopical 2-syzygies [12] or as Igusa's pictures [12, 10]. One can also interpret identities among relations as the critical pairs of a group presentation by a convergent word rewriting system [7]. This point of view yields an algorithm based on Knuth-Bendix's completion procedure that computes a family of generators of the module of identities among relations [9].

In this work, we define the notion of identities among relations for n-categories presented by higher-dimensional rewriting systems called *polygraphs* [6], using notions introduced in [8]. Given an n-polygraph Σ, we consider the free *track n-category* $\Sigma^{\top}$ generated by Σ, that is, the free $(n-1)$-category enriched in groupoid on Σ. We define *identities among relations for* Σ as the elements of a *natural system* $\Pi(\Sigma)$ on the n-category $\overline{\Sigma}$ it presents. For that, we extend a result proved by Baues and Jibladze [3] for the case $n = 2$.

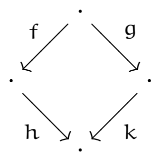
Theorem 2.2.3. *A track n-category $\mathcal{T}$ is abelian if and only if there exists a unique (up to isomorphism) natural system $\Pi(\mathcal{T})$ on $\overline{\mathcal{T}}$ such that $\widehat{\Pi(\mathcal{T})}$ is isomorphic to* $\mathrm{Aut}^{\mathcal{T}}$.

We define $\Pi(\Sigma)$ as the natural system associated by that result to the abelianized track n-category $\Sigma^{\top}_{\mathrm{ab}}$. In Section 2.2, we give an explicit description of the natural system $\Pi(\Sigma)$.

In Section 2.3, we interpret generators of $\Pi(\Sigma)$ as elements of a *homotopy basis* of the track n-category $\Sigma^{\top}$, see [8]. More precisely, we prove:

Theorem 2.3.7. *If an n-polygraph Σ has finite derivation type then the natural system $\Pi(\Sigma)$ is finitely generated.*

From this result, we deduce a way to compute generators of $\Pi(\Sigma)$ from the critical pairs of a convergent polygraph Σ. Indeed, there exists, for every critical pair (f, g) of Σ, a confluence diagram:



An $(n+1)$-cell filling such a diagram is called a *generating confluence* of Σ. It is proved in [8] that the generating confluences of Σ form a homotopy basis of $\Sigma^{\top}$. We show here that they also form a generating set for the natural system $\Pi(\Sigma)$ of identities among relations.

1. Preliminaries

In this section we recall several notions from [8]: presentations of n-categories by polygraphs (1.1), rewriting properties of polygraphs (1.2), track n-categories and homotopy bases (1.3).

1.1. Higher-dimensional categories and polygraphs

We fix an n-category $\mathcal{C}$ throughout this section.

1.1.1. Notations. We denote by $\mathcal{C}_k$ the set (and the k-category) of k-cells of $\mathcal{C}$. If f is in $\mathcal{C}_k$, then $s_i(f)$ and $t_i(f)$ respectively denote the i-source and i-target of f; we drop the suffix i when $i = k-1$. The source and target maps satisfy the *globular relations*:

$$s_i \circ s_{i+1} \;=\; s_i \circ t_{i+1} \qquad \text{and} \qquad t_i \circ s_{i+1} \;=\; t_i \circ t_{i+1}. \tag{1}$$

If f and g are i-composable k-cells, that is when $t_i(f) = s_i(g)$, we denote by $f \star_i g$ their i-composite k-cell. The compositions satisfy the *exchange relations* given, for every $i \neq j$ and every possible cells f, g, h and k, by:

$$(f \star_i g) \star_j (h \star_i k) \;=\; (f \star_j h) \star_i (g \star_j k). \tag{2}$$

If f is a k-cell, we denote by 1_f its identity $(k+1)$-cell and, by abuse, all the higher-dimensional identity cells it generates. When 1_f is composed with cells of dimension $k+1$ or higher, we simply denote it by f. A k-cell f with $s(f) = t(f) = u$ is called a *closed* k-cell with *base point* u.

1.1.2. Spheres. Let $\mathcal{C}$ be an n-category and let $k \in \{0, \dots, n\}$. A k-*sphere of* $\mathcal{C}$ is a pair $\gamma = (f, g)$ of parallel k-cells of $\mathcal{C}$, that is, with $s(f) = s(g)$ and $t(f) = t(g)$; we call f the *source* of γ and g its *target*. We denote by $\mathbf{S}_k\mathcal{C}$ (resp. $\mathbf{S}\mathcal{C}$) the set of k-spheres (resp. n-spheres) of $\mathcal{C}$. An n-category is *aspherical* when all of its n-spheres have shape (f, f).

1.1.3. Cellular extensions. A *cellular extension of* $\mathcal{C}$ is a pair $\Gamma = (\Gamma_{n+1}, \partial)$ made of a set Γ_{n+1} and a map $\partial : \Gamma_{n+1} \to \mathbf{S}\mathcal{C}$. By considering all the formal compositions of elements of Γ, seen as $(n+1)$-cells with source and target in $\mathcal{C}$, one builds the *free* $(n+1)$*-category generated by* Γ, denoted by $\mathcal{C}[\Gamma]$.

The *quotient of* $\mathcal{C}$ *by* Γ, denoted by $\mathcal{C}/\Gamma$, is the n-category one gets from $\mathcal{C}$ by identification of n-cells $s(\gamma)$ and $t(\gamma)$, for every n-sphere γ of Γ. We usually denote by $\overline{f}$ the equivalence class of an n-cell f of $\mathcal{C}$ in $\mathcal{C}/\Gamma$. We write $f \equiv_\Gamma g$ when $\overline{f} = \overline{g}$ holds.

1.1.4. Polygraphs. We define n-polygraphs and the free n-category generated by an n-polygraph by induction on n. A 1*-polygraph* is a graph, with the usual notion of free category.

An n*-polygraph* is a pair $\Sigma = (\Sigma_n, \Sigma_{n+1})$ made of an n-polygraph Σ_n and a cellular extension Σ_{n+1} of the free n-category generated by Σ_n. The *free* $(n+1)$*-category generated by* Σ and the n*-category presented by* Σ are respectively denoted by Σ^* and $\overline{\Sigma}$ and defined by:

$$\Sigma^* \;=\; \Sigma_n^*[\Sigma_{n+1}] \qquad \text{and} \qquad \overline{\Sigma} \;=\; \Sigma_n^*/\Sigma_{n+1}.$$

An n-polygraph Σ is *finite* when each set Σ_k is finite, $0 \leq k \leq n$. Two n-polygraphs whose presented $(n-1)$-categories are isomorphic are *Tietze-equivalent*. A property on n-polygraphs that is preserved up to Tietze-equivalence is *Tietze-invariant*.

An n-category $\mathcal{C}$ is *presented* by an $(n+1)$-polygraph Σ when it is isomorphic to $\overline{\Sigma}$. It is *finitely generated* when it is presented by an $(n+1)$-polygraph Σ whose underlying n-polygraph Σ_n is finite. It is *finitely presented* when it is presented by a finite $(n+1)$-polygraph.

