Nash Equilibria in Non-Cooperative Games: How to Compute Them?

Régis Sabbadin and Paul Jourdan

Algorithmic Decision Theory,
November, 2, 2021,
Toulouse

RÉPUBLIQUE
FRANÇAISE
Liberté
Égalité
Fraternité

INRAE

Game theory

Definition (Game theory)

- Game theory is a theoretical framework for conceiving social situations among competing players
- Game theory is the science of strategy, or at least the optimal decision-making of independent and competing actors in a strategic setting

There are different features in game theory. A game can be:

- Cooperative / competitive
- Static / repeated
- Normal form / extensive form / succinct form
- Complete /incomplete information

We consider competitive complete information normal form games
Results can be extended to succinct / incomplete information games

Game theory

Example (Prisoner's dilemma)

Two criminals arrested for robbing a bank. The police do not have any proof against them. But, to get their confession, both prisoners are put in separate cells, where they cannot talk with each other:

1. If both confess robbing the bank, they would each get a prison sentence of four years
2. If prisoner 1 confesses, but prisoner two does not, then the first one will receive a prison sentence of one year. The other one will get a nine-year sentence
3. If prisoner 2 confesses but not prisoner 1, then prisoner one will get a ten-year prison sentence. The other one will get a prison sentence of one year
4. If none confess, both will get a prison sentence of two years

Game theory

How are self-interested robbers supposed to act?

But also:

- Board games players
- Firms entering a new martket
- Nations facing a conflict
- Animals competing for a resource
- Law enforcers and criminals
- etc.

Game theory can help! It provides modeling frameworks, solution concepts and algorithmic tools to analyze situations of interaction between self-interested agents

Outline of the talk

Basics of game theory

Nash equilibrium computation
- Polynomial complementarity problems
- Solving polynomial complementarity problems

Concluding remarks

Table of Contents

Game components and representation

The basic elements of a game-theoretic situation are the following:

- Players: A set $P = \{1, ..., N\}$ of players indices.
- Strategies:
    - Finite sets $S_1, \dots, S_N$ of strategies for every players.
    - $\omega \in \Omega = S_1 \times \dots \times S_N$ is a pure "outcome" of the game.
- Utilities: Tables $u_n : \Omega \to \mathbb{R}$. $u_n(\omega)$ is the utility to player $n \in \{1, \dots, N\}$ of pure joint strategy (outcome) ω.

Example (Prisonner's dilemma)

- Players: $P = \{1, 2\}$
- Strategies: $S_1 = S_2 = \{C(onfess), D(oes\ not\ confess)\}$
- Utilities:

		Player 2	
		C	D
Player 1	C	$-4, -4$	$-1, -9$
	D	$-10, -1$	$-2, -2$

Game components and representation

Example (A three-players game)

- Players: $P = \{1, 2, 3\}$
- Strategies: $S_1 = S_2 = S_3 = \{0, 1\}$
- Utilities:

ω_1	ω_2	ω_3	$u_1(\omega)$	$u_2(\omega)$	$u_3(\omega)$
0	0	0	6	0	4
0	0	1	7	3	7
0	1	0	0	7	5
0	1	1	5	4	0
1	0	0	1	6	3
1	0	1	4	5	2
1	1	0	2	1	6
1	1	1	3	2	1

Solution concepts: Dominated strategies

Example (Another two-player game)

		Player 2		
		0	1	2
Player 1	0	$3, 6$	$7, 1$	$4, 8$
	1	$5, 1$	$8, 2$	$6, 1$
	2	$6, 0$	$6, 2$	$3, 2$

Solution concepts: Dominated strategies

Example (Another two-player game)

		Player 2		
		0	1	2
Player 1	0	3,6	7,1	4,8
	1	5,1	8,2	6,1
	2	6,0	6,2	3,2

- Strategy 0 of player 1 is strictly dominated by strategy 1

Solution concepts: Dominated strategies

Example (Another two-player game)

		Player 2		
		0	1	2
Player 1	1	5,1	8,2	6,1
	2	6,0	6,2	3,2

- Strategy 0 of player 1 is strictly dominated by strategy 1

Solution concepts: Dominated strategies

Example (Another two-player game)

		Player 2		
		0	1	2
Player 1	1	5,1	8,2	6,1
	2	6,0	6,2	3,2

- Strategy 0 of player 1 is strictly dominated by strategy 1
- Strategy 0 of player 2 is strictly dominated by strategy 1 (strategy 2 is only weakly dominated)

Solution concepts: Dominated strategies

Example (Another two-player game)

		Player 2	
		1	2
Player 1	1	8,2	6,1
	2	6,2	3,2

- Strategy 0 of player 1 is strictly dominated by strategy 1
- Strategy 0 of player 2 is strictly dominated by strategy 1 (strategy 2 is only weakly dominated)

Solution concepts: Dominated strategies

Example (Another two-player game)

		Player 2	
		1	2
Player 1	1	8,2	6,1
	2	6,2	3,2

- Strategy 0 of player 1 is strictly dominated by strategy 1
- Strategy 0 of player 2 is strictly dominated by strategy 1 (strategy 2 is only weakly dominated)
- Strategy 2 of player 1 is strictly dominated by strategy 1

Solution concepts: Dominated strategies

Example (Another two-player game)

		Player 2	
		1	2
Player 1	1	8,2	6,1

- Strategy 0 of player 1 is strictly dominated by strategy 1
- Strategy 0 of player 2 is strictly dominated by strategy 1 (strategy 2 is only weakly dominated)
- Strategy 2 of player 1 is strictly dominated by strategy 1
- Strategy 2 of player 2 is strictly dominated by strategy 1

Solution concepts: Dominated strategies

Example (Another two-player game)

		Player 2		
		0	1	2
	0	3,6	7,1	4,8
Player 1	1	5,1	8,2	6,1
	2	6,0	6,2	3,2

- Strategy 0 of player 1 is strictly dominated by strategy 1
- Strategy 0 of player 2 is strictly dominated by strategy 1 (strategy 2 is only weakly dominated)
- Strategy 2 of player 1 is strictly dominated by strategy 1
- Strategy 2 of player 2 is strictly dominated by strategy 1
- Joint strategy (1,1) is obtained through iterated removal of dominated strategies

Solution concepts: pure Nash equilibrium

Definition (Nash equilibrium)

A (pure) Nash equilibrium of game $\Gamma = \langle P, \Omega, \{u_n\} \rangle$ is a joint strategy $\omega^* = (\omega_1^*, \ldots, \omega_N^*)$ such that $\omega_n^* \in BR_n(\omega_{-n}^*), \forall n \in P$.

