BOOTSTRAP FOR MULTIFRACTAL ANALYSIS

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Summary

Multifractal analysis, which mainly consists in **estimating scaling exponents**, has become a popular tool for empirical data analysis. Although widely used in different applications, the statistical performance and the reliability of the estimation procedures are still poorly known. Notably, little is known about confidence intervals, though they are of first importance in applications. The present work investigates the **potential uses of bootstrap for multifractal estimation**:

1) Can the bootstrap be used to improve current estimation procedures?

2) Can the bootstrap be used to obtain reliable confidence intervals for scaling exponents?

Comparing the statistical performance of different estimators, our major result is to show that **bootstrap** based procedures provide us both with **accurate estimates** and **reliable confidence intervals**.

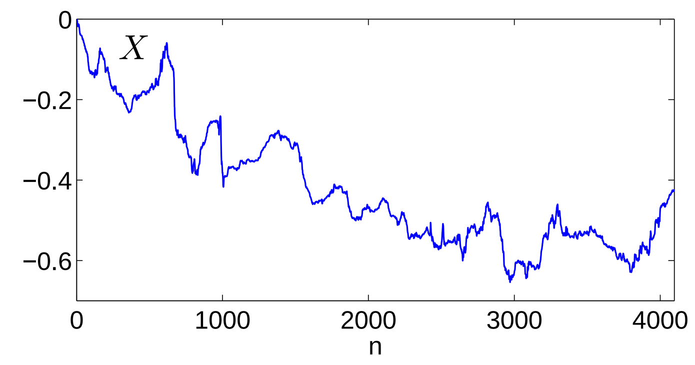
Scaling

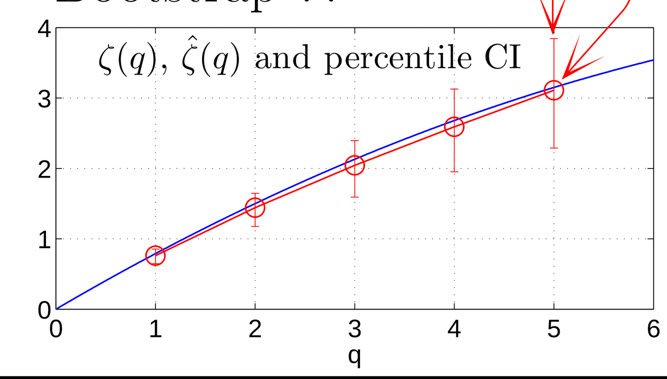
$\boxed{\frac{1}{n_j}\sum_{k=1}^{n_j}|d_X(j,k)|^q \simeq c_q|a|^{\zeta(q)}}$ for statistical orders $q \in [q_*^-, q_*^+]$
for scales $a = 2^j \in [a_m, a_M]$, $a_M/a_m >> 1$

$X(t),\ t \in [0,n)$ - Process under analysis
$d_X(j,k) = \langle \psi_{j,k} | X \rangle$ - Discrete wavelet transform (DWT) of X
$S(j,q) = \frac{1}{n_j}\sum_{k=1}^{n_j}|d_X(j,k)|^q$ - Structure functions
$\zeta(q)$ - Scaling exponents.

Goals:

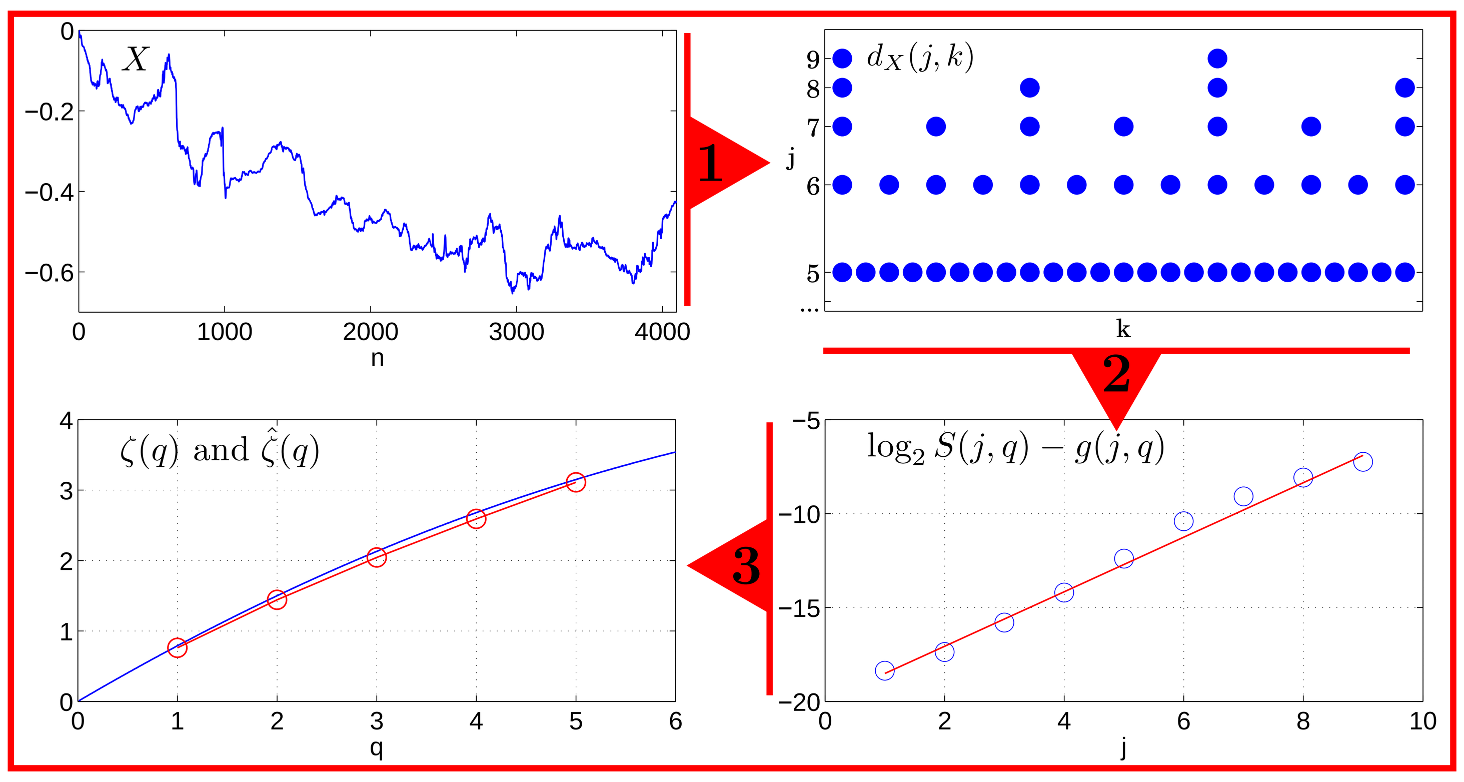
1. Estimate scaling exponents $\zeta(q)$
2. Estimate confidence intervals for $\zeta(q)$
   X mono-fractal Gaussian $\rightarrow$ Gaussian expansion
   X multi-fractal non Gaussian $\rightarrow$ Bootstrap ??





Estimation of Scaling Exponents $\zeta(q)$

1. Calculate DWT $d_X(j,k)$ of X
2. Structure functions $S(j,q) = \frac{1}{n_j}\sum_{k=1}^{n_j}|d_X(j,k)|^q$
3. Weighted linear regression in $\log_2 S(j,q)$ vs. $log_2 a = j$ diagrams:
   $\boxed{\hat\zeta(q) = \sum_{j=j_1}^{j_2} w_{j,q}\left(\log_2 S(j,q) - g(j,q)\right)}$ - bias correction $g(j,q)$



Bias and Confidence Limits:

Bias

$g(j,q) = \mathbb{E}\log_2 S(j,q) - \log_2 \mathbb{E}S(j,q)$
Estimate of $g(j,q)$: Asymptotic expansion or bootstrap

