# MULTIFRACTAL ANALYSIS OF SELF-SIMILAR PROCESSES

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**ABSTRACT**

Scale invariance and multifractal analysis are nowadays widely used in applications. For modeling scale invariance in data, two classes of processes are classically in competition: self-similar processes and multiplicative cascades. They imply fundamentally different underlying (additive or multiplicative) mechanisms, hence the crucial practical need for data driven model selection. Such identification relies on properties often associated with the former: self-similarity, monofractality, linear scaling function, null $c_2$ parameter. By performing a wavelet leader based analysis of the multifractal properties of a large variety of self-similar processes, the present work contributes to a better disentangling of these different properties, sometimes confused one with another. Also, it leads to the formulation of conjectures regarding the scaling and multifractal properties of self-similar processes.

***Index Terms*—** self-similar; multifractal analysis; wavelet leader; monofractal; scaling function

## 1. INTRODUCTION

**Scale invariance.** Scale invariance or scaling is a paradigm that is nowadays widely and fruitfully used to analyze and model real-world data in applications of very different natures (cf. e.g., [1, 2] for a review). In essence, it amounts to assuming that data are not characterized by well-defined scales (of time) playing dominant roles, but instead that their dynamics involve all scales, which are hence equally important. Analyzing such data should therefore not rely on singling out specific scales but, instead, on identifying the mechanisms relating scales ones to the others.

**Scale invariance analysis.** From a data analysis point of view, *multifractal analysis* is now considered as the reference for scaling analysis. A recent formulation is based on specific multiresolution quantities, the wavelet leaders $L_X$ of the data $X$ [3, 4]. It states that the sums over time of the $q$-th power of $L_X(j,k)$ at fixed analysis scale $a = 2^j$ behave as a power-law w.r.t. scale in the limit of small scales, $a \to 0$: $(1/n_j)\sum_k L_X^q(j,k) = F_q 2^{j\zeta(q)}$ (with $n_j$ the number of $L_X$ actually computed at scale $2^j$). The $\zeta(q)$ are termed *scaling exponents* or *scaling function* and are by construction a concave function of $q$ [3]. By taking the Legendre transform of $\zeta(q)$, one obtains (an upper bound of) the *multifractal spectrum* $\mathcal{D}(h)$. Together, the functions $\zeta(q)$ and $\mathcal{D}(h)$ convey rich information on the scaling properties and the local regularity fluctuations of $X$ (cf. e.g., [5, 4]). Wavelet leader multifractal analysis is detailed in Section 2.

**Scale invariance modeling.** For data modeling, two categories of stochastic processes are most prominent and in competition to model scale invariance. The first, multiplicative cascade (or martingale) constructions (hereafter referred to as *MC*), have the following scaling properties: for some range of orders $q$ which includes $q = 0$ and for which $\mathbb{E}|X(1)|^q < +\infty$

$$\mathbb{E}|X(t+\tau) - X(t)|^q = \lambda_q^{MC}|\tau|^{\eta(q)}, \quad (1)$$

where $\eta(q)$ is a strictly concave function that is specified by the $q$-th order moments of the positive random variables entering the construction (cf. [6, 7, 8, 9] for details). Second, self-similar processes with stationary increments (below referred to as $H$-sssi) have scaling properties conveyed by the fact that $\forall q$ for which $\mathbb{E}|X(1)|^q < +\infty$

