Processes

Fractional Brownian Motion (FBM): **mono-fractal**

- Only Gaussian self-similar process with stationary increments
- $X(t) \overset{\text{fdd}}{=} a^H X(t/a)$ for all $a > 0$
- $\zeta(q) = qH$ for $q \in [0, \infty)$

Multifractal Random Walk (MRW): **multi-fractal**

- Non Gaussian processes with stationary increments
- Multifractal properties mimic Mandelbrot's multiplicative log-normal cascades
- $X(k) = \sum_{k=1}^{n} G_H(k) e^{\omega(k)}$

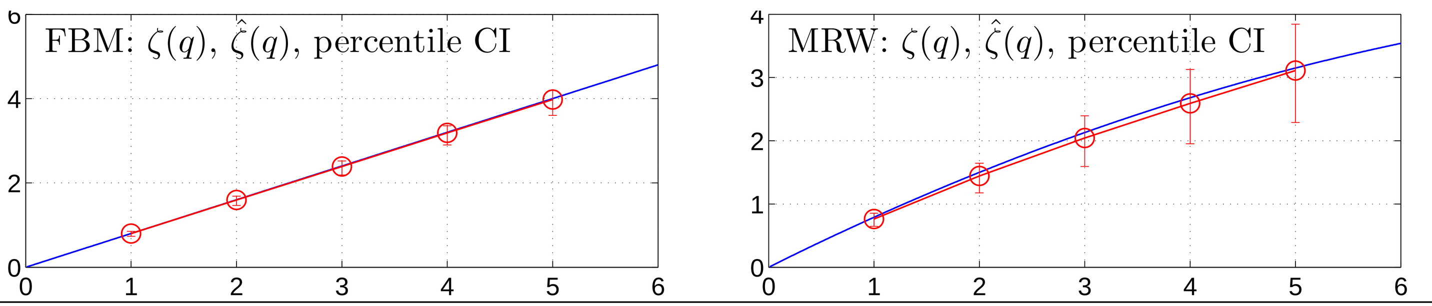
  $G_H(k)$: increments of FBM with parameter H.

  ω : independent of G_H, Gaussian, with non trivial covariance:

  $\text{cov}(\omega(k_1), \omega(k_2)) = c_2 \ln\left(\frac{L}{|k_1-k_2|+1}\right)$ for $|k_1 - k_2| < L$; 0 otherwise
- $\zeta(q) = (H + c_2)q - c_2 q^2/2$ for $q \in \left[0, \sqrt{2/c_2}\right]$

$\Rightarrow$ departure from linear behavior in q fully controlled by c_2

Scaling Exponents:
FBM
MRW
(CI: percentile)



Estimators and Confidence Limits

Estimators

$$\hat\zeta_X(q) = \sum_{j_1}^{j_2} w_j \left(\log_2 S(j,q) - \hat g_X(j,q)\right)$$

X: G - Gaussian, A - Asymptotic, B - Bootstrap

Confidence Intervals

1. Asymptotic $(1-\alpha)$ Confidence Intervals:

$$\text{CI}_{V_X}(q) = \left(\hat\zeta(q) - p\left(\tfrac{\alpha}{2}\right)\sqrt{V_X(q)}\,,\ \hat\zeta(q) - p\left(\tfrac{\alpha}{2}\right)\sqrt{V_X(q)}\right)$$

$V_X(q) = \sum_{j=j_1}^{j_2} w_{j,q}^2 \hat\sigma_X{}^2(j,q)$

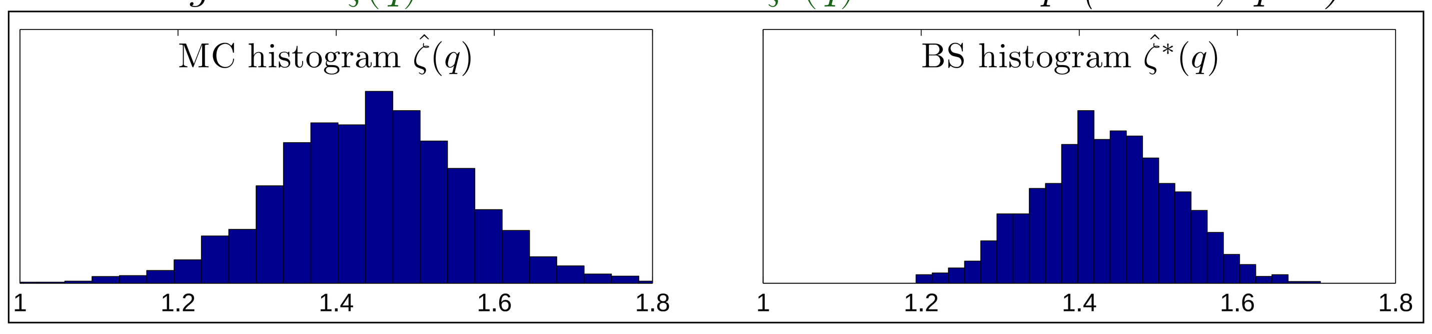
$p\left(\frac{\alpha}{2}\right)$: Quantile of standard Normal distribution