Confidence limits

Asymptotic expansion or bootstrap

1. Asymptotic Limits:
   $$\mathrm{Var}\hat\zeta(q) \simeq \sum_{j=j_1}^{j_2} w_{j,q}^2 \sigma^2(j,q)$$
   where $\sigma^2(j,q) = \mathrm{Var}\ \log_2 S(j,q)$
   Wavelet coefficients display only weak interscale correlation
   Estimate of $\sigma^2(j,q)$: Asymptotic expansion or bootstrap
2. Bootstrap Limits:
   From simulated distribution $\hat\zeta^*(q)$ of $\hat\zeta(q)$

Asymptotic Expansion

Asymptotic Expansion: Wavelet coefficients assumed uncorrelated at each scale
- Approximate formulae for change of variable $\Rightarrow$ Estimates $\hat g_A(j,q)$ and $\hat\sigma_A^2(j,q)$

Gaussian Expansion: Wavelet coefficients assumed uncorrelated and Gaussian
- Analytic expressions $\hat g_G(j,q)$ and $\hat\sigma_G^2(j,q)$. No quantity needs to be estimated.

Asymptotic Expansion:

$$\hat g_A(j,q) = -\frac{\log_2 e}{2n_j}\frac{\widehat{\mathrm{Var}}\ |d_X(j,\cdot)|^q}{\left(\widehat{\mathbb{E}}|d_X(j,\cdot)|^q\right)^2};\quad \hat\sigma_A^2(j,q) = \frac{(\log_2 e)^2}{n_j}\frac{\widehat{\mathrm{Var}}\ |d_X(j,\cdot)|^q}{\left(\widehat{\mathbb{E}}|d_X(j,\cdot)|^q\right)^2}$$

Gaussian Expansion:

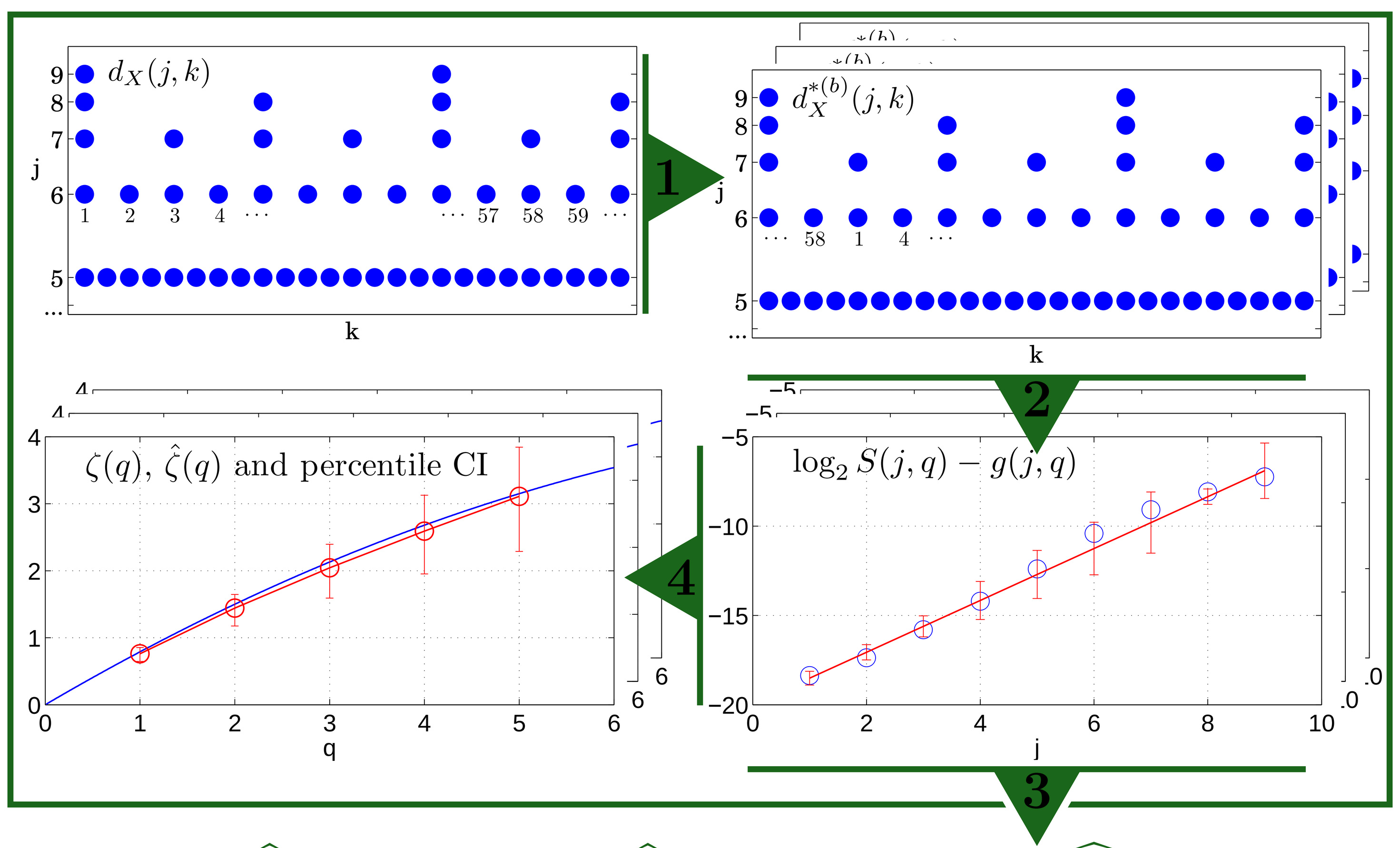
$$\hat g_G(j,q) = -\frac{\log_2 e}{2n_j}\left(\sqrt{\pi}\frac{\Gamma\left(q+\frac{1}{2}\right)}{\Gamma\left(\frac{q+1}{2}\right)^2} - 1\right);\quad \hat\sigma_G^2(j,q) = \frac{(\log_2 e)^2}{n_j}\left(\sqrt{\pi}\frac{\Gamma\left(q+\frac{1}{2}\right)}{\Gamma\left(\frac{q+1}{2}\right)^2} - 1\right)$$