1.1.5. Example. Let us consider the monoid $\mathbf{As} = \{a_0, a_1\}$ with a_0 being the unit and with product given by $a_1 a_1 = a_1$. We see $\mathbf{As}$ as a 1-category with one 0-cell a_0 and one non-identity 1-cell $a_1 : a_0 \to a_0$. This monoid is presented by the 2-polygraph Σ_2 with one 0-cell a_0, one 1-cell $a_1 : a_0 \to a_0$ and one 2-cell $a_2 : a_1 a_1 \Rightarrow a_1$, where we write $a_1 a_1$ for $a_1 \star_0 a_1$. Thus $\mathbf{As}$ is finitely generated and presented. In what follows, we use graphical notations for those cells, where the 1-cell a_1 is pictured as a vertical "string" | and the 2-cell a_2 as  .

1.1.6. Contexts and whiskers. A *context of* $\mathcal{C}$ is a pair (x, C) made of an $(n-1)$-sphere x of $\mathcal{C}$ and an n-cell C in $\mathcal{C}[x]$ such that C contains exactly one occurence of x. We simply denote by C such a context. If f is an n-cell which is parallel to x, then $C[f]$ is the n-cell of $\mathcal{C}$ one gets by replacing x by f in C.

Every context C of $\mathcal{C}$ has a decomposition

$$C \;=\; f_n \star_{n-1} (f_{n-1} \star_{n-2} \cdots (f_1 \star_0 x \star_0 g_1) \cdots \star_{n-2} g_{n-1}) \star_{n-1} g_n,$$

where, for every k in $\{1, \ldots, n\}$, f_k and g_k are n-cells of $\mathcal{C}$. Moreover, one can choose those cells so that f_k and g_k are k-cells. A *whisker of* $\mathcal{C}$ is a context that admits such a decomposition with f_n and g_n being identities.

If Γ is a cellular extension of $\mathcal{C}$, then every $(n+1)$-cell f of $\mathcal{C}[\Gamma]$ has a decomposition

$$f \;=\; C_1[\varphi_1] \star_n \cdots \star_n C_k[\varphi_k],$$

where, for every i in $\{1, \ldots, k\}$, φ_i is in Γ and C_i is a context of $\mathcal{C}$.

The *category of contexts of* $\mathcal{C}$ is denoted by $\mathbf{C}\mathcal{C}$, its objects are the n-cells of $\mathcal{C}$ and its morphisms from f to g are the contexts C of $\mathcal{C}$ such that $C[f] = g$ holds. We denote by $\mathbf{W}\mathcal{C}$ the subcategory of $\mathbf{C}\mathcal{C}$ with the same objects and with whiskers as morphisms.

1.1.7. Natural systems of abelian groups. A functor D from $\mathbf{C}\mathcal{C}$ to the category $\mathbf{Ab}$ of abelian groups is called a *natural system (of abelian groups) on* $\mathcal{C}$. We denote by D_u and $\mathrm{D}(\mathrm{C})$ the images of an n-cell u and of a context C of $\mathcal{C}$ by the functor D. When no confusion may occur, we write $\mathrm{C}[a]$ instead of $\mathrm{D}(\mathrm{C})(a)$. The category of natural systems on $\mathcal{C}$ is denoted by $\mathsf{Nat}(\mathcal{C})$.

1.2. Rewriting properties of polygraphs

We fix an $(n+1)$-polygraph Σ throughout this section.

1.2.1. Termination. One says that an n-cell u of Σ_n^* *reduces* into an n-cell v when Σ^* contains a non-identity $(n+1)$-cell with source u and target v. One says that u is a *normal form* when it does not reduce into an n-cell. A *normal form of* u is an n-cell v which is a normal form and such that u reduces into v. A *reduction sequence* is a countable family $(u_n)_{n\in I}$ of n-cells such that each u_n reduces into u_{n+1}; it is *finite* or *infinite* when the indexing set I is.

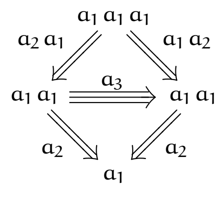
One says that Σ *terminates* when it does not generate any infinite reduction sequence. In that case, every n-cell has at least one normal form and one can use *Noetherian induction*: one can prove properties on n-cells by induction on the length of reduction sequences.

1.2.2. Confluence. A *branching* (resp. *confluence*) is a pair (f, g) of $(n+1)$-cells of Σ^* with same source (resp. target). A branching (f, g) is *local* when f and g contain exactly one generating $(n+1)$-cell of Σ. It is *confluent* when there exists a confluence (f', g') with $t(f) = s(f')$ and $t(g) = s(g')$. A local branching (f, g) is *critical* when the common source of f and g is a minimal overlapping of the sources of the $(n+1)$-cells contained in f and g. A *confluence diagram* of a branching (f, g) is an $(n+1)$-sphere with shape $(f \star_n f', g \star_n g')$, where (f', g') is a confluence. A confluence diagram of a critical branching is called a *generating confluence of* Σ.

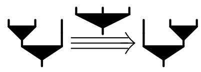
One says that Σ is *(locally) confluent* when each of its (local) branchings is confluent. A local branching (f, g) is *critical* when the common source of f and g is a minimal overlapping of the sources of the generating $(n+1)$-cells of f and g. In a confluent $(n+1)$-polygraph, every n-cell has at most one normal form. For terminating $(n+1)$-polygraphs, Newman's lemma ensures that local confluence and confluence are equivalent properties [13].

1.2.3. Convergence. One says that Σ is *convergent* when it terminates and it is confluent. In that case, every 1-cell u has a unique normal form, denoted by $\hat{u}$. Moreover, we have $u \equiv_{\Sigma_{n+1}} v$ if and only if $\hat{u} = \hat{v}$. As a consequence, a finite and convergent $(n+1)$-polygraph yields a representation of the n-cells of the category it presents, together with a decision procedure for the corresponding word problem.

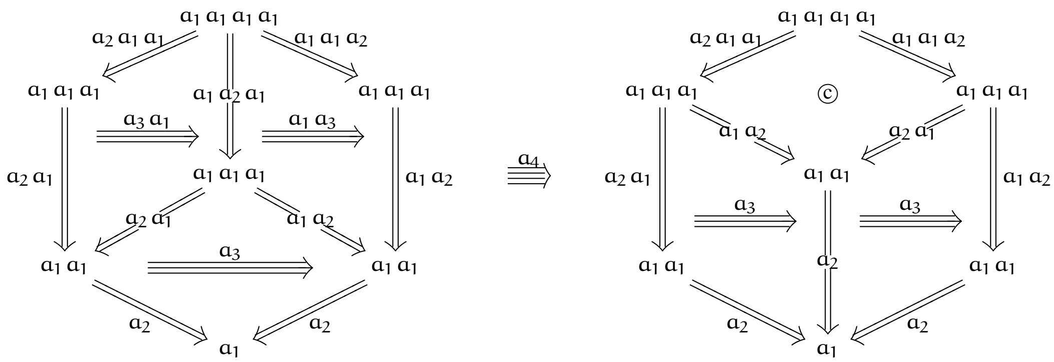
1.2.4. Example. The 2-polygraph $\Sigma_2 = (a_0, a_1, a_2)$ presenting $\mathbf{As}$ is convergent and has exactly one critical pair, with corresponding generating confluence a_3:



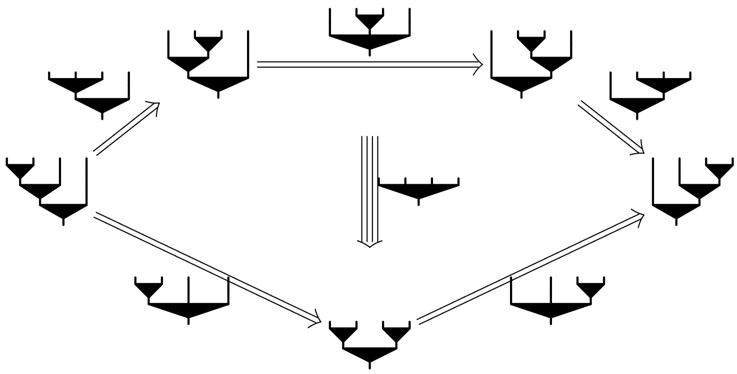
Alternatively, this 3-cell a_3 can be pictured as follows:



In turn, the 3-polygraph $\Sigma_3 = (a_0, a_1, a_2, a_3)$, which is a part of a presentation of the theory of monoids, is convergent and has exactly one critical pair, with corresponding generating confluence a_4:



In fact, this 4-cell a_4 is Mac Lane's pentagon [8]:



1.3. Track n-categories and homotopy bases

1.3.1. Track n-categories. A *track* n*-category* is an n-category $\mathcal{T}$ whose n-cells are invertible, that is, an $(n-1)$-category enriched in groupoid. In a track n-category, we denote by f^- the inverse of the n-cell f. A track n-category is *acyclic* when, for every $(n-1)$-sphere (u, v), there exists an n-cell f with source u and target v.

The n*-category presented* by a track $(n+1)$-category $\mathcal{T}$ is the n-category $\overline{\mathcal{T}} = \mathcal{T}_n/\mathcal{T}_{n+1}$. Two track $(n+1)$-categories are *Tietze-equivalent* if the n-categories they present are isomorphic. Given an n-category $\mathcal{C}$ and a cellular extension Γ of $\mathcal{C}$, the *track* $(n+1)$*-category generated by* Γ is denoted by $\mathcal{C}(\Gamma)$ and defined as follows:

$$\mathcal{C}(\Gamma) \;=\; \mathcal{C}\left[\Gamma,\ \Gamma^-\right] / \operatorname{Inv}(\Gamma)$$

where Γ^- contains the same $(n+1)$-cells as Γ, with source and target reversed, and $\operatorname{Inv}(\Gamma)$ is made of the $(n+2)$-cells $(\gamma \star_n \gamma^-, 1_{s\gamma})$ and $(\gamma^- \star_n \gamma, 1_{t\gamma})$, where γ ranges over Γ. Let us note that, when f

and g are n-cells of $\mathcal{C}$, we have $f \equiv_\Gamma g$ if and only if there exists an $(n+1)$-cell A with source f and target g in $\mathcal{C}(\Gamma)$. When Σ is an $(n+1)$-polygraph, one writes $\Sigma^\top$ instead of $\Sigma_n^*(\Sigma_{n+1})$.

1.3.2. Homotopy bases. Let $\mathcal{C}$ be an n-category. A *homotopy basis of* $\mathcal{C}$ is a cellular extension Γ of $\mathcal{C}$ such that the track $(n+1)$-category $\mathcal{C}(\Gamma)$ is acyclic or, equivalently, when the quotient n-category $\mathcal{C}/\Gamma$ is aspherical or, again equivalently, when every sphere (f, g) of $\mathcal{C}$ satisfies $f \equiv_\Gamma g$.

1.3.3. Lemma (Squier's fundamental confluence lemma). *Let Σ be a convergent n-polygraph. The generating confluences of Σ form a homotopy basis of $\Sigma^\top$.*

Remark. A complete proof of Lemma 1.3.3 is given in [8]. Squier has proved the same result for presentations of monoids by word rewriting systems [18, 19]. When formulated in terms of homotopy bases, Squier's result is a subcase of the case $n = 2$ of Lemma 1.3.3.

1.3.5. Example. The 2-polygraph $\Sigma_2 = (a_0, a_1, a_2)$ presenting **As** has exactly one generating confluence a_3 and, thus, this 3-cell forms a homotopy basis of the track 2-category $\Sigma_2^\top$. The 3-polygraph $\Sigma_3 = (a_0, a_1, a_2, a_3)$ also has exactly one generating confluence a_4, with Mac Lane's pentagon as shape, which forms a homotopy basis of the track 3-category $\Sigma_3^\top$.

The resulting 4-polygraph $\Sigma_4 = (a_0, a_1, a_2, a_3, a_4)$ is a part of a presentation of the theory of monoidal categories. In [8], Mac Lane's coherence theorem is reformulated in terms of homotopy bases and proved by an application of Lemma 1.3.3 to a convergent 3-polygraph containing Σ_3.

1.3.6. Lemma. *Let $\mathcal{T}$ be a track n-category and let $\mathcal{B}$ be a family of closed n-cells of $\mathcal{T}$. The following assertions are equivalent:*

1. *The cellular extension $\widetilde{\mathcal{B}} = \left\{ \widetilde{\beta} : \beta \to 1_{s\beta},\ \beta \in \mathcal{B} \right\}$ is a homotopy basis of $\mathcal{T}$.*

2. *Every closed n-cell f in $\mathcal{T}$ can be written*

$$f \;=\; \left(g_1 \star_{n-1} C_1\left[\beta_1^{\varepsilon_1}\right] \star_{n-1} g_1^-\right) \star_{n-1} \cdots \star_{n-1} \left(g_k \star_{n-1} C_k\left[\beta_k^{\varepsilon_k}\right] \star_{n-1} g_k^-\right) \qquad (3)$$

where, for every $i \in \{1, \ldots, k\}$, we have $\beta_i \in \mathcal{B}$, $\varepsilon_i \in \{-,+\}$, $C_i \in \mathbf{W}\mathcal{T}$ and $g_i \in \mathcal{T}_n$.

Proof. Let us assume that $\widetilde{\mathcal{B}}$ is a homotopy basis of $\mathcal{T}$ and let us consider a closed n-cell $f : w \to w$ in $\mathcal{T}$. Then, by definition of a homotopy basis, there exists an $(n+1)$-cell $A : f \to 1_w$ in $\mathcal{T}(\widetilde{\mathcal{B}})$. By construction of $\mathcal{T}(\widetilde{\mathcal{B}})$, the $(n+1)$-cell A decomposes into

$$A \;=\; A_1 \star_n \cdots \star_n A_k,$$

where each A_i is an $(n+1)$-cell of $\mathcal{T}(\widetilde{\mathcal{B}})$ that contains exactly one generating $(n+1)$-cell of $\mathcal{B}$. As a consequence, each A_i has shape

$$g_i \star_{n-1} C_i\left[\widetilde{\beta}_i^{\varepsilon_i}\right] \star_{n-1} h_i$$

with C_i in $\mathbf{W}\mathcal{T}$, g_i and h_i in $\mathcal{T}_n$, β_i in $\mathcal{B}$ and ε_i in $\{-,+\}$. By hypothesis on A, we have $f = s(A)$, hence:

$$f \;=\; g_1 \star_{n-1} C_1[s(\beta_1^{\varepsilon_1})] \star_{n-1} h_1.$$

We proceed by case analysis on ε_1. If $\varepsilon_1 = +$, then we have:

$$\begin{aligned} f &= g_1 \star_{n-1} C_1[\beta_1] \star_{n-1} h_1 \\ &= \big(g_1 \star_{n-1} C_1[\beta_1] \star_{n-1} g_1^-\big) \star_{n-1} (g_1 \star_{n-1} h_1) \\ &= \big(g_1 \star_{n-1} C_1[\beta_1] \star_{n-1} g_1^-\big) \star_{n-1} s(A_2). \end{aligned}$$