Definition (Best response)

$$BR_n(\omega_{-n}^*) = \left\{ \omega_n \in S_n \text{ s.t. } u_n(\omega_n, \omega_{-n}^*) \geq u_n(\omega_n', \omega_{-n}^*), \forall \omega_n' \in S_n \right\}$$

- When iterated removal of strictly dominated strategies leads to a single pure joint strategy, this is a Nash equilibrium

		Player 2		
		0	1	2
Player 1	0	3,6	7,1	4,8
	1	5,1	8,2	6,1
	2	6,0	6,2	3,2

Limitations of pure Nash equilibrium

Problems:

- Iterated removal of dominated strategies does not always succeed in finding a single joint strategy
- Games may admit a single pure NE or several NE or none

Example (Games with pure NE)

		Player 2	
		0	1
Player 1	0	0, 1	6, 0
	1	2, 0	5, 2
	2	3, 4	3, 3

		Player 2	
		0	1
Player 1	0	2, 1	0, 0
	1	0, 0	1, 2

Limitations of pure Nash equilibrium

Problems:

- Iterated removal of dominated strategies does not always succeed in finding a single joint strategy
- Games may admit a single pure NE or several NE or none

Example (A game with no pure NE (Cissors,Paper,Rock))

		Player 2		
		C	P	R
Player 1	C	0,0	1,-1	-1,1
	P	-1,1	0,0	1,-1
	R	1,-1	-1,1	0,0

Solution concepts: Mixed strategy

Definition (Mixed strategy)

Let a game $\Gamma = \langle P, \Omega = S_1 \times \ldots \times S_N, \{u_n\}\rangle$ be given
For any player n, $\xi^n = \left(\xi^n_{\omega_n}\right)$ denotes player n's mixed strategy over S_n. ξ^n is a probability distribution.
Then, $\xi = (\xi^n)_{n=1..N}$ denotes a joint mixed strategy

Definition (Mixed strategy expected utility)

Let Γ, ξ and player n be given. Then, the expected utility of ξ to player n is:

$$EU_n[\xi] =_{def} \sum_{\omega=(\omega_1,\ldots,\omega_N)\in\Omega} u^n(\omega) \prod_{i=1}^{N} \xi^i_{\omega_i}$$

Solution concepts: Mixed Nash equilibrium

Definition (Mixed Nash equilibrium)

Let a game $\Gamma = \langle P, \Omega = S_1 \times \ldots \times S_N, \{u_n\}\rangle$ be given.
Mixed strategy $\bar{\xi}$ is a Nash equilibrium of Γ if and only if:

$$EU_n[\bar{\xi}] \geq EU_n[\xi^n, \bar{\xi}^{-n}], \forall n \in P, \forall \xi^n.$$

Theorem (Alternative definitions)

Let game Γ, mixed Nash equilibrium $\bar{\xi}$ and player $n \in P$ be given:

1. $\bar{\xi}^n_{\omega_n} > 0$ *if and only if* $\omega_n \in BR_n(\bar{\xi}^{-n})$
2. $EU_n[\bar{\xi}] \geq EU_n[\omega_n, \bar{\xi}^{-n}]$ $\forall \omega_n \in S_n$, *where* $(\omega_n, \bar{\xi}^{-n})$ *is* $\bar{\xi}$ *modified such that player n plays* ω_n *with probability 1*
3. $EU_n[\omega_n, \bar{\xi}^{-n}] = EU_n[\bar{\xi}],\ \forall \omega_n \in BR_n(\bar{\xi}^{-n})$

Solution concepts: Mixed Nash equilibrium

Example (Mixed Nash equilibrium)

		Player 2	
		0	1
Player 1	0	0, 1	6, 0
	1	2, 0	5, 2
	2	3, 4	3, 3

$$\bar{\xi}^1 = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix} \qquad \bar{\xi}^2 = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$\begin{aligned}
EU_1[\xi] &= 2\xi^1_1\xi^2_0 + 3\xi^1_2\xi^2_0 + 6\xi^1_0\xi^2_1 + 5\xi^1_1\xi^2_1 + 3\xi^1_2\xi^2_1 \\
EU_2[\xi] &= \xi^1_0\xi^2_0 + 4\xi^1_2\xi^2_0 + 2\xi^1_1\xi^2_1 + 3\xi^1_2\xi^2_1 \\
EU_1[\xi^1, \bar{\xi}^2] &= \frac{2}{3}\xi^1_1 + \xi^1_2 + 4\xi^1_0 + \frac{10}{3}\xi^1_1 + 2\xi^1_2 = 4\xi^1_0 + 4\xi^1_1 + 3\xi^1_2 \\
EU_2[\bar{\xi}^1, \xi^2] &= \frac{2}{3}\xi^2_0 + \frac{2}{3}\xi^2_1
\end{aligned}$$

Solution concepts: Mixed Nash equilibrium

		Player 2	
		0	1
Player 1	0	0, 1	6, 0
	1	2, 0	5, 2
	2	3, 4	3, 3

$$\bar{\xi}^1 = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix} \qquad \bar{\xi}^2 = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$\begin{aligned} EU_1[\xi^1, \bar{\xi}^2] &= 4\xi^1_0 + 4\xi^1_1 + 3\xi^1_2 \\ EU_2[\bar{\xi}^1, \xi^2] &= \frac{2}{3}\xi^2_0 + \frac{2}{3}\xi^2_1 \end{aligned}$$

Nash equilibrium conditions are satisfied:

- $EU_1[\bar{\xi}] = 4 \geq EU_1[\xi^1, \bar{\xi}^2]$ and $EU_2[\bar{\xi}] = \frac{2}{3} \geq EU_2[\bar{\xi}^1, \xi^2]$
- $BR_1(\bar{\xi}^2) = \{0, 1\}$ and $BR_2(\bar{\xi}^1) = \{0, 1\}$
  (and $\bar{\xi}^1_0 > 0$, $\bar{\xi}^1_1 > 0$, $\bar{\xi}^2_0 > 0$, $\bar{\xi}^2_1 > 0$)
- $EU_1[0, \bar{\xi}^2] = EU_1[1, \bar{\xi}^2] = EU_1[\bar{\xi}]$ and
  $EU_2[\bar{\xi}^1, 0] = EU_2[\bar{\xi}^2, 1] = EU_2[\bar{\xi}]$

Nash equilibrium computation

Theorem (Nash, by Brouwer's fixed-point theorem)

A game $\Gamma = \langle P, \Omega, \{u_n\} \rangle$ *admits at least one mixed equilibrium (which may be pure)*

Nash equilibrium computation

Theorem (Nash, by Brouwer's fixed-point theorem)

A game $\Gamma = \langle P, \Omega, \{u_n\}\rangle$ *admits at least one mixed equilibrium (which may be pure)*

OK,fine, but how hard is it to compute a mixed NE?