$$\mathbb{E}|X(t+\tau) - X(t)|^q = \lambda_q^{SS}|\tau|^{qH}, \quad (2)$$

where $H > 0$ is the so-called self-similarity parameter of $X$ (cf. e.g., [10] for a review). It is often considered to be of crucial importance to decide whether data are better modeled by one process class or the other: Indeed, $H$-sssi relies on the single parameter $H$ for describing scaling properties and is deeply related with random walks (hence, with additive structure), while *MC* involve the entire non-linear function $\eta(q)$ to describe scaling and are tied to a multiplicative structure. The investigation of this fundamental issue has often been based on the case example of fractional Brownian motion (fBm), the only jointly Gaussian $H$-sssi process. Its scaling function $\zeta(q)$, measured from the wavelet leaders, has been shown to take a particularly simple form, $\forall q \in \mathbb{R}: \zeta(q) = qH$ [3]. Equivalently, its multifractal spectrum reduces to a single point, $\mathcal{D}(h = H) = 1$ and $\mathcal{D}(h \neq H) = -\infty$, and fBm is hence referred to as *monofractal*. Because fBm is considered a (if not *the*) canonical process to model scale invariance and is widely used in applications, discrimination between $H$-sssi and *MC* has often been based on heuristically identifying the former with being monofractal with $\zeta(q)$ linear in $q$, and the latter with being multifractal with strictly concave $\zeta(q)$. This heuristic does not hold in general, as the known counter-example of Levy-stable processes shows: These are theoretically shown to be exactly $H$-sssi but are *not* monofractal [11] (i.e., their $\mathcal{D}(h)$ does not collapse on a single point). Furthermore, for a large variety of $H$-sssi processes, the multifractal properties (i.e., $\zeta(q)$ and $\mathcal{D}(h)$) remain to be studied theoretically.

**Contributions.** In this context, the present work aims at contributing to the multifractal analysis of $H$-sssi processes by disentangling the three properties $H$-sssi, monofractal and linear scaling function $\zeta(q)$, as well as clarifying their possibly intertwined relations. To that end, multifractal analysis and the wavelet leader multifractal formalism (detailed in Section 2) is applied to large classes of $H$-sssi processes (defined in Section 3). Results, discussed in Section 4, enable us to shed new and interesting lights on the multifractal properties of $H$-sssi processes.

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## 2. MULTIFRACTAL ANALYSIS

**Local Regularity and Hölder Exponent.** Let $X(t)$ denote the bounded function to analyze. Its local regularity at time $t_0$ is measured by the Hölder exponent $h(t_0)$, defined as the largest $\alpha > 0$ such that there exist a constant $C > 0$ and a polynomial $P_{t_0}$ of degree less than $\alpha$ such that $|X(t) - P_{t_0}(t)| \leq C|t - t_0|^{\alpha}$ in a neighborhood of $t_0$.

**Multifractal Spectrum.** Although based on a local regularity measure, multifractal analysis provides a global description of the variability of the local regularity of the data: It characterizes the geometrical structure of the subsets $E_h$ of points $t_i$ on the real line where $h(t_i) = h$. Because such geometrical structures are inherited from the time evolution of the data, multifractal analysis hence measures globally the local dynamics (or variability) of the regularity of $X$. This measure is based on the Haussdorf dimension of $E_h$, denoted by $\mathcal{D}(h)$, and is referred to as the multifractal spectrum.

**Wavelet Leader Multifractal Formalism.** The spectrum $\mathcal{D}(h)$ can in practice not be measured directly from a given time series $X$. Instead, one has to resort to a multifractal formalism. Use is made here of the recently proposed formalism based on wavelet leaders [5, 4]. Let $\psi$ denote a mother wavelet and $\psi_{j,k}(t) = 2^{-j}\psi(2^{-j}t-k)$ its dilated and translated versions. Let $d_X(j,k)$ be the ($L^1$-normalized) discrete wavelet transform coefficients of $X$, where $j$ refers to the analysis scale ($a = 2^j$) and $k$ to time ($t = 2^j k$). Wavelet leaders $L_X(j,k)$ are defined (for locally bounded processes only) as the local supremum of wavelet coefficients taken within a spatial neighborhood over all finer scales [4]: $L_X(j,k)=\sup_{\lambda' \subset 3\lambda_{j,k}} |d_X(\lambda')|$, where $\lambda_{j,k}=[k2^j,(k+1)2^j)$ and $3\lambda_{j,k}=\bigcup_{m\in\{-1,0,1\}} \lambda_{j,k+m}$. The scaling function $\zeta(q)$ is defined as

$$\zeta(q) = \liminf_{2^j \to 0} \log_2 S(2^j, q)/\log_2 2^j, \tag{3}$$

where $S(2^j,q) = \frac{1}{n_j}\sum_k L_X(j,k)^q$. Its Legendre transform

$$\mathcal{L}(h) = \min_q(1 + qh - \zeta(q)) \geq \mathcal{D}(h) \tag{4}$$

provides an upper bound for the multifractal spectrum $\mathcal{D}(h)$. For further theoretical details on multifractal analysis, multifractal formalism and wavelet leaders, the reader is referred to, e.g., [5, 4].