And, if $\varepsilon_1 = -$, we get:

$$\begin{aligned} f &= g_1 \star_{n-1} h_1 \\ &= \big(g_1 \star_{n-1} C_1[\beta_1^-] \star_{n-1} g_1^-\big) \star_{n-1} (g_1 \star_{n-1} C_1[\beta_1] \star_{n-1} h_1) \\ &= \big(g_1 \star_{n-1} C_1[\beta_1^-] \star_{n-1} g_1^-\big) \star_{n-1} s(A_2). \end{aligned}$$

An induction on the natural number k proves that f has a decomposition as in (3).

Conversely, we assume that every closed n-cell f in $\mathcal{T}$ has a decomposition as in (3). Then we have $f \equiv_{\widetilde{\mathcal{B}}} 1_{s(f)}$ for every closed n-cell f in $\mathcal{T}$. Let us consider two parallel n-cells f and g in $\mathcal{T}$. Then $f \star_{n-1} g^-$ is a closed n-cell, yielding $f \star_{n-1} g^- \equiv_{\widetilde{\mathcal{B}}} 1_{s(f)}$. We compose both members by g on the right hand to get $f \equiv_{\widetilde{\mathcal{B}}} g$. Thus $\widetilde{\mathcal{B}}$ is a homotopy basis of $\mathcal{T}$. □

1.3.7. Finite derivation type. One says that an n-polygraph Σ has *finite derivation type* when it is finite and when the track n-category $\Sigma^\top$ admits a finite homotopy basis. This property is Tietze-invariant for finite n-polygraphs, so that one says that an n-category has *finite derivation type* when it admits a presentation by an $(n+1)$-polygraph with finite derivation type.

1.3.8. Lemma. *Let $\mathcal{T}$ be a track n-category and let Γ be a cellular extension of $\mathcal{T}$. If $\mathcal{T}$ has finite derivation type, then so does $\mathcal{T}/\Gamma$.*

Proof. Let $\mathcal{B}$ be a finite homotopy basis of $\mathcal{T}$. Let us denote by $\overline{\mathcal{B}}$ the cellular extension of $\mathcal{T}/\Gamma$ made of one $(n+1)$-cell $\overline{A}$ with source $\overline{f}$ and target $\overline{g}$ for each $(n+1)$-cell A from f to g in $\mathcal{B}$. Then $\overline{\mathcal{B}}$ is a homotopy basis of $\mathcal{T}/\Gamma$. □

2. Identities among relations

2.1. Abelian track n-categories

2.1.1. Definition. Let $\mathcal{T}$ be a track n-category. For every $(n-1)$-cell u in $\mathcal{T}$, we denote by $\mathrm{Aut}^{\mathcal{T}}_u$ the group of closed n-cells of $\mathcal{T}$ with base u. This mapping extends to a natural system of groups $\mathrm{Aut}^{\mathcal{T}}$ on the $(n-1)$-category $\mathcal{T}_{n-1}$, sending a whisker C of $\mathcal{T}$ to the morphism of groups that maps f to $C[f]$.

A track n-category $\mathcal{T}$ is *abelian* when, for every $(n-1)$-cell u of $\mathcal{T}$, the group $\mathrm{Aut}^{\mathcal{T}}_u$ is abelian. The *abelianized* of a track n-category $\mathcal{T}$ is the track n-category denoted by $\mathcal{T}_{\mathrm{ab}}$ and defined as the quotient of $\mathcal{T}$ by the n-spheres $(f \star_{n-1} g, g \star_{n-1} f)$, where f and g are closed n-cells with the same base.

2.1.2. Lemma. *A track n-category $\mathcal{T}$ is abelian if and only if $\mathrm{Aut}^{\mathcal{T}}$ is a natural system of abelian groups on $\mathcal{T}_{n-1}$. In particular, each $\mathrm{Aut}^{\mathcal{T}_{ab}}_u$ is the abelianized group of $\mathrm{Aut}^{\mathcal{T}}_u$.*

2.1.3. Lemma. *Let $\mathcal{T}$ be a track n-category. For every n-cell $g : v \to u$, the mapping $(\cdot)^g$ from $\mathrm{Aut}^{\mathcal{T}}_u$ to $\mathrm{Aut}^{\mathcal{T}}_v$ and sending f to*

$$f^g \;=\; g^- \star_{n-1} f \star_{n-1} g$$

is an isomorphism of groups. Moreover, if $\mathcal{T}$ is abelian and $g, h : v \to u$ are n-cells of $\mathcal{T}$, then the isomorphisms $(\cdot)^g$ and $(\cdot)^h$ are equal.

Proof. We have:

$$(1_u)^g \;=\; g^- \star_{n-1} 1_u \star_{n-1} g \;=\; 1_v.$$

Let f_1 and f_2 be closed n-cells of $\mathcal{T}$ with base u. Then:

$$\begin{aligned}(f_1 \star_{n-1} f_2)^g \;&=\; g^- \star_{n-1} f_1 \star_{n-1} f_2 \star_{n-1} g\\ &=\; g^- \star_{n-1} f_1 \star_{n-1} g \star_{n-1} g^- \star_{n-1} f_2 \star_{n-1} g\\ &=\; f_1^g \star_{n-1} f_2^g.\end{aligned}$$

Hence $(\cdot)^g$ is a morphism of groups and it admits $(\cdot)^{g^-}$ as inverse. Now, if $\mathcal{T}$ is abelian and $g, h : v \to u$ are parallel n-cells, we have:

$$\begin{aligned}f^g \;&=\; g^- \star_{n-1} f \star_{n-1} g\\ &=\; (g^- \star_{n-1} h) \star_{n-1} (h^- \star_{n-1} f \star_{n-1} h \star_{n-1} h^- \star_{n-1} g)\\ &=\; (h^- \star_{n-1} f \star_{n-1} h) \star_{n-1} (g^- \star_{n-1} h \star_{n-1} h^- \star_{n-1} g)\\ &=\; f^h.\end{aligned}$$

□

2.1.4. Proposition. *If a track n-category $\mathcal{T}$ has finite derivation type, then so does $\mathcal{T}_{ab}$.*

Proof. We apply Lemma 1.3.8 to the quotient $\mathcal{T}_{\mathrm{ab}}$ of $\mathcal{T}$. □

2.2. Defining identities among relations

2.2.1. Definition. Let $\mathcal{T}$ be a track n-category and let D be a natural system of groups on $\overline{\mathcal{T}}$. We denote by $\widehat{D}$ the natural system on $\mathcal{T}_{n-1}$ defined by $\widehat{D}_u = D_{\overline{u}}$. A track n-category $\mathcal{T}$ is *linear* when there exists a natural system $\Pi(\mathcal{T})$ on $\overline{\mathcal{T}}$ such that $\widehat{\Pi(\mathcal{T})}$ is isomorphic to $\mathrm{Aut}^{\mathcal{T}}$.