- Finding a Nash equilibrium is hard. Namely, PPAD-complete (Why not NP-complete?)
- In other words, polynomial time algorithms are unlikely to exist
- An exception is 2-players zero-sum games (and polymatrix games with zero-sum local utility functions) → Polynomial-time computation through linear programming

Nash equilibrium computation

Theorem (Nash, by Brouwer's fixed-point theorem)

A game $\Gamma = \langle P, \Omega, \{u_n\} \rangle$ *admits at least one mixed equilibrium (which may be pure)*

Still, there are some exact and approximate solution algorithms[1]

- 2-player: Lemke-Howson (exact, worst-case exponential)
- N-player: often incomplete, approximate and slow
  - Simplicial subdivision (path-following)
  - Govindan and Wilson (continuation method)
  - Support enumeration + Polynomial systems solving
  - Function minimization
  - Polynomial complementarity (extends LH)

---

[1] See the Gambit library: `http://gambit-project.org/`

Nash equilibrium computation

Theorem (Nash, by Brouwer's fixed-point theorem)

A game $\Gamma = \langle P, \Omega, \{u_n\} \rangle$ *admits at least one mixed equilibrium (which may be pure)*

Still, there are some exact and approximate solution algorithms[1]

- 2-player: Lemke-Howson (exact, worst-case exponential)
- N-player: often incomplete, approximate and slow
  - Simplicial subdivision (path-following)
  - Govindan and Wilson (continuation method)
  - Support enumeration + Polynomial systems solving
  - Function minimization
  - Polynomial complementarity (extends LH)

---

[1] See the Gambit library: `http://gambit-project.org/`

Table of Contents

Nash equilibrium in terms of disutility

$$\text{Recall that: } EU_n[\xi] = \sum_{\omega \in \Omega} u^n(\omega) \prod_{n'=1}^{N} \xi^{n'}_{\omega_{n'}}$$

$$\text{Let us write: } a^n(\omega) = 1 + \left(\max_{\omega \in \Omega} u^n(\omega) \right) - u^n(\omega)$$

$a^n(\omega)$ is a positive disutility of ω to player n

$$\text{Then, we write: } ED_n[\xi] = \sum_{\omega \in \Omega} a^n(\omega) \prod_{n'=1}^{N} \xi^{n'}_{\omega_{n'}}$$

the expected disutility of ξ to player n.

Proposition (Disutility minimization)

$$\bar{\xi} \textit{ is a NE of } \Gamma \Leftrightarrow ED_n[\bar{\xi}] \leq ED_n[i, \bar{\xi}^{-n}], \forall n \in P, \forall i \in S_n$$

Polynomial Complementarity Problem

Proposition ([Wilson, 1971])

A Nash equilibrium can be obtained as a solution of a Polynomial Complementarity Problem (PCP):

$$\forall n \in P, i \in S_n, \begin{cases} x_i^n & \geq & 0 \\ A_i^n(x^{-n}) & \geq & 1 \\ x_i^n(A_i^n(x^{-n}) - 1) & = & 0 \end{cases} \quad (\mathcal{S})$$

$$\textit{where } A_i^n(x^{-n}) = \sum_{\substack{\omega \in \Omega \\ \omega_n = i}} a^n(\omega) \prod_{\substack{\omega_{n'} = j \\ n' \neq n}} x_j^{n'}$$

Proof: Uses the facts that $\bar{\xi}_i^n > 0 \Leftrightarrow ED_n[i, \bar{\xi}^{-n}] = ED_n[\bar{\xi}]$ and that the x^n are "unnormalized" versions of the ξ^n

$\Rightarrow$ $\bar{x}$ is a solution of $\mathcal{S}$ iff $\bar{\xi}$ is a NE of Γ, where $\bar{\xi}_i^n = \frac{\bar{x}_i^n}{\sum_{i' \in S_n} \bar{x}_{i'}^n}$

PCP-based NE computation methods

Several approaches to Nash equilibrium computation exploit the PCP representation:

- [Lemke and Howson, 1964] proposed a pivoting algorithm for solving 2-players games, following a path of almost complementary points of a Linear Complementarity Problem
- [Wilson, 1971] extended LH to N-player games and PCP
  ⇒ Paul's thesis is based on and extends Wilson's results
- [Porter et al., 2008] is also based on systems of polynomial equations and can be cast as a PCP solution method
- [Howson, 1972] extended LH to polymatrix games
  ⇒ Paul's thesis extends the PCP approach to hypergraphical and bayesian games

⇒ We are going to describe these works in a more or less detailed way

PCP-based NE computation methods

$$A_i^n(x^{-n}) = \sum_{\substack{\omega\in\Omega \\ \omega_n=i}} a^n(\omega) \prod_{\substack{\omega_{n'}=j \\ n'\neq n}} x_j^{n'}$$

Example (Polynomials associated to a game)

ω_1	ω_2	ω_3	a^1	a^2	a^3
0	0	0	2	8	4
0	0	1	1	5	1
0	1	0	8	1	3
0	1	1	3	4	8
1	0	0	7	2	5
1	0	1	4	3	6
1	1	0	6	7	2
1	1	1	5	6	7

$$\begin{aligned}
A_0^1\left(x^2,x^3\right) &= 2x_0^2x_0^3 + x_0^2x_1^3 + 8x_1^2x_0^3 + 3x_1^2x_1^3 \\
A_1^1\left(x^2,x^3\right) &= 7x_0^2x_0^3 + 4x_0^2x_1^3 + 6x_1^2x_0^3 + 5x_1^2x_1^3 \\
A_0^2\left(x^1,x^3\right) &= 8x_0^1x_0^3 + 5x_0^1x_1^3 + 2x_1^1x_0^3 + 3x_1^1x_1^3 \\
A_1^2\left(x^1,x^3\right) &= x_0^1x_0^3 + 4x_0^1x_1^3 + 7x_1^1x_0^3 + 6x_1^1x_1^3 \\
A_0^3\left(x^1,x^2\right) &= 4x_0^1x_0^2 + 3x_0^1x_1^2 + 5x_1^1x_0^2 + 2x_1^1x_1^2 \\
A_1^3\left(x^1,x^2\right) &= x_0^1x_0^2 + 8x_0^1x_1^2 + 6x_1^1x_0^2 + 7x_1^1x_1^2
\end{aligned}$$

PCP-based NE computation methods

$$A_i^n(x^{-n}) = \sum_{\substack{\omega\in\Omega \\ \omega_n=i}} a^n(\omega) \prod_{\substack{\omega_{n'}=j \\ n'\neq n}} x_j^{n'}$$

Example (Polynomials associated to a game)

ω_1	ω_2	ω_3	a^1	a^2	a^3
0	0	0	2	8	4
0	0	1	1	5	1
0	1	0	8	1	3
0	1	1	3	4	8
1	0	0	7	2	5
1	0	1	4	3	6
1	1	0	6	7	2
1	1	1	5	6	7

$$\begin{aligned}
A_0^1\left(x^2,x^3\right) &= 2x_0^2x_0^3 + 1x_0^2x_1^3 + 8x_1^2x_0^3 + 3x_1^2x_1^3 \\
A_1^1\left(x^2,x^3\right) &= 7x_0^2x_0^3 + 4x_0^2x_1^3 + 6x_1^2x_0^3 + 5x_1^2x_1^3 \\
A_0^2\left(x^1,x^3\right) &= 8x_0^1x_0^3 + 5x_0^1x_1^3 + 2x_1^1x_0^3 + 3x_1^1x_1^3 \\
A_1^2\left(x^1,x^3\right) &= x_0^1x_0^3 + 4x_0^1x_1^3 + 7x_1^1x_0^3 + 6x_1^1x_1^3 \\
A_0^3\left(x^1,x^2\right) &= 4x_0^1x_0^2 + 3x_0^1x_1^2 + 5x_1^1x_0^2 + 2x_1^1x_1^2 \\
A_1^3\left(x^1,x^2\right) &= x_0^1x_0^2 + 8x_0^1x_1^2 + 6x_1^1x_0^2 + 7x_1^1x_1^2
\end{aligned}$$