**Cumulant expansion.** It has been proposed to re-express the scaling function as a polynomial expansion [12]: $\zeta(q) = \sum_{p\geq 1} c_p q^p/p!$. For the range of orders $q$ for which the time averages $S(2^j,q)$ converge to the finite ensemble averages $\mathbb{E}L_X(j,\cdot)$, simple calculations involving the second characteristic function of the random variable $\ln L_X(j,\cdot)$ show that the coefficients $c_p$ in this expansion are deeply tied to the scaling properties of $X$ via the behavior across scales of the cumulants $C_p(2^j)$ of orders $p \in \mathbb{N}^*$ of $\ln L_X(j,\cdot)$

$$C_p(2^j) = c_p^0 + c_p \ln 2^j. \tag{5}$$

Furthermore, the polynomial expansion above shows that $c_1$ and $c_2$ measure the first and second derivatives of $\zeta(q)$ at $q = 0$. This implies that when $\zeta(q) = qH$, as is the case for fBm (cf. Section 3), $c_1 \equiv H$ and $c_2 \equiv 0$. An open question, addressed in this contribution, is whether this holds for all $H$-sssi processes or not.

**Minimal Regularity and Fractional Integration.** Multifractal analysis is theoretically defined for bounded functions only, i.e., essentially for functions with positive minimal regularity, defined as

$$h_m = \liminf_{2^j \to 0} \frac{\ln \sup_k |d_X(j,k)|}{\ln 2^j}. \tag{6}$$

To overcome this limitation, it has been proposed to first measure $h_m$ according to Eq. (6), and second, if $h_m < 0$, to fractionally integrate the data with an integration order $\gamma > -h_m$ [4]. Performing fractional integration can in practice be avoided by applying the wavelet leader formalism to the pseudo-wavelet coefficients $d_X^{\gamma}(j,k) = 2^{j\gamma} d_X(j,k)$ [4]. Then, one uses the relations

$$\mathcal{L}(h) = \mathcal{L}^{\gamma}(h-\gamma),\ \zeta(q) = \zeta^{\gamma}(q) - \gamma q,\ c_1 = c_1^{\gamma} - \gamma,\ c_2 = c_2^{\gamma} \tag{7}$$

as long as $\gamma > \max(0, -h_m)$. Note that these relations are known not to hold for all processes (cf. [1, 2] and references therein). They were proven to be valid for fBm, and it remains an open issue whether they hold for all $H$-sssi processes.

**Practical Estimation Procedures.** Eqs. (3-6) above indicate that the practical estimation of $\zeta(q)$, $\mathcal{L}(h)$, $c_p$ and $h_m$ can be performed using linear regressions. This has been detailed elsewhere and is not recalled here [4]. MATLAB routines implementing the estimation procedures are available upon request. They are complemented by a non-parametric time-scale bootstrap procedure, enabling in addition the construction of confidence intervals and hypothesis tests [4].