Remark. If such a natural system D exists, then it is unique up to isomorphism. Indeed, by definition of $\widehat{D}$, we have $\widehat{D}_u = \widehat{D}_v$ whenever u and v are $(n-1)$-cells of $\mathcal{T}$ such that $\overline{u} = \overline{v}$ holds. Thus, if u is an $(n-1)$-cell of $\overline{\mathcal{T}}$, then $D_u = \widehat{D}_w$ for every $(n-1)$-cell w of $\mathcal{T}$ with $\overline{w} = u$. As a consequence, if D and E are natural systems on $\overline{\mathcal{T}}$ such that both $\widehat{D}$ and $\widehat{E}$ are isomorphic to $\mathrm{Aut}^{\mathcal{T}}$, then D and E are isomorphic.

2.2.3. Theorem. *A track n-category is abelian if and only if it is linear.*

Proof. If $\mathcal{T}$ is linear, then each group $\mathrm{Aut}^{\mathcal{T}}_u$ is isomorphic to an abelian group. Thus $\mathcal{T}$ is abelian.

Conversely, let us assume that $\mathcal{T}$ is abelian and let us define the natural system $\Pi(\mathcal{T})$ on $\overline{\mathcal{T}}$. If w is an $(n-1)$-cell of $\overline{\mathcal{T}}$, then $\Pi(\mathcal{T})_w$ is defined as the quotient

$$\Pi(\mathcal{T})_w \;=\; \left(\bigoplus_{\overline{a}=w} \mathrm{Aut}^{\mathcal{T}}_a \right) \Big/ \simeq$$

with $\simeq$ generated by the following two families of relations, where $\lfloor f \rfloor$ denotes the equivalence class of a closed n-cell f of $\mathcal{T}$ with base a such that $\overline{a} = w$:

i) $\lfloor f \star_{n-1} g \rfloor = \lfloor f \rfloor + \lfloor g \rfloor$, for any closed n-cells f and g in $\mathcal{T}$ with same base a such that $\overline{a} = u$,

ii) $\lfloor f \star_1 g \rfloor = \lfloor g \star_1 f \rfloor$, for any n-cells $f : a \to b$ and $g : b \to a$ in $\mathcal{T}$ with $\overline{a} = \overline{b} = u$.

If u and v are $(n-1)$-cells of $\overline{\mathcal{T}}$ and if C is a context of $\overline{\mathcal{T}}$ from u to u', then the action

$$\Pi(\mathcal{T})(C) \;:\; \Pi(\mathcal{T})_u \;\longrightarrow\; \Pi(\mathcal{T})_{u'}$$

is defined, on a generator $\lfloor f \rfloor$, with f a closed n-cell of $\mathcal{T}$ with base a such that $\overline{a} = u$, by

$$C\lfloor f \rfloor \;=\; \lfloor B[f] \rfloor,$$

where B is a context of $\mathcal{T}_{n-1}$, from a to some a' with $\overline{a'} = u'$, such that $\overline{B} = C$ holds. We note that $B[f]$ is a closed n-cell of $\mathcal{T}$ with base some a' such that $\overline{a}' = u'$, so that $\lfloor B[f] \rfloor$ is a generating element of $\Pi(\mathcal{T})_{u'}$. Now, let us check that this action is well-defined, that is, it does not depend on the choice of the representatives f and B.

For f, we check that $\Pi(\mathcal{T})(C)$ is compatible with the relations defining $\Pi(\mathcal{T})_u$. If f and g are closed n-cells of $\mathcal{T}$ with base a such that $\overline{a} = u$, then we have:

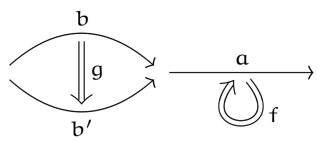
$$\lfloor B[f \star_{n-1} g] \rfloor \;=\; \lfloor B[f] \star_{n-1} B[g] \rfloor \;=\; \lfloor B[f] \rfloor + \lfloor B[g] \rfloor.$$

And, for n-cells $f : a \to b$ and $g : b \to a$, with $\overline{a} = \overline{b} = u$, we have:

$$\lfloor B[f \star_{n-1} g] \rfloor \;=\; \lfloor B[f] \star_{n-1} B[g] \rfloor \;=\; \lfloor B[g] \star_{n-1} B[f] \rfloor \;=\; \lfloor B[g \star_{n-1} f] \rfloor.$$

For B, we decompose C in $v \star_{n-2} C' \star_{n-2} w$, where v and w are $(n-1)$-cells of $\overline{\mathcal{T}}$ and C' is a whisker of $\overline{\mathcal{T}}$. Since $\overline{\mathcal{T}}$ and $\mathcal{T}_{n-1}$ coincide up to dimension $n-2$, any representative B of C can be written $B = b \star_{n-2} C' \star_{n-2} c$, where b and c are respective representatives of v and w in $\mathcal{T}_{n-1}$. As a consequence, it is sufficient (and, in fact, equivalent) to prove that the definition of $\Pi(\mathcal{T})(C)$ is invariant with respect to the choice of the representative B of C when C has shape $v \star_{n-2} x$ or $x \star_{n-2} w$.

We examine the case $C = v \star_{n-2} x$, the other one being symmetric. We consider two representatives b and b' of v in $\mathcal{T}_{n-1}$. By definition of $\overline{\mathcal{T}}$, there exists an n-cell $g : b \to b'$ in $\mathcal{T}$, as in the following diagram, drawn for the case $n = 2$:



Thanks to the exchange relation, we have:

$$(g \star_{n-2} a) \star_{n-1} (b' \star_{n-2} f) \;=\; g \star_{n-2} f \;=\; (b \star_{n-2} f) \star_{n-1} (g \star_{n-2} a).$$

Hence:

$$b' \star_{n-2} f \;=\; (g^{-} \star_{n-2} a) \star_{n-1} (b \star_{n-2} f) \star_{n-1} (g \star_{n-2} a).$$

As, a consequence, one gets, using the second defining relation of $\Pi(\mathcal{T})_{v\star_{n-2}u}$:

$$\begin{aligned}\lfloor b' \star_{n-2} f \rfloor &= \lfloor (g^{-} \star_{n-2} a) \star_{n-1} (b \star_{n-2} f) \star_{n-1} (g \star_{n-2} a) \rfloor \\ &= \lfloor (b \star_{n-2} f) \star_{n-1} (g \star_{n-2} a) \star_{n-1} (g^{-} \star_{n-2} a) \rfloor \\ &= \lfloor b \star_{n-2} f \rfloor.\end{aligned}$$

Now, let us prove that the natural systems $\overline{\Pi}(\mathcal{T})$ and $\mathrm{Aut}^{\mathcal{T}}$ are isomorphic. For an $(n-1)$-cell u of $\mathcal{T}$, we define $\Phi_u : \Pi(\mathcal{T})_{\overline{u}} \to \mathrm{Aut}^{\mathcal{T}}_u$ as the morphism of groups given on generators by

$$\Phi_u(\lfloor f \rfloor) = f^g,$$

where f is a closed n-cell of $\mathcal{T}$ with base v such that $\overline{v} = \overline{u}$ and g is any n-cell of $\mathcal{T}$ with source v and target u. Let us check that Φ_u is well-defined. We already know that Φ_u is independent of the choice of g. Let us prove that this definition is compatible with the relations defining $\Pi(\mathcal{T})_{\overline{u}}$.