PCP-based NE computation methods

Inequations domain

$$\mathcal{D}^N = \left\{ x = (x_i^n)_{(n,i)\in I_N}, x_i^n \geq 0,\ A_i^n(x^{-n}) \geq 1, \forall (n,i) \right\}.$$

PCP rewriting

For any $Z, W \subseteq I_N = \{(n,i), n \in P, i \in S_n\}$, define:

$$\left\{ \begin{array}{rl} & x \in \mathcal{D}^N, \\ x_i^n = 0, & \forall (n,i) \in Z, \\ A_i^n(x^{-n}) = 1, & \forall (n,i) \in W. \end{array} \right. \qquad (\mathcal{S}^{Z,W})$$

Nash equilibrium computation

- Find $Z, W \subseteq I_N$, $Z \cap W = \emptyset$ and $Z \cup W = I_N$ (Compl) Such that $Sol\left(\mathcal{S}^{Z,W}\right) \neq \emptyset$
- When the PCP is non-degenerate, $Sol\left(\mathcal{S}^{Z,W}\right)$ is a singleton corresponding to a Nash equilibrium

Complementary node and Nash equilibrium

Example (Polynomial system associated to a game)

$x_i^n\left(A_i^n(x^{-n})-1\right)=0, \forall (n,i)\in I_N$,where:

$$\begin{aligned}
A_0^1\left(x^2,x^3\right) &= 2x_0^2x_0^3+x_0^2x_1^3+8x_1^2x_0^3+3x_1^2x_1^3\\
A_1^1\left(x^2,x^3\right) &= 7x_0^2x_0^3+4x_0^2x_1^3+6x_1^2x_0^3+5x_1^2x_1^3\\
A_0^2\left(x^1,x^3\right) &= 8x_0^1x_0^3+5x_0^1x_1^3+2x_1^1x_0^3+3x_1^1x_1^3\\
A_1^2\left(x^1,x^3\right) &= x_0^1x_0^3+4x_0^1x_1^3+7x_1^1x_0^3+6x_1^1x_1^3\\
A_0^3\left(x^1,x^2\right) &= 4x_0^1x_0^2+3x_0^1x_1^2+5x_1^1x_0^2+2x_1^1x_1^2\\
A_1^3\left(x^1,x^2\right) &= x_0^1x_0^2+8x_0^1x_1^2+6x_1^1x_0^2+7x_1^1x_1^2
\end{aligned}$$

If $Z=\{(3,1)\}$ and $W=\{(1,0),(1,1),(2,0),(2,1),(3,0)\}$, we can show that $Sol\left(\mathcal{S}^{Z,W}\right)$ is a singleton,

Complementary node and Nash equilibrium

Example (Polynomial system associated to a game)

$$
\begin{array}{rcl}
x_1^3 & = & 0 \\
A_0^1(x^2,x^3) & = & 2x_0^2x_0^3 + x_0^2x_1^3 + 8x_1^2x_0^3 + 3x_1^2x_1^3 = 1 \\
A_1^1(x^2,x^3) & = & 7x_0^2x_0^3 + 4x_0^2x_1^3 + 6x_1^2x_0^3 + 5x_1^2x_1^3 = 1 \\
A_0^2(x^1,x^3) & = & 8x_0^1x_0^3 + 5x_0^1x_1^3 + 2x_1^1x_0^3 + 3x_1^1x_1^3 = 1 \\
A_1^2(x^1,x^3) & = & x_0^1x_0^3 + 4x_0^1x_1^3 + 7x_1^1x_0^3 + 6x_1^1x_1^3 = 1 \\
A_0^3(x^1,x^2) & = & 4x_0^1x_0^2 + 3x_0^1x_1^2 + 5x_1^1x_0^2 + 2x_1^1x_1^2 = 1
\end{array}
$$

- After simplifications...

Complementary node and Nash equilibrium

Example (Polynomial system associated to a game)

$$\begin{aligned}
x_1^3 &= 0\\
2x_0^2x_0^3 + 8x_1^2x_0^3 &= 1\\
7x_0^2x_0^3 + 6x_1^2x_0^3 &= 1\\
8x_0^1x_0^3 + 2x_1^1x_0^3 &= 1\\
x_0^1x_0^3 + 7x_1^1x_0^3 &= 1\\
4x_0^1x_0^2 + 3x_0^1x_1^2 + 5x_1^1x_0^2 + 2x_1^1x_1^2 &= 1
\end{aligned}$$

Computing a solution by hand is possible here:

$$x_1^1 = \frac{7}{5}x_0^1 \ ; \ x_1^2 = \frac{5}{2}x_0^2 \ ; \ x_0^1x_0^2 = \frac{2}{51} \ ; \ x_0^2x_0^3 = \frac{1}{22} \ ; \ x_0^1x_0^3 = \frac{5}{54}.$$

$$x_0^1 = \sqrt{\frac{110}{1377}} \ ; \ x_1^1 = \sqrt{\frac{1078}{6885}} \ ; \ x_0^2 = \sqrt{\frac{18}{935}} \ ; \ x_1^2 = \sqrt{\frac{45}{374}} \ ; \ x_0^3 = \sqrt{\frac{85}{792}}$$

How to solve polynomial complementarity problems?

There are three main approaches:

- Directly solving the system

$$x_i^n\left(A_i^n(x^{-n})-1\right)=0,\forall(n,i)\in I_N\to\text{Inefficient!}$$

- Enumerating the sets $W\subseteq I_N$ and trying to solve the systems $\mathcal{S}^{\overline{W},W}\Rightarrow$ Can be efficient! [Porter et al., 2008]
- Following a "well-chosen" finite path of arcs and nodes $(Z_t,W_t)_{t=1..T}\Rightarrow$ Can also be efficient! [Wilson, 1971, Lemke and Howson, 1964]

Support enumeration method

The algorithm of [Porter et al., 2008] consists in enumerating supports $W \subseteq I_N$ until a system $\mathcal{S}^{\overline{W},W}$ becomes solvable

```
Data: A non-degenerate game $\Gamma = \langle P, \Omega, \{a_n\} \rangle$
Result: A Nash equilibrium $\bar{\xi}$
1 for $W \subseteq I_N$ do
2     if $\neg IRDA(\Gamma^W)$ then
3         if $Sol\left(\mathcal{S}^{\overline{W},W}\right) \neq \emptyset$ then
4             return $normalize\left(Sol\left(\mathcal{S}^{\overline{W},W}\right)\right)$
```