## 3. SELF SIMILAR PROCESSES

**Fractional Brownian motion.** Fractional Brownian motion (fBm) [13, 10] constitutes the canonical reference for scale invariance modeling. It is the only Gaussian $H$-sssi process, with parameter $0 < H < 1$, and is defined as a fractional integration of white Gaussian noise $dB(s)$: $B_H(t) = \int_{\mathbb{R}} f(t,s)dB(s)$. The integration kernel is specified as $f(t,s) = K_{H,2}(t,s)$, with $K_{H,\alpha}(t,s) = (t-s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha}$, and $(t)_+ = t$ if $t \geq 0$ and 0 elsewhere. For fBm, one has: $\forall q > -1$, $\mathbb{E}|B_H(t+\tau) - X(t)|^q = \mathbb{E}|B_H(1)|^q|\tau|^{qH}$. It has recently been shown that $\forall q \in \mathbb{R}$: $\zeta(q) = qH$ [3] and $\mathcal{D}(h = H) = 1$, $\mathcal{D}(h \neq H) = -\infty$. fBm is hence $H$-sssi, monofractal and has a strictly linear scaling function $\zeta(q) = qH$ over all $q \in \mathbb{R}$, hence $c_2 \equiv 0$.

There exist two ways to depart from the reference $H$-sssi fBm (and hence from Gaussian) while remaining in the class of $H$-sssi processes: Hermite processes, preserving finite variance (and finite moments of all orders $q > -1$); and stable processes, whose definition involves an extra parameter – the stability exponent $\alpha \in (0,2)$ – and whose moments are finite only within the range $-1 < q < \alpha$.

**Stable processes.** There exists a large variety of different stable $H$-sssi processes (cf. e.g., [10]). Here, we focus on the so-called Levy-stable process ($L_\alpha$) and the linear fractional stable motion ($L_{\alpha,H}$) as representative and well documented examples.

Let $M(ds)$ denote a symmetric $\alpha$-stable measure (with scale parameter $\sigma$; also denoted $S\alpha S(\sigma)$) [10]. $L_\alpha$ are defined as $L_\alpha(t) = \int_{\mathbb{R}} f(t,s)M(ds)$, with $f(t,s) = 1(t-s>0) - 1(-s>0)$, and are $H$-sssi, with $H = 1/\alpha$. It was shown that $L_\alpha$ are *not* monofractal [11], as their multifractal spectrum reads

$$\mathcal{D}(h) = \begin{cases} h\alpha & \text{if } 0 \leq h \leq 1/\alpha, \\ -\infty & \text{elsewhere.} \end{cases} \tag{8}$$

$L_{\alpha,H}$ are defined as $L_{\alpha,H}(t) = \int_{\mathbb{R}} f(t,s)M(ds)$, with $f(t,s) = K_{H,\alpha}(t,s)$, $H \in (0,1)$, $\alpha \in (1/2, 2)$ [10]. They are $H$-sssi with self-similarity parameter $H$ and satisfy $-1 < q < \alpha$: $\mathbb{E}|L_{\alpha,H}(t+\tau) - L_{\alpha,H}(t)|^q = \mathbb{E}|L_{\alpha,H}(1)|^q|\tau|^{qH}$. Their multifractal properties have yet not been studied theoretically.