For the first relation, let f_1 and f_2 be closed n-cells of $\mathcal{T}$ with base v such that $\overline{v} = \overline{u}$ and let $g : v \to u$ be an n-cell of $\mathcal{T}$. Then:

$$\begin{aligned}\Phi_u(\lfloor f_1 \star_{n-1} f_2 \rfloor) &= (f_1 \star_{n-1} f_2)^g \\ &= f_1^g \star_{n-1} f_2^g \\ &= \Phi_u(\lfloor f_1 \rfloor) \star_{n-1} \Phi_u(\lfloor f_2 \rfloor) \\ &= \Phi_u(\lfloor f_1 \rfloor + \lfloor f_2 \rfloor).\end{aligned}$$

For the second relation, we fix n-cells $f_1 : v_1 \to v_2$, $f_2 : v_2 \to v_1$ and $g : v_1 \to u$, with $\overline{v_1} = \overline{v_2} = \overline{u}$. Then:

$$\begin{aligned}\Phi_u(\lfloor f_1 \star_{n-1} f_2 \rfloor) &= (f_1 \star_{n-1} f_2)^g \\ &= (g^{-} \star_{n-1} f_1) \star_{n-1} (f_2 \star_{n-1} f_1) \star_{n-1} (f_1^{-} \star_{n-1} g) \\ &= (f_2 \star_{n-1} f_1)^{g^{-} \star_{n-1} f_1} \\ &= \Phi_u(\lfloor f_2 \star_{n-1} f_1 \rfloor).\end{aligned}$$

Thus Φ_u is a morphism of groups from $\Pi(\mathcal{T})_{\overline{u}}$ to $\mathrm{Aut}^{\mathcal{T}}_u$. Moreover, it admits $f \mapsto \lfloor f \rfloor$ as inverse and, as a consequence, is an isomorphism.

Finally, let us prove that Φ_u is natural in u. Let C be a context of $\mathcal{T}_{n-1}$ from u to v. Let us check that the morphisms of groups $\Phi_v \circ \Pi(\mathcal{T})(\overline{C})$ and $\mathrm{Aut}^{\mathcal{T}}(C) \circ \Phi_u$ coincide. Let f be a closed n-cell of $\mathcal{T}$ with base point u' such that $\overline{u}' = \overline{u}$. We fix an n-cell $g : u' \to u$ in $\mathcal{T}$ and we note that $C[g]$ is an n-cell of $\mathcal{T}$ with source $C[u']$ and target $C[u] = v$. Then we have:

$$\begin{aligned}\Phi_v \circ \Pi(\mathcal{T})(\overline{C})(\lfloor f \rfloor) &= (C[f])^{C[g]} \\ &= C[g^{-}] \star_{n-1} C[f] \star_{n-1} C[g] \\ &= C\left[g^{-} \star_{n-1} f \star_{n-1} g\right] \\ &= C[f^g] \\ &= \mathrm{Aut}^{\mathcal{T}}(C) \circ \Phi_u(\lfloor f \rfloor).\end{aligned}$$

□

Remark. Theorem 2.2.3 is proved in [2, 3] for the case $n = 2$.

2.2.5. Definition. Let Σ be an n-polygraph. The *natural system of identities among relations of* Σ is the natural system $\Pi(\Sigma^{\top}_{ab})$, which we simply denote by $\Pi(\Sigma)$. If w is an $(n-1)$-cell of $\overline{\Sigma}$, an element of the abelian group $\Pi(\Sigma)_w$ is called an *identity among relations associated to* w.

2.3. Generating identities among relations

2.3.1. Lemma. *Let Σ be an n-polygraph and let w be an $(n-1)$-cell of $\overline{\Sigma}$. The identities among relations associated to w are the sums*

$$\lfloor f \rfloor \;=\; \varepsilon_1 C_1 \lfloor \varphi_1 \rfloor \;+\; \cdots \;+\; \varepsilon_k C_k \lfloor \varphi_k \rfloor, \tag{4}$$

where, for every $i \in \{1, \ldots, k\}$, φ_i is an n-cell of Σ, C_i is a whisker of Σ^* and ε_i is in $\{-,+\}$, such that the composite

$$f \;=\; C_1[\varphi_1^{\varepsilon_1}] \star_{n-1} \cdots \star_{n-1} C_k[\varphi_k^{\varepsilon_k}] \tag{5}$$

exists and is a closed n-cell of $\Sigma^{\top}_{ab}$ with base u such that $\overline{u} = w$.

Proof. By definition, the abelian group $\Pi(\Sigma)_w$ is generated by the $\lfloor f \rfloor$ where f is a closed n-cells of $\Sigma^{\top}_{ab}$ with base u such that $\overline{u} = w$. By construction of $\Sigma^{\top}_{ab}$, the n-cell f has a decomposition such as in (5). We apply the morphism of groups $\lfloor \cdot \rfloor$ to this decomposition to get (4). □

2.3.2. Lemma. *Let Σ and Υ be two Tietze-equivalent n-polygraphs. Then there exist n-functors*

$$F : \Sigma^{\top}_{ab} \to \Upsilon^{\top}_{ab} \qquad \text{and} \qquad G : \Upsilon^{\top}_{ab} \to \Sigma^{\top}_{ab}$$

such that the following two diagrams commute:

$$\begin{array}{ccc} \Sigma^{\top}_{ab} & \xrightarrow{F} & \Upsilon^{\top}_{ab} \\ {\scriptstyle \pi_\Sigma}\downarrow & © & \downarrow{\scriptstyle \pi_\Upsilon} \\ \overline{\Sigma} & = & \overline{\Upsilon} \end{array} \qquad\qquad \begin{array}{ccc} \Upsilon^{\top}_{ab} & \xrightarrow{G} & \Sigma^{\top}_{ab} \\ {\scriptstyle \pi_\Upsilon}\downarrow & © & \downarrow{\scriptstyle \pi_\Sigma} \\ \overline{\Upsilon} & = & \overline{\Sigma} \end{array}$$

Proof. Let us build F, the construction of G being symmetric. First, we define an n-functor F from $\Sigma^{\top}$ to $\Upsilon^{\top}$. On i-cells, with $i \leq n-2$, F is the identity. If a is an $(n-1)$-cell in Σ, we arbitrarily choose an $(n-1)$-cell in $\pi_{\Upsilon}^{-1}\pi_{\Sigma}(a)$ for $F(a)$. Then, F is extended to any $(n-1)$-cell of $\Sigma^{\top}$ by functoriality. Let $\varphi : u \to v$ be an n-cell of Σ. We have, by definition of $F(u)$ and $F(v)$:

$$\pi_\Upsilon \circ F(u) \;=\; \pi_\Sigma(u) \;=\; \pi_\Sigma(v) \;=\; \pi_\Upsilon \circ F(v).$$

Thus, there exists an n-cell from $F(u)$ to $F(v)$ in $\Sigma^{\top}$. We arbitrarily choose $F(\varphi)$ to be one of those n-cells and, then, we extend F to any n-cell of $\Sigma^{\top}$ by functoriality.