Line 1 If $W \subsetneq W'$, W is encountered before W'

Line 2 Γ^W is the game where only strategies in W are considered

Line 2 $IRDA(\Gamma^W) =$`True` iff some alternative is removed by IRDA

Line 3 If Γ is non-degenerate and $Sol\left(\mathcal{S}^{\overline{W},W}\right) \neq \emptyset$,returns a NE

Support enumeration method

Example (Polynomial system)

$x_i^n \left(A_i^n(x^{-n}) - 1\right) = 0, \forall (n,i) \in I_N$,where:

$$\begin{aligned}
A_0^1\left(x^2, x^3\right) &= 2x_0^2x_0^3 + x_0^2x_1^3 + 8x_1^2x_0^3 + 3x_1^2x_1^3\\
A_1^1\left(x^2, x^3\right) &= 7x_0^2x_0^3 + 4x_0^2x_1^3 + 6x_1^2x_0^3 + 5x_1^2x_1^3\\
A_0^2\left(x^1, x^3\right) &= 8x_0^1x_0^3 + 5x_0^1x_1^3 + 2x_1^1x_0^3 + 3x_1^1x_1^3\\
A_1^2\left(x^1, x^3\right) &= x_0^1x_0^3 + 4x_0^1x_1^3 + 7x_1^1x_0^3 + 6x_1^1x_1^3\\
A_0^3\left(x^1, x^2\right) &= 4x_0^1x_0^2 + 3x_0^1x_1^2 + 5x_1^1x_0^2 + 2x_1^1x_1^2\\
A_1^3\left(x^1, x^2\right) &= x_0^1x_0^2 + 8x_0^1x_1^2 + 6x_1^1x_0^2 + 7x_1^1x_1^2
\end{aligned}$$

You can check that $Sol\left(\mathcal{S}^{\overline{W'},W'}\right)$ is empty when $W' \subsetneq W = \{(1,0),(1,1),(2,0),(2,1),(3,0)\}$ (there is a solution to the system of equations, but it does not belong to $\mathcal{D}^N$)

Exercise: Try it, for example for $W' = \{(1,0),(2,0),(3,0)\}$

Support enumeration method

Example (Polynomial system)

Exercise: Try it, for example for $W' = \{(1,0),(2,0),(3,0)\}$
$\Rightarrow Z' = \{(1,1),(2,1),(3,1)\}$
So, $x^1_1 = x^2_1 = x^3_1 = 0$ and

$$\begin{aligned}
A^1_0\left(x^2, x^3\right) &= 2x^2_0 x^3_0 = 1 \\
A^1_1\left(x^2, x^3\right) &= 7x^2_0 x^3_0 \geq 1 \\
A^2_0\left(x^1, x^3\right) &= 8x^1_0 x^3_0 = 1 \\
A^2_1\left(x^1, x^3\right) &= x^1_0 x^3_0 \geq 1 \\
A^3_0\left(x^1, x^2\right) &= 4x^1_0 x^2_0 = 1 \\
A^3_1\left(x^1, x^2\right) &= x^1_0 x^2_0 \geq 1
\end{aligned}$$

This system doesn't have a solution since, for example, we cannot have both $8x^1_0 x^3_0 = 1$ and $x^1_0 x^3_0 \geq 1$

Path-following methods

- The path-following methods of [Lemke and Howson, 1964] (in the 2-player case) and [Wilson, 1971] (in the N-player case) also explore a sequence of systems $\mathcal{S}^{Z,W}$
- However, these systems are not formed from complementary pairs $(\overline{W},, W)$, but from (n,i)-almost complementary pairs (Z, W)
- Furthermore, [Wilson, 1971] uses sub-PCP $\mathcal{S}_k$ at level $1 \leq k \leq N$, built from the initial PCP $\mathcal{S}$ and from an arbitrary pure joint strategy ω^0
- The approach of [Lemke and Howson, 1964] can be seen as a particular case of [Wilson, 1971], where only linear systems have to be solved at each step (and not polynomial systems)

Complementary and almost-complementary nodes, arcs and paths

Complementary node

Let PCP $\mathcal{S}$ be given. If non-empty, $Sol\left(\mathcal{S}^{Z,W}\right)$, where $Z \cup W = I_N, Z \cap W = \emptyset$, is called a complementary node of $\mathcal{S}$

(n, i)-almost complementary node

Let PCP $\mathcal{S}$, $1 \leq n \leq N$ and $i \in S_n$ be given. If non-empty, $Sol\left(\mathcal{S}^{Z,W}\right)$, where $|Z| + |W| = |I_N|$ and $(I_N \setminus \{(n, i)\}) \subseteq (Z \cup W)$, is called a (n, i)-almost complementary node, denoted $\rho(Z, W)$.

(n, i)-almost complementary arc

Let PCP $\mathcal{S}$, $1 \leq n \leq N$ and $i \in S_n$ be given. If non-empty, $Sol\left(\mathcal{S}^{Z,W}\right)$, where $Z \cap W = \emptyset$ and $Z \cup W = I_N \setminus \{(n, i)\}$, is a (n, i)-almost complementary arc, denoted $\gamma(Z, W)$

In non-degenerate games, $\rho(Z, W)$ is a point and $\gamma(Z, W)$ an arc

Geometrical interpretation

Complementary node ⇔ PCP Solution ⇔ Nash equilibrium

Neighbourhood of an almost-complementary node

Let $\rho(Z, W)$ be a (n, i)-AC node defined from $Sol\left(\mathcal{S}^{Z,W}\right)$

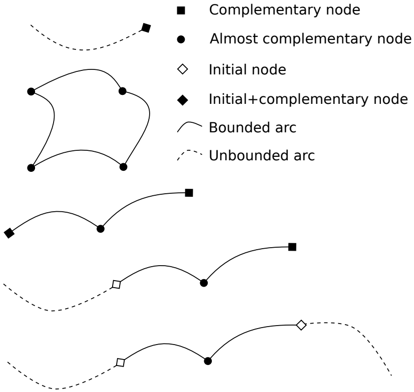
- Complementary → neighbours a single (n, i)-AC arc, $\gamma(Z \setminus \{(n, i)\}, W)$ or $\gamma(Z, W \setminus \{(n, i)\})$, since $Z \cap W = \emptyset$
- Almost-complementary: neighbours two (n, i)-AC arcs, $\gamma(Z \setminus W, W)$ and $\gamma(Z, W \setminus Z)$

Neighbourhood of an almost-complementary arc

Let $\gamma(Z, W)$ be a (n, i)-AC arc defined from $Sol\left(\mathcal{S}^{Z,W}\right)$

- There exist one or two distinct pairs (m, j) such that $Sol\left(\mathcal{S}^{Z\cup\{(m,j)\},W}\right)$ or $Sol\left(\mathcal{S}^{Z,W\cup\{(m,j)\}}\right)$ is non-empty
- These correspond to (n, i)-AC nodes

Geometrical interpretation



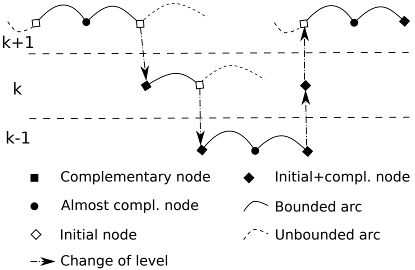
- This suggests a path-following algorithm, starting from one end of an AC-path and ending in a complementary node
- But... It is a bit more complex: How to find the first node of the path?
- Idea : Going through different "levels" of the game!