| Process | $\gamma$ | $c_1$ | $\hat{c}_1^\gamma - \gamma$ | $c_2$ | $\hat{c}_2^\gamma$ |
|---|---|---|---|---|---|
| $\mathrm{fBm}_{H=0.7}$ | 0 | 0.7 | $0.690 \pm 0.017$ | 0 | $0.004 \pm 0.005$ |
| $\mathcal{H}_{p,H=(2,0.7)}$ | 0 | 0.7 | $0.689 \pm 0.042$ | 0 | $-0.001 \pm 0.015$ |
| $\mathcal{H}_{p,H=(3,0.7)}$ | 0 | 0.7 | $0.695 \pm 0.055$ | 0 | $-0.011 \pm 0.032$ |
| $\mathcal{H}_{p,H=(4,0.7)}$ | 0 | 0.7 | $0.687 \pm 0.074$ | 0 | $-0.005 \pm 0.043$ |
| $L_{\alpha=1.25}$ | 0 | 0.8 | $0.820 \pm 0.090$ | 0 | $0.002 \pm 0.192$ |
| $L_{\alpha=0.8}$ | 0 | 1.25 | $1.293 \pm 0.156$ | 0 | $-0.026 \pm 0.462$ |
| $L_{\alpha,H=(1.75,0.85)}$ | 0 | 0.85 | $0.801 \pm 0.044$ | 0 | $-0.003 \pm 0.060$ |
| $L_{\alpha,H=(1.75,0.85)}$ | 1.0 | 0.85 | $0.824 \pm 0.039$ | 0 | $-0.005 \pm 0.050$ |
| $L_{\alpha,H=(1.50,0.85)}$ | 0 | 0.85 | $0.823 \pm 0.069$ | 0 | $-0.016 \pm 0.114$ |
| $L_{\alpha,H=(1.50,0.85)}$ | 1.0 | 0.85 | $0.839 \pm 0.060$ | 0 | $-0.016 \pm 0.098$ |
| $L_{\alpha,H=(1.50,0.70)}$ | 0 | 0.7 | $0.712 \pm 0.067$ | 0 | $-0.013 \pm 0.109$ |
| $L_{\alpha,H=(1.50,0.70)}$ | 1.0 | 0.7 | $0.708 \pm 0.061$ | 0 | $-0.014 \pm 0.116$ |
| $L_{\alpha,H=(1.25,0.60)}$ | 0.5 | 0.6 | $0.635 \pm 0.084$ | 0 | $-0.028 \pm 0.151$ |
| $L_{\alpha,H=(1.25,0.60)}$ | 1.5 | 0.6 | $0.634 \pm 0.079$ | 0 | $-0.032 \pm 0.158$ |

**Table 1**. Estimates (with 95% confidence intervals) $\hat{c}_1^\gamma - \gamma$ and $\hat{c}_2^\gamma$ and theoretical values $c_1$, $c_2$ for fBm (top), $\mathcal{H}_{p,H}$ (second row block), $L_\alpha$ (third row block) and $L_{\alpha,H}$ (bottom rows), for different choices of process parameters and integration parameter $\gamma$.

**Hermite processes.** Hermite processes $\mathcal{H}_{p,H}(t)$, with $H \in (1/2, 1)$ and $p \in \mathbb{N}^*$, are defined as $\mathcal{H}_{p,H}(t) =$

$$\int_{-\infty}^{+\infty} dB(v_1) \int_{-\infty}^{v_1} dB(v_2) \ldots \int_{-\infty}^{v_{p-1}} dB(v_p) \int_0^t \Pi_{k=1}^p (u - v_k)_+^{(\frac{H-1}{p} - \frac{1}{2})}, \quad (9)$$

where the $\ldots, dB(v_k), \ldots$ are independent realizations of Gaussian white noise. As soon as $p \geq 2$, they are non Gaussian $H$-sssi, with self-similarity parameter $H$ and finite moments above $\forall q > -1$, hence satisfying $\forall q > -1$, $\mathbb{E}|\mathcal{H}_{p,H}(t+\tau) - \mathcal{H}_{p,H}(t)|^q = \mathbb{E}|\mathcal{H}_{p,H}(1)|^q |\tau|^{qH}$. The case $p = 1$ exactly reduces to fBm, while $p = 2$ is commonly referred to as Rosenblatt process. For $p \geq 2$, their multifractal properties have not yet been studied theoretically.

**Simulation procedures.** Sample paths of all $H$-sssi processes described above were numerically simulated by MATLAB routines implemented by ourselves, available upon request.

## 4. RESULTS

**Numerical simulations.** For each of the above $H$-sssi processes, we apply the wavelet leader multifractal formalism to 1000 realizations with $2^{15}$ samples each, using Daubechies' wavelet with $N_\psi = 3$ vanishing moments and weighted regressions from scale $2^3$ to the coarsest available scale $2^{12}$. Estimated means over realizations and 95% confidence intervals for $\zeta(q)$, $\mathcal{L}(h)$, $c_1$ and $c_2$ are plotted in Fig. 1. Estimated $c_1$ and $c_2$ for a larger selection of process parameters are reported In Tab. 1. Further results are available on request.