Let f and g be closed n-cells in $\Sigma^{\top}$. We have $F(f \star_{n-1} g) = F(f) \star_{n-1} F(g)$ by definition of F. As a consequence, F induces a n-functor from $\Sigma^{\top}_{ab}$ to $\Upsilon^{\top}_{ab}$ that satisfies, by construction, the relation $\pi_\Upsilon \circ F = \pi_\Sigma$. □

2.3.3. Notation. We fix two Tietze-equivalent n-polygraphs Σ and Υ, together with n-functors F and G as in Lemma 2.3.2. We denote by $\widetilde{G}$ the morphism of natural systems on $\overline{\Sigma} = \overline{\Upsilon}$, from $\Pi(\Upsilon)$ to $\Pi(\Sigma)$, defined by $\widetilde{G}(\lfloor f \rfloor) = \lfloor G(f) \rfloor$.

For every $(n-1)$-cell w in $\Sigma^{\top}_{ab}$, we define an n-cell from w to $GF(w)$ in $\Sigma^{\top}_{ab}$, by structural induction on w. If w is an identity, then Λ_w is 1_w. Now, let w be a generating $(n-1)$-cell in Σ_{n-1}. By hypothesis on F and G, we have:

$$\pi_\Sigma \circ GF(w) \;=\; \pi_\Upsilon \circ F(w) \;=\; \pi_\Sigma(w).$$

As a consequence, there exists an n-cell from w to $GF(w)$ in $\Sigma^{\top}_{ab}$ and we arbitrarily choose Λ_w to be such an n-cell. Finally, if $w = w_1 \star_i w_2$, for some $i \in \{0, \ldots, n-2\}$, then Λ_w is defined as $\Lambda_{w_1} \star_i \Lambda_{w_2}$.

If $f : u \to v$ is an n-cell of $\Sigma^{\top}_{ab}$, we denote by Λ_f the closed n-cell with basis u defined by:

$$\Lambda_f \;=\; f \star_{n-1} \Lambda_v \star_{n-1} GF(f)^- \star_{n-1} \Lambda_u^-.$$

Finally, we define:

$$\Lambda_\Sigma \;=\; \left\{ \; \lfloor \Lambda_\varphi \rfloor \;\middle|\; \varphi \in \Sigma_n \; \right\}.$$

2.3.4. Lemma. *Let f be an n-cell in $\Sigma^{\top}_{ab}$ with a decomposition*

$$f \;=\; C_1[\varphi_1^{\varepsilon_1}] \star_{n-1} \cdots \star_{n-1} C_k[\varphi_k^{\varepsilon_k}],$$

with φ_i in Σ_n, C_i a whisker of Σ^* and ε_i in $\{-, +\}$. Then we have:

$$\lfloor \Lambda_f \rfloor \;=\; \sum_{i=1}^{k} \varepsilon_i C_i \lfloor \Lambda_{\varphi_i} \rfloor. \tag{6}$$

Proof. Let $f : u \to v$ and $g : v \to w$ be n-cells in $\Sigma^{\top}_{ab}$. We have:

$$\begin{aligned}
\Lambda_{f \star_{n-1} g} \;&=\; (f \star_{n-1} g) \star_{n-1} \Lambda_w \star_{n-1} GF(f \star_{n-1} g)^- \star_{n-1} \Lambda_u^- \\
&=\; f \star_{n-1} \big(g \star_{n-1} \Lambda_w \star_{n-1} GF(g)^- \star_{n-1} \Lambda_v^-\big) \star_{n-1} \Lambda_v \star_{n-1} GF(f)^- \star_{n-1} \Lambda_u^- \\
&=\; f \star_{n-1} \Lambda_g \star_{n-1} \Lambda_v \star_{n-1} GF(f)^- \star_{n-1} \Lambda_u^- \\
&=\; f \star_{n-1} \Lambda_g \star_{n-1} f^- \star_{n-1} \Lambda_f.
\end{aligned}$$

Hence:

$$\lfloor \Lambda_{f \star_{n-1} g} \rfloor \;=\; \lfloor f \star_{n-1} \Lambda_g \star_{n-1} f^- \star_{n-1} \Lambda_f \rfloor \;=\; \lfloor \Lambda_f \rfloor + \lfloor \Lambda_g \rfloor. \tag{7}$$

Now, let $f : w \to w'$ be an n-cell and u be an i-cell, $i \leq n-1$, of $\Sigma^{\top}_{ab}$ such that $u \star_i w$ is defined. Then we have:

$$\begin{aligned}
\Lambda_{u \star_i f} \;&=\; (u \star_i f) \star_{n-1} \Lambda_{u \star_i w'} \star_{n-1} GF(u \star_i f)^- \star_{n-1} \Lambda^-_{u \star_i w} \\
&=\; (u \star_i f) \star_{n-1} (\Lambda_u \star_i \Lambda_{w'}) \star_{n-1} (GF(u) \star_i GF(f)^-) \star_{n-1} (\Lambda_u^- \star_i \Lambda_w^-) \\
&=\; (u \star_{n-1} \Lambda_u \star_{n-1} GF(u) \star_{n-1} \Lambda_u^-) \star_i (f \star_{n-1} \Lambda_{w'} \star_{n-1} GF(f)^- \star_{n-1} \Lambda_w^-) \\
&=\; u \star_i \Lambda_f.
\end{aligned}$$

Similarly, we prove that $\Lambda_{f \star_i v} = \Lambda_f \star_i v$ if v is an i-cell, $i \leq n-1$, such that $w \star_i v$ is defined. As a consequence, we get $\Lambda_{C[f]} = C[\Lambda_f]$, for every whisker C of Σ^*, hence:

$$\lfloor \Lambda_{C[f]} \rfloor \;=\; C \lfloor \Lambda_f \rfloor. \tag{8}$$

We prove (6) by induction on k, using (7) and (8). □

2.3.5. Lemma. *Let $\mathcal{B}$ be a generating set for the natural system $\Pi(\Upsilon)$. Then the set $\Lambda_\Sigma \amalg \widetilde{G}(\mathcal{B})$ is a generating set for the natural system $\Pi(\Sigma)$.*

Proof. Let f be a closed n-cell with basis w in $\Sigma^\top$. By definition of Λ_f, we have:

$$\lfloor f \rfloor \;=\; \lfloor \Lambda_f \star_{n-1} \Lambda_w \star_{n-1} GF(f) \star_{n-1} \Lambda_w^- \rfloor \;=\; \lfloor \Lambda_f \rfloor + \lfloor GF(f) \rfloor.$$

On the one hand, we consider a decomposition of f in generating n-cells of Σ_n:

$$f \;=\; C_1[\varphi_1^{\varepsilon_1}] \star_{n-1} \cdots \star_{n-1} C_k[\varphi_k^{\varepsilon_k}].$$