2. Bootstrap $(1-\alpha)$ Percentile Confidence Interval:

$\text{CI}_B(q) = \left(\hat\zeta^{*(b_1)}(q),\ \hat\zeta^{*(b_2)}(q)\right)$ $\hat\zeta^*(q)$: bootstrap distribution

$b_1 = \frac{B\alpha}{2}$, $b_2 = \frac{B(1-\alpha)}{2}$: Quantiles of bootstrap distribution

Histograms: $\hat\zeta(q)$ Monte Carlo $\hat\zeta^*(q)$ Bootstrap (MRW, q=2)



Performance Assessment

Monte Carlo Simulation:

Estimation procedures applied to N_{MC} realizations of FBM and MRW of length n:

Biases $\hat\beta(q) = \frac{\widehat{\mathbb{E}}\hat\zeta(q) - \zeta(q)}{q}$

Standard deviations $\hat s(q) = \sqrt{\widehat{\text{Var}}\hat\zeta(q)/q}$

Mean-square errors $\text{MSE}(q) = \sqrt{\hat s(q)^2 + \hat\beta(q)^2}$

Empirical coverages $\text{Cemp}(q) = \frac{\sum_{i=1}^{N_{MC}} \varepsilon(\zeta(q), \text{CI}_i(q))}{N_{MC}}$

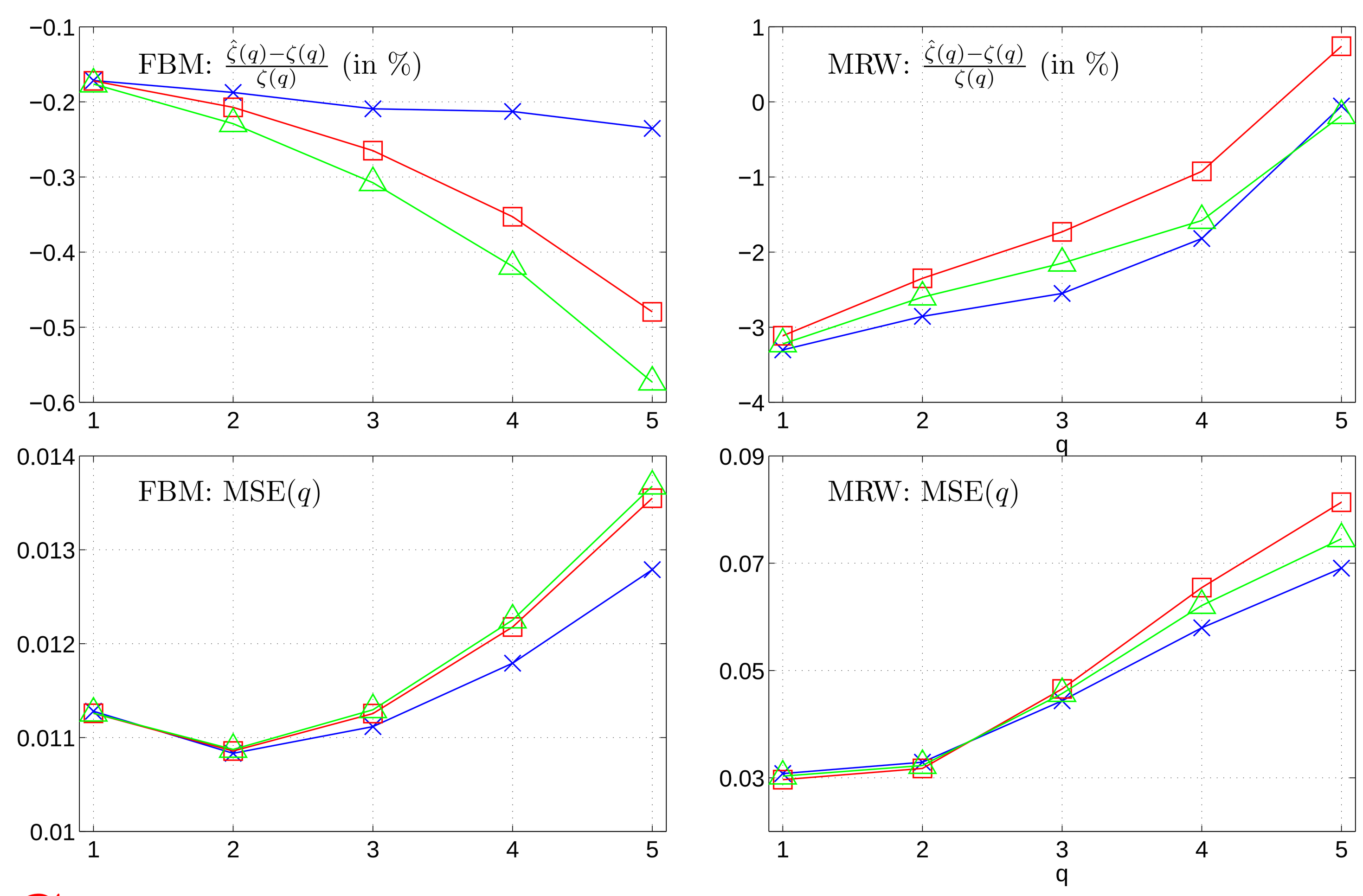
$\varepsilon(\zeta(q), \text{CI}(q)) = 1$ if $\zeta(q) \in \text{CI}(q)$ and 0 otherwise