**fBm.** For sample paths of fBm, the estimates (Fig. 1 and Tab. 1, top rows) are found to be in excellent agreement with theory. This validates that the wavelet leader multifractal formalism is practically operational and effective and shows its excellent practical accuracy [3, 4]. If these properties were not already known theoretically, the numerical results would unambiguously suggest that fBm is characterized by a perfectly linear scaling function $\zeta(q)$, $\forall q \in \mathbb{R}$, by $c_1 = H$ and $c_2 = 0$, and by a multifractal spectrum that collapses to a point (and is, therefore, monofractal). From such observations, it could hence be conjectured that the properties $H$-sssi, linear $\zeta(q)$, $c_2 \equiv 0$, and monofractal are equivalent and that linear $\zeta(q)$ and

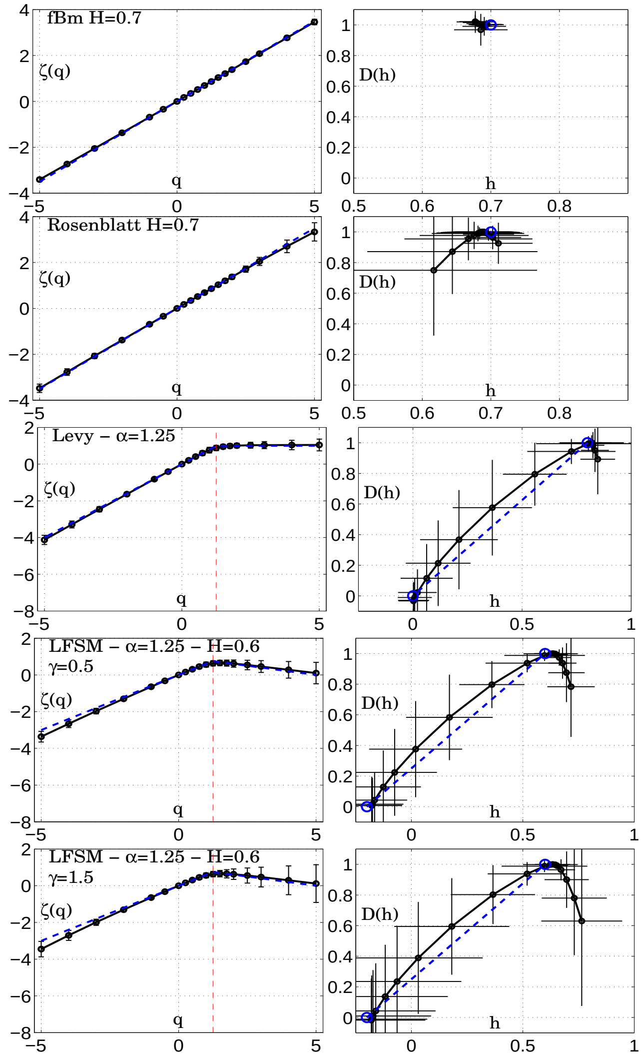


**Fig. 1**. Multifractal analysis of fBm (top row), $\mathcal{H}_{2,H}$ (2nd row), $L_\alpha$ (center row) and $L_{\alpha,H}$ (bottom rows): $\zeta(q)$ (left column) and $\mathcal{L}(h)$ (right column). Blue dashed line: theoretical values; Black solid dotted line: estimates with 95% confidence intervals.

$c_2 \equiv 0$ are discriminative against $MC$. We will, however, see in the next paragraphs that this is not in general the case.

**Hermite processes.** Numerical estimates for $\zeta(q)$, $\mathcal{L}(h)$ estimated from sample path of $\mathcal{H}_{p=2,H=0.7}$ (Rosenblatt process) are reported in Fig. 1, second row, and for $c_p$ in Tab. 1. They show that for this finite variance non Gaussian $H$-sssi process the scaling function $\zeta(q)$ clearly displays a linear behavior $\forall q \in \mathbb{R}$, that $c_2 = 0$, that the spectrum $\mathcal{L}(h)$ collapses to the single point $(h = H, \mathcal{D} = 1)$ and that the process is hence monofractal. Plots for $\mathcal{H}_{p,H}$ with $p > 2$ are not reported here for space reasons, estimates $\hat{c}_p$ are however reported in Tab. 1 for processes of orders $p = \{2, 3, 4\}$. They clearly suggest that, systematically and independently of $p$, $\hat{c}_1$ equals $H$, and $\hat{c}_2$ is identically zero. Therefore, similar to fBm, this process class is jointly characterized by the properties $H$-sssi and $c_2 \equiv 0$.