Hence:

$$\lfloor \Lambda_f \rfloor \;=\; \sum_{i=1}^{k} \varepsilon_i C_i \lfloor \Lambda_{\varphi_i} \rfloor.$$

On the other hand, the natural system $\Pi(\Upsilon)$ is generated by $\mathcal{B}$, so that $\lfloor F(f) \rfloor$ admits a decomposition $\lfloor F(f) \rfloor = \sum_{j \in J} \eta_j B_j \lfloor g_j \rfloor$, with $\lfloor g_j \rfloor \in \mathcal{B}$. Hence:

$$\lfloor GF(f) \rfloor \;=\; \sum_{j \in J} B_j \lfloor G(g_j) \rfloor \;=\; \sum_{j \in J} B_j [\widetilde{G}(\lfloor g_j \rfloor)].$$

Thus, $\lfloor f \rfloor$ can be written as a linear combination of elements of Λ_Σ and of $\mathcal{B}$, proving the result. □

2.3.6. Proposition. *Let Σ and Υ be two Tietze-equivalent n-polygraphs such that Σ_n and Υ_n are finite. Then the natural system $\Pi(\Sigma)$ is finitely generated if and only if the natural system $\Pi(\Upsilon)$ is finitely generated.*

Proof. We use Lemma 2.3.5 with $\mathcal{B}$ and Σ_n finite. □

2.3.7. Theorem. *If an n-polygraph Σ has finite derivation type then the natural system $\Pi(\Sigma)$ is finitely generated.*

Proof. Let us assume that the n-polygraph Σ has finite derivation type. By Proposition 2.1.4, the abelian track category $\Sigma^\top_{ab}$ has finite derivation type. Let $\mathcal{B}$ be a finite homotopy basis of $\Sigma^\top_{ab}$ and let $\widetilde{B}$ be the set of closed n-cells of $\Sigma^\top_{ab}$ defined by:

$$\widetilde{\mathcal{B}} \;=\; \left\{ \; s(\beta) \star_{n-1} t(\beta)^- \;\middle|\; \beta \in \mathcal{B} \; \right\}.$$

By Lemma 1.3.6, any closed n-cell f in $\Sigma^\top_{ab}$ can be written

$$f \;=\; \left(g_1 \star_{n-1} C_1[\beta_1^{\varepsilon_1}] \star_{n-1} g_1^-\right) \star_{n-1} \cdots \star_{n-1} \left(g_k \star_{n-1} C_k[\beta_k^{\varepsilon_k}] \star_{n-1} g_k^-\right),$$

where, for every i in $\{1, \ldots, k\}$, β_i is in $\widetilde{\mathcal{B}}$, ε_i is in $\{-, +\}$, C_i is a whisker of Σ^* and g_i is an n-cell of Σ^*. As a consequence, for any identity among relations $\lfloor f \rfloor$ in $\Pi(\Sigma)$, we have:

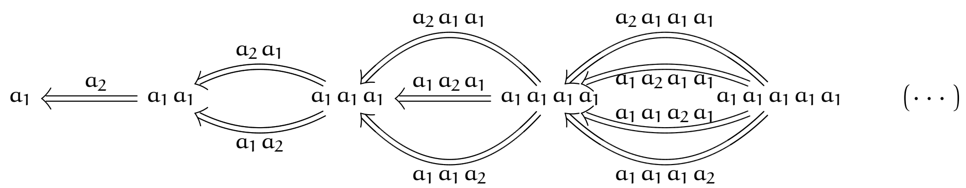
$$\lfloor f \rfloor \;=\; \sum_{i=1}^{k} \varepsilon_i \lfloor g_i \star_{n-1} C_i[\beta_i] \star_{n-1} g_i^- \rfloor \;=\; \sum_{i=1}^{k} \varepsilon_i C_i \lfloor \beta_i \rfloor.$$

Thus, the elements of $\lfloor \widetilde{B} \rfloor$ form a generating set for $\Pi(\Sigma)$. □

2.3.8. Proposition. *For a convergent n-polygraph Σ, the natural system $\Pi(\Sigma)$ is generated by the generating confluences of Σ.*

Proof. By Squier's confluence lemma (Lemma 1.3.3), the set of generating confluences of Σ forms a homotopy basis of $\Sigma^{\top}$. Following the proof of Theorem 2.3.7, we transform it into a generating set for the natural system $\Pi(\Sigma)$. □

2.3.9. Example. We consider the 2-polygraph $\Sigma = (a_0, a_1, a_2)$ presenting the monoid **As**. Here is a part of the free 2-category Σ^*:



The 2-polygraph Σ is convergent and has exactly one generating confluence

$$a_3 \,:\, a_2a_1 \star_1 a_2 \Rrightarrow a_1a_2 \star_1 a_2.$$

Thus the natural system $\Pi(\Sigma)$ on the category $\overline{\overline{\Sigma}}$ is generated by the element:

$$\lfloor s(a_3) \star_1 t(a_3)^- \rfloor \;=\; \lfloor (a_2a_1 \star_1 a_2) \star_1 (a_2^- \star_1 a_1a_2^-) \rfloor \;=\; \lfloor a_2a_1 \star_1 a_1a_2^- \rfloor \;=\; \lfloor a_2a_2^- \rfloor.$$

One can prove the same result by a combinatorial analysis. Indeed, one can note that the minimal 2-cells from a_1^{n+1} to a_1^n are the $a_1^i a_2 a_1^{n-1-i}$, for i in $\{0, \ldots, n-1\}$. Thus, the natural system $\Pi(\Sigma)$ is generated by the following elements, for $n \geq 2$ and $0 \leq i < j \leq n-1$:

$$\lfloor g_{i,j} \rfloor \;=\; \lfloor a_1^i a_2 a_1^{n-i-1} \star_1 a_1^j a_2^- a_1^{n-j-1} \rfloor.$$

Then, one uses the exchange relations to get:

$$g_{i,j} \;=\; \begin{cases} a_1^i(a_2a_1 \star_1 a_1a_2^-)a_1^{n-i-1} & \text{if } j = i+1 \\ a_1^i a_2 a_1^{j-i-2} a_2^- a_1^{n-j-1} & \text{if } j > i+2. \end{cases}$$

Hence, if $j = i+1$, we have, using the relations defining $\Pi(\Sigma)$ and $\lfloor a_1 \rfloor = 0$:

$$\lfloor g_{i,i+1} \rfloor \;=\; i\lfloor a_1 \rfloor + \lfloor a_2a_1 \star_1 a_1a_2^- \rfloor + (n-i-1)\lfloor a_1 \rfloor \;=\; \lfloor a_2a_2^- \rfloor.$$

And, if $j > i+2$, we get:

$$\lfloor g_{i,j} \rfloor \;=\; i\lfloor a_1 \rfloor + \lfloor a_2 \rfloor + (j-i-2)\lfloor a_1 \rfloor - \lfloor a_2 \rfloor + (n-j-1)\lfloor a_1 \rfloor \;=\; 0.$$

Thus, the natural system $\Pi(\Sigma)$ is generated by one element: $\lfloor a_2a_2^- \rfloor$.

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