Defining PCPs at different levels

- Let us assume that an arbitrary joint strategy ω^0 is fixed, and that the k first players respond to this strategy, played by the remaining players
- If we solve the corresponding PCP, we will find a complementary node "at level k"
- This will serve as a starting AC-node at level $k+1$ from which we will follow a path toward a complementary node at level $k+1$
- So doing, we will eventually find a Nash equilibrium. But...
- In the process we may have to decrease level from time to time

$\Rightarrow$ But it works!

Graphically



Sub-PCPs, initial nodes

- Let us consider a PCP $\mathcal{S}$, an arbitrary pure joint strategy $\omega^0 = (\omega_1^0, \ldots, \omega_N^0) \in \pi$, two integers $1 \le n \le k \le N$ and a pure strategy $i \in S_n$.
- Let $A_i^{n,k}\left(x^{\{1,..,k\}\setminus\{n\}}\right)$ be the multivariate polynomial:

$$A_i^{n,k}\left(x^{\{1,..,k\}\setminus\{n\}}\right) = \sum_{\substack{\omega\in\pi,\omega_n=i\\ \omega_m=\omega_m^0,\forall m>k}} a_\omega^n \prod_{\substack{\nu\le k,\\ \nu\neq n}} x_{\omega_\nu}^\nu .$$

- $A_i^{n,k}$ is obtained from A_i^n by fixing the values of variables $x_{\omega_m^0}^m$ to 1 and x_j^m to 0 whenever $m > k$ and $j \neq \omega_m^0$.
- If ξ is a joint mixed strategy, $A_i^{n,k}\left(\xi^{\{1,\ldots,k\}\setminus\{n\}}\right)$ is the expected disutility of player n playing action $i \in S_n$ when players $1, \ldots, k$ except n play their mixed strategy in ξ, while players $k+1, \ldots, N$ play their pure strategy in ω^0.

Sub-PCPs, initial nodes

Let us also define $I_k = \{(n,i), 1 \leq n \leq k, i \in S_n\}$.

Definition (Sub-PCP)

Let $\mathcal{S}$ be a given PCP and ω^0 a fixed pure joint strategy.
For $2 \leq k < N$, we define sub-PCP $\mathcal{S}_k$:

$$\forall (n,i) \in I_k, \left\{ \begin{array}{rcl} x_i^n & \geq & 0 \\ A_i^{n,k}\left(x^{\{1,..,k\}\setminus\{n\}}\right) & \geq & 1 \\ x_i^n \cdot \left(A_i^{n,k}\left(x^{\{1,..,k\}\setminus\{n\}}\right) - 1\right) & = & 0 \end{array} \right. \quad (\mathcal{S}_k)$$

For $n = 1$, PCP $\mathcal{S}_1$ is slightly different:

$$\left\{ \begin{array}{rcl} x_i^1 & \geq & 0, \forall i \in S_1 \\ x_i^1 \cdot \left(\frac{a^1_{(i,\omega^0_{-1})}}{\min_{j \in S_1} a^1_{(j,\omega^0_{-1})}} - 1 \right) & = & 0, \forall i \in S_1 \\ \sum_{i \in S_1} x_i^1 & = & 1 \end{array} \right. \quad (\mathcal{S}_1)$$

Sub-PCPs: Change of level

Definition (Polynomial subsystems)

We define polynomial systems $\mathcal{S}_k^{Z,W}$ for any $Z, W \subseteq I_k$:

$$\left\{ \begin{array}{rl} & x \in \mathcal{D}^k, \\ x_i^n = 0, & \forall (n,i) \in Z, \\ A_i^{n,k}(x^{\{1..k\} \setminus \{n\}}) = 1, & \forall (n,i) \in W, \end{array} \right. \qquad \left(\mathcal{S}_k^{Z,W}\right)$$

Proposition (Complementary node lifting)

- *Assume $\mathcal{S}_k^{Z,W}$ defines a complementary node of $\mathcal{S}_k$*
- *Then, $\mathcal{S}_{k+1}^{Z',W}$ defines a $(k+1, \omega_{k+1}^0)$-almost complementary arc of sub-PCP $\mathcal{S}^{k+1}$, where $Z' = Z \cup \left\{(k+1, j), j \neq \omega_{k+1}^0\right\}$.*
- *Furthermore, $\mathcal{S}_{k+1}^{Z',W}$ neighbours a single $(k+1, \omega_{k+1}^0)$-almost complementary node at level $k+1$ (it is an unbounded arc).*

Sub-PCPs: Change of level

Definition (Initial node)

- An initial node at level $k+1$ is a $(k+1, \omega_{k+1}^0)$-almost complementary node, solution to $\mathcal{S}_{k+1}^{Z,W}$, with $Z, W \subseteq I_{k+1}$, such that only one of its neighbouring arcs is bounded.
- It also satisfies:
  $(k+1, \omega_{k+1}^0) \notin Z$ and $(k+1, j) \in Z, \forall j \neq \omega_{k+1}^0$.

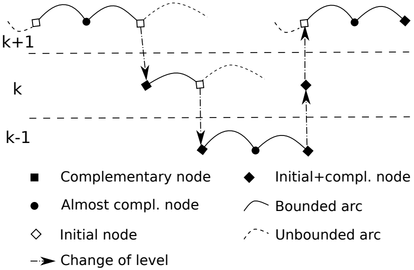
Proposition (Initial node descent)

- *Let $\mathcal{S}_k^{Z,W}$ define an initial node at level $k > 1$.*
  Let $Z' = Z \cap I_{k-1}$ and $W' = W \cap I_{k-1}$.
  Then, either $Z' \cap W' = \{(\nu, j)\}$ or $Z' \cap W' = \emptyset$.
- *In the first case, either $\mathcal{S}_{k-1}^{Z' \setminus \{(\nu,j)\}, W'}$ or $\mathcal{S}_{k-1}^{Z', W' \setminus \{(\nu,j)\}}$ defines a complementary node at level $k-1$.*
- *In the second case, $\mathcal{S}_{k-1}^{Z',W'}$ defines a complementary node.*

Path-following algorithm

Now, we have all the ingredients to build a path-following procedure to find a NE of non-degenerate game Γ:

1. Build PCP $\mathcal{S}$, choose ω^0 and build sub-PCPs $\mathcal{S}_k, 1 \leq k \leq N$
2. Find a solution to $\mathcal{S}_1$ (complementary node at level 1)
   $\Rightarrow$ lift to level 2
3. Follow the path of AC-nodes and arcs (using lifts and descents if necessary) until a complementary node at level N is reached