Results obtained using Daubechies wavelets with $N = 3$ vanishing moments

FBM: $H = 0.8$ $n = 2^{15}$ Resamples $B = 200$

MRW: $(H, c_2) = (0.75, 0.08)$ $N_{MC} = 3000$ Blocklength $L = 6$

Performance of Estimators



Coverage of Confidence Intervals

$n = 2^{15}$: Empirical Coverage (in %): Nominal 95%									
		FBM				MRW			
q	X	CI$_B$	CI$_{V_G}$	CI$_{V_B}$	CI$_{V_A}$	CI$_B$	CI$_{V_G}$	CI$_{V_B}$	CI$_{V_A}$
1	*G*	89.9	88.2	**90.9**	88.0	72.5	30.0	**80.6**	52.0
	B	89.9	88.5	**91.0**	88.2	74.6	32.7	**82.2**	55.8
	A	89.9	88.3	**91.1**	88.1	73.4	31.0	**81.3**	53.8
2	*G*	88.1	86.8	**90.6**	85.9	82.2	35.2	**89.8**	74.1
	B	87.9	86.8	**90.3**	86.0	84.9	38.7	**90.9**	76.3
	A	87.9	86.9	**90.4**	86.0	83.6	37.3	**90.4**	75.5
3	*G*	86.7	87.7	**90.0**	86.0	88.9	32.7	**92.7**	82.4
	B	86.0	87.6	**89.4**	85.5	89.5	32.2	**91.5**	81.6
	A	85.7	87.6	**89.4**	85.4	89.0	32.7	**91.9**	81.8
4	*G*	86.3	89.2	**89.3**	86.3	**92.6**	28.4	91.4	80.5
	B	83.9	87.9	**88.4**	85.0	**91.4**	26.1	87.4	76.3
	A	83.4	87.5	**88.0**	84.2	**91.6**	26.6	89.3	78.1
5	*G*	84.6	**90.4**	89.4	85.8	**92.1**	26.2	86.3	73.9
	B	79.9	**88.3**	87.1	83.6	**88.9**	24.2	81.2	67.1
	A	79.1	**88.0**	86.8	82.8	**90.9**	24.3	84.1	70.9

Conclusions

Estimators

$g_B(j,q)$ has smallest bias and MSE for both Gaussian mono-fractal and non Gaussian multi-fractal process