Non Parametric Bootstrap Approach

Wavelet coefficients at each scale only weakly correlated $\Rightarrow$ Moving blocks bootstrap:

1. At each scale j, draw B bootstrap resamples blockwise, with replacement from sample of original coefficients (Block length: L):
   $$\{d_X(j,1),\cdots,d_X(j,n_j)\} \xrightarrow[\text{draw blockwise, with replacement}]{\text{repeat } B \text{ times}} \left\{d_X^{*(1)}(j,\cdot),\cdots,d_X^{*(n_j)}(j,\cdot)\right\}$$
2. Calculate, for each of the B resamples, and for each scale j and moment q of interest:
   $S^{*(b)}(j,q) = \frac{1}{n_j}\sum_{k=1}^{n_j}|d_X^{*(p)}(j,k)|^q,\quad b = 1,\cdots,B$ - structure functions BS replica
3. Calculate the bootstrap estimates $\hat g_B(j,q)$and $\hat\sigma_B^2(j,q)$ for each j and q of interest:
   $\hat g_B(j,q) = \widehat{\mathbb{E}}\log_2 S^*(j,q) - \log_2\widehat{\mathbb{E}}S^*(j,q);\quad \hat\sigma_B^2(j,q) = \widehat{\mathrm{Var}}\log_2 S^*(j,q).$
4. Calculate the B bootstrap estimates $\hat\zeta^{*(b)}(q)$ for each moment q of interest:
   $\hat\zeta^{*(b)}(q) = \sum_{j_1}^{j_2} w_j\left(\log_2 S^{*(b)}(j,q) - \hat g(j,q)\right),\ b = 1,\cdots,B$

Bootstrap:



$\hat g_B(j,q) = \widehat{\mathbb{E}}\log_2 S^*(j,q) - \log_2\widehat{\mathbb{E}}S^*(j,q);\quad \hat\sigma_B^2(j,q) = \widehat{\mathrm{Var}}\log_2 S^*(j,q).$