Path-following example

Example (Same game as before)

ω_1	ω_2	ω_3	u^1	u^2	u^3	a^1	a^2	a^3
0	0	0	6	0	4	2	8	4
0	0	1	7	3	7	1	5	1
0	1	0	0	7	4	8	1	3
0	1	1	5	4	0	3	4	8
1	0	0	1	6	3	7	2	5
1	0	1	4	5	2	4	3	6
1	1	0	2	1	6	6	7	2
1	1	1	3	2	1	5	6	7

We will assume that $\omega^0 = (0, 0, 0)$

Path-following example

Example (Sub-PCP polynomials)

$$
\begin{array}{rcl}
A_0^{1,1} = 1 & ; & A_1^{1,1} = \dfrac{7}{2} \\
A_0^{1,2}\left(x^2\right) = 2x_0^2 + 8x_1^2 & ; & A_1^{1,2}\left(x^2\right) = 7x_0^2 + 6x_1^2 \\
A_0^{2,2}\left(x^1\right) = 8x_0^1 + 2x_1^1 & ; & A_1^{2,2}\left(x^1\right) = x_0^1 + 7x_1^1 \\
A_0^{1,3}\left(x^2, x^3\right) & = & 2x_0^2x_0^3 + x_0^2x_1^3 + 8x_1^2x_0^3 + 3x_1^2x_1^3 \\
A_1^{1,3}\left(x^2, x^3\right) & = & 7x_0^2x_0^3 + 4x_0^2x_1^3 + 6x_1^2x_0^3 + 5x_1^2x_1^3 \\
A_0^{2,3}\left(x^1, x^3\right) & = & 8x_0^1x_0^3 + 5x_0^1x_1^3 + 2x_1^1x_0^3 + 3x_1^1x_1^3 \\
A_1^{2,3}\left(x^1, x^3\right) & = & x_0^1x_0^3 + 4x_0^1x_1^3 + 7x_1^1x_0^3 + 6x_1^1x_1^3 \\
A_0^{3,3}\left(x^1, x^2\right) & = & 4x_0^1x_0^2 + 3x_0^1x_1^2 + 5x_1^1x_0^2 + 2x_1^1x_1^2 \\
A_1^{3,3}\left(x^1, x^2\right) & = & x_0^1x_0^2 + 8x_0^1x_1^2 + 6x_1^1x_0^2 + 7x_1^1x_1^2
\end{array}
$$

Path-following example

Example (Initialization)

$$(\mathcal{S}_1)\begin{cases} x_0^1, x_1^1 & \geq & 0 \\ x_0^1 \cdot \left(A_0^{1,1} - 1\right) & = & 0 \\ x_1^1 \cdot \left(A_1^{1,1} - 1\right) & = & 0 \\ x_0^1 + x_1^1 = 1. \end{cases}$$

Z	W	*Remark*
$\{(1,1)\}$	$\{(1,0)\}$	*complementary*

Lifting of the (1,0)-complementary node ⇒ (2,0)-almost complementary arc: $Z^1 = \{(1,1),(2,1\}$; $W^1 = \{(1,0)\}$

Path-following example

Example (Complementary node lifting)

We try to solve $\left(\mathcal{S}_2^{Z,W}\right)$ for the following pairs, (Z, W) until we get a "solvable" system:

Z	W	*Remark*
$(1,1),(2,1)$	$(1,0),(1,1)$	$(2,0)$ – *almost complementary*
$(1,1),(2,1)$	$(1,0),(2,0)$	*complementary*
$(1,1),(2,1)$	$(1,0),(2,1)$	$(2,0)$ – *almost complementary*

Path-following example

Example (Complementary node lifting)

We try to solve $\left(\mathcal{S}_2^{Z,W}\right)$ for the following pairs, (Z, W) until we get a "solvable" system:

Z	W	*Remark*
$(1,1),(2,1)$	$(1,0),(1,1)$	$(2,0)$ – *almost complementary*
$(1,1),(2,1)$	$(1,0),(2,0)$	*complementary*
$(1,1),(2,1)$	$(1,0),(2,1)$	$(2,0)$ – *almost complementary*

This is the only feasible AC-node (have to try to solve every systems)...

Path-following example

Example (Path-following at level 2)

The "other arc" leaving current (2,0)-AC node is $Z^2 = \{(1,1)\}$, $W^2 = \{(1,0),(2,1)\}$

The potential $(2,0)$-AC nodes neighbouring arc (Z^2, W^2) are:

Z	W	*Remark*
$(1,0),(1,1)$	$(1,0),(2,1)$	*incoherent*
$(1,1),(2,0)$	$(1,0),(2,1)$	*complementary*
$(1,1),(2,1)$	$(1,0),(2,1)$	$previous\ (2,0) - AC\ node$
$(1,1)$	$(1,0),(1,1),(2,1)$	*almost complementary*
$(1,1)$	$(1,0),(2,0),(2,1)$	*complementary*

Path-following example

Example (Path-following at level 2)

The "other arc" leaving current (2,0)-AC node is $Z^2 = \{(1,1)\}$, $W^2 = \{(1,0),(2,1)\}$
The potential $(2,0)$-AC nodes neighbouring arc (Z^2, W^2) are:

Z	W	*Remark*
$(1,0),(1,1)$	$(1,0),(2,1)$	*incoherent*
$(1,1),(2,0)$	$(1,0),(2,1)$	*complementary*
$(1,1),(2,1)$	$(1,0),(2,1)$	*previous* $(2,0)-$ *AC node*
$(1,1)$	$(1,0),(1,1),(2,1)$	*almost complementary*
$(1,1)$	$(1,0),(2,0),(2,1)$	*complementary*

Only feasible node

Path-following example

Example (Path-following at level 2)

The "other arc" leaving current (2,0)-AC node is $Z^3 = \emptyset$, $W^3 = \{(1,0),(1,1),(2,1)\}$
The potential $(2,0)$-AC nodes neighbouring arc (Z^3, W^3) are:

Z	W	*Remark*
$\{(1,1)\}$	$\{(1,0),(1,1),(2,1)\}$	*previous* $(2,0)-$*AC node*
$\emptyset$	$\{(1,0),(1,1),(2,0),(2,1)\}$	*complementary*
$\{(2,0)\}$	$\{(1,0),(1,1),(2,1)\}$	*complementary*
$\{(1,0)\}$	$\{(1,0),(1,1),(2,1)\}$	*almost complementary*
$\{(2,1)\}$	$\{(1,0),(1,1),(2,1)\}$	*almost complementary*

Path-following example

Example (Path-following at level 2)

The "other arc" leaving current (2,0)-AC node is $Z^3 = \emptyset$, $W^3 = \{(1,0),(1,1),(2,1)\}$
The potential $(2,0)$-AC nodes neighbouring arc (Z^3, W^3) are:

Z	W	*Remark*
$\{(1,1)\}$	$\{(1,0),(1,1),(2,1)\}$	*previous* $(2,0) - AC$ *node*
$\emptyset$	$\{(1,0),(1,1),(2,0),(2,1)\}$	*complementary*
$\{(2,0)\}$	$\{(1,0),(1,1),(2,1)\}$	*complementary*
$\{(1,0)\}$	$\{(1,0),(1,1),(2,1)\}$	*almost*
$\{(2,1)\}$	$\{(1,0),(1,1),(2,1)\}$	*complementary*

Only feasible node

Path-following example

Example (Node lifting)

First, a $(3,0)$-almost complementary arc[2] is computed at level 3:
$Z^4 = \{(3,1)\}$, $W^4 = \{(1,0),(1,1),(2,0),(2,1)\}$
The potential (3,0)-AC nodes neighbouring arc (Z^4, W^4) are:

Z	W	*Remark*
$\{(3,0),(3,1)\}$	$\{(1,0),(1,1),(2,0),(2,1)\}$	*incoherent*
$\{(3,1)\}$	$\{(1,0),(1,1),(2,0),(2,1),(3,0)\}$	*complementary*
$\{(3,1)\}$	$\{(1,0),(1,1),(2,0),(2,1),(3,1)\}$	*almost comp.*

---

[2] Initial nodes ate level 3 are characterized by $(3,i) \in Z, \forall i \neq 0$

Path-following example

Example (Node lifting)

First, a $(3,0)$-almost complementary arc[3] is computed at level 3:
$Z^4 = \{(3,1)\}$, $W^4 = \{(1,0),(1,1),(2,0),(2,1)\}$
The potential (3,0)-AC nodes neighbouring arc (Z^4, W^4) are:

Z	W	*Remark*
$\{(3,0),(3,1)\}$	$\{(1,0),(1,1),(2,0),(2,1)\}$	*incoherent*
$\{(3,1)\}$	$\{(1,0),(1,1),(2,0),(2,1),(3,0)\}$	*complementary*
$\{(3,1)\}$	$\{(1,0),(1,1),(2,0),(2,1),(3,1)\}$	*almost comp.*

Only feasible node

Solution:

$$x_0^1 = \sqrt{\frac{110}{1377}}\ ;\ x_1^1 = \sqrt{\frac{1078}{6885}}\ ;\ x_0^2 = \sqrt{\frac{18}{935}}\ ;\ x_1^2 = \sqrt{\frac{45}{374}}\ ;\ x_0^3 = \sqrt{\frac{85}{792}}$$

---

[3] Initial nodes at a level 3 are characterized by $(3,i) \in Z, \forall i \neq 0$

Remarks

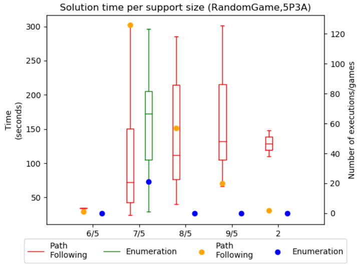
In the above example:

- Support enumeration explored $3^3 - 1 = 26$ joint strategy supports (but did not solve systems for all of them)
- Path-following solved:
  - 4 systems with 2 unknowns
  - 5 systems with 3 unkniwns
  - 1 system with 4 unknowns
  - 2 systems with 5 unknowns

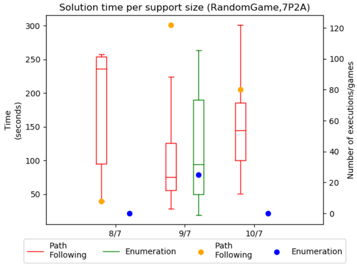
Path-following may encounter degenerate problems

- Problems where more then one (Z, W) correspond to an AC-node
- These can be solved by maintaining a list of potential AC-nodes and by performing depth-first search

Empirical comparison between support enumeration and path-following



Solution time per support size (RandomGame,5P3A)



Solution time per support size (RandomGame,7P2A)

- Enumeration finds equilibria with smaller size than path-following, in general[4]
- Enumeration better for "smaller" games, path-following for larger ones, but with more variability (time, supports size...)

---

[4] Support enumeration may miss smaller *unbalanced* supports' sizes in case of degenerate games

Table of Contents

Summary

- PCP approaches are powerful ways to represent NE search in games
- They come up with various algorithms to explore the set of potential complementary points
- They rely on a basic building block: polynomial systems solver

What we did not describe here:

- How to handle degenerate games
- The ability to model other kinds of games: Hypergraphical games, bayesian games can be modelled as PCP!
- Other approaches to Nash equilibrium computation

Related approaches to NE computation

- Homotopy approaches. These are arc-following approaches using *homotopy methods* [Govindan and Wilson, 2003], [Govindan and Wilson, 2004], [Blum et al., 2006], [Herings and Peeters, 2010].
  - Following a parametrized continuum of games, joining an arbitrary game with known equilibrium, to the game of interest. The "arc" followed is an arc of equilibria of the parametrized games.
  - Prone to numerical errors and potential non-convergence
- Discretized strategy space. Uniform strategy enumeration methods [Lipton et al., 2003], [Babichenko et al., 2014], [Berg and Sandholm, 2017]
  - Suggest to enumerate a space of discretized mixed strategies in order to find a ε-approximate strategy
  - Scalability with the dimension of the strategy space

Good time to stop, no?

And have a drink!

References I

Babichenko, Y., Barman, S., and Peretz, R. (2014).
Simple approximate equilibria in large games.
In *EC'14*, pages 753–770. ACM.

Berg, K. and Sandholm, T. (2017).
Exclusion method for finding nash equilibrium in multiplayer games.
In *AAAI'17*. AAAI Press.

Blum, B., Koller, D., and Shelton, C. (2006).
A continuation method for Nash equilibria in structured games.
Journal of Artificial Intelliegence Research, 25:457–502.

Govindan, S. and Wilson, R. (2003).
A global newton method to compute Nash equilibria.
Journal of Economic Theory, 110(1):65–86.

Govindan, S. and Wilson, R. (2004).
Computing Nash equilibria by iterated polymatrix approximation.
Journal of Economic Dynamics and Control, 28(7):1229–1241.

Herings, J. and Peeters, R. (2010).
Homotopy methods to compute equilibria in game theory.
Economic Theory, 42:119–156.

Howson, J. (1972).
Equilibria of polymatrix games.
Management Science, 18(5-part-1):312–318.

References II

Lemke, C. and Howson, J. (1964).
Equilibrium points of bimatrix games.
Journal of the Society for industrial and Applied Mathematics, 12(2):413–423.

Lipton, R., Markakis, E., and Mehta, A. (2003).
Playing large games using simple strategies.
In *EC'03*, pages 36–41. ACM.

Porter, R., Nudelman, E., and Shoham, Y. (2008).
Simple search methods for finding a Nash equilibrium.
Games and Economic Behavior, 63(2):664–669.

Wilson, R. (1971).
Computing equilibria of n-person games.
SIAM Journal on Applied Mathematics, 21(1):80–87.