Confidence Intervals

FBM: - CI_{V_B} very good
- CI_{V_G} and CI_B good

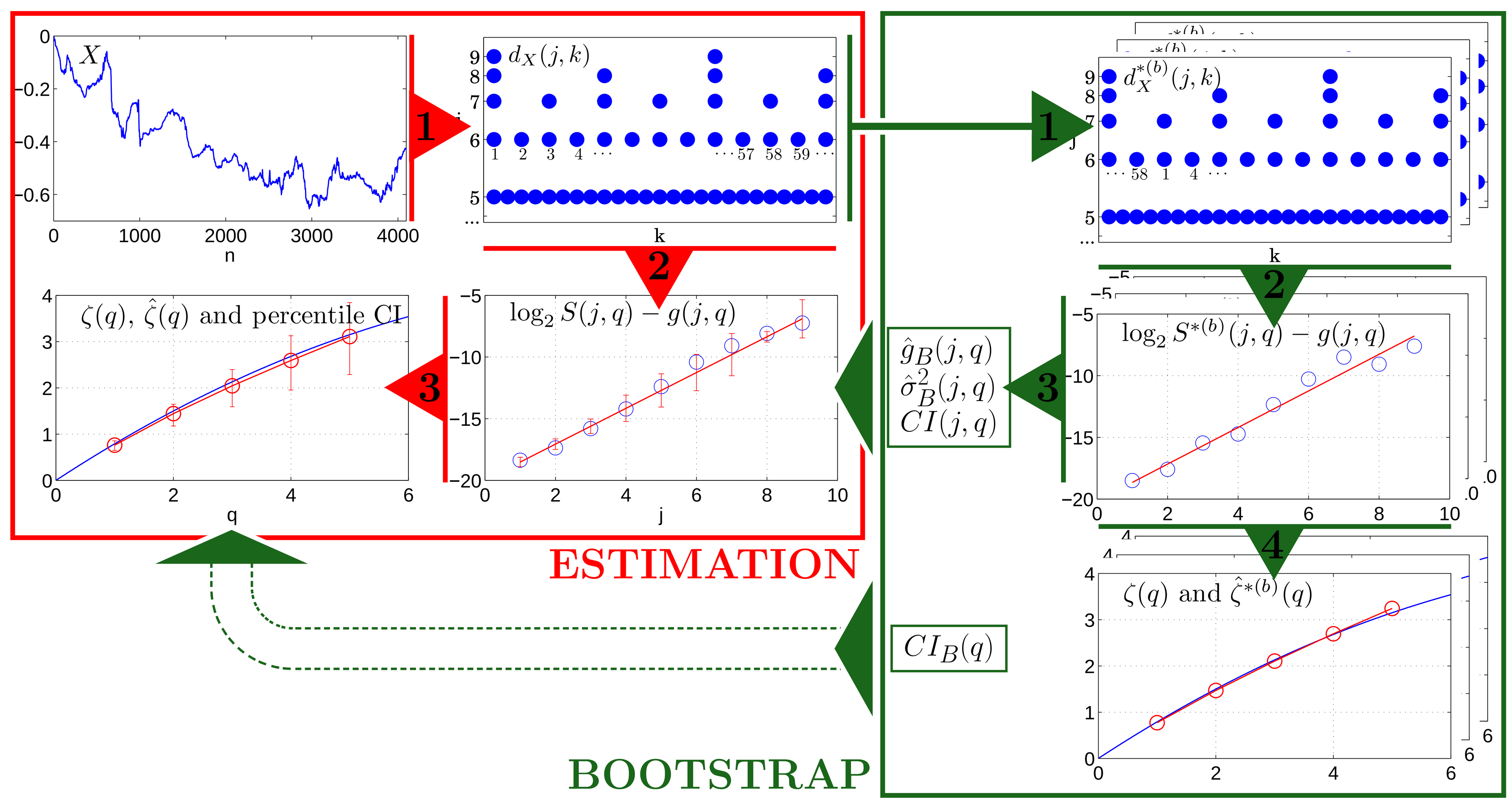
MRW: - CI_B and CI_{V_B} good
- CI_{V_G} has empirical coverage below 45% !

- Confidence intervals are the main benefit of bootstrap in Multifractal Analysis
- Bootstrap CI_B and CI_{V_B} highly relevant for both Gaussian mono-fractal and non Gaussian multi-fractal process
- "Optimal" block length is $L = 2N$ (wavelet time support)

Perspectives

- Double Bootstrap Techniques ?
- Time-Scale Bootstrap ?
- Bootstrap Hypothesis Test (mono- vs. multi.fractal) ?

Estimation from Single Realisation



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