**Stable processes.** Let us begin with $L_\alpha$, for which the multifractal properties are theoretically known [11]. The multifractal parameters estimated with the wavelet leader multifractal formalism (cf. Fig. 1 and Tab. 1 (center rows) ) are found to be in excellent agreement with theory: $\mathcal{L}(h)$ (providing a convex upper bound for the multifractal spectrum $\mathcal{D}(h)$, cf. Eq. (4)) tightly envelopes the theoretical line segment of $\mathcal{D}(h)$, in agreement with Eq. (8). Again, this illustrates the practical accuracy of the wavelet leader based estimation procedure, even when applied to infinite variance processes. The fact that the estimated spectrum does not collapse on a single $h$ clearly confirms that $L_\alpha$ is multifractal. Nevertheless, and in contradiction to common fBm-based heuristics associating multifractal with non-zero $c_2$, we observe that $\hat{c}_2^\gamma \equiv 0$. Estimates of the scaling function confirm this observation: $\zeta(q)$ is linear in a neighborhood including $q = 0$, but not for all $q$, instead it is found to be piece-wise linear: $\zeta(q)$ is linear with slope $c_1 = H = 1/\alpha$ for $q < \alpha$, and $\zeta(q) = 1$ for $q > \alpha$.

Let us now turn our attention to $L_{\alpha,H}$, whose sample paths are multifractal, though their exact multifractal spectrum has not yet been established theoretically. However, results reported in Fig. 1 and Tab. 1 (bottom rows) enable us to propose the following conjecture for the multifractal spectrum of $L_{\alpha,H}$

$$\mathcal{D}(h) = \begin{cases} 1 + \alpha(h - H) & \text{if } H - 1/\alpha \leq h \leq H, \\ -\infty & \text{elsewhere.} \end{cases} \quad (10)$$

This conjecture is theoretically sound as long as $H \geq 1/\alpha$, since the definition of the Hölder exponent implies $h \geq 0$, and receives the following interpretation: The fact that the multifractal spectrum of $L_{\alpha,H}$ matches that of $L_\alpha$ up to a horizontal shift by $H - 1/\alpha$ stems from $L_{\alpha,H}$ being obtained as a fractional integration (derivation) of order $H - 1/\alpha$ of $L_\alpha$ (compare the integration kernels $f(t, s)$ in Section 3). This conjecture confirms that $L_{\alpha,H}$ is multifractal, However, as is the case for $L_\alpha$, the $\zeta(q)$ are found to be clearly linear in $q$ around 0, and $c_2$ are estimated to 0, again showing that $c_2 = 0$ does not necessarily imply monofractality. The $\zeta(q)$ are found to be piecewise linear (above and below $q = \alpha$). We also note that there is no different linear behavior below $q < -1$ despite infinite moments of $L_{\alpha,H}$.

**Fractional integration.** In contrast to $L_\alpha$, the minimum regularity (cf. Eq. (6)) of $L_{\alpha,H}$ can be negative, $h_m < 0$ when $H < \frac{1}{\alpha}$, as sample paths are then not locally bounded. Consequently, data may need to be fractionally integrated (of order $\gamma > \frac{1}{\alpha} - H$) before performing multifractal analysis. Results reported in Fig. 1 and Tab. 1 (bottom rows) suggest that the relations Eq. (7) hold: estimates of $\zeta(q)$, $\mathcal{L}(h)$ and $c_p$ obtained for different values of $\gamma$ are consistent. Results obtained for fBm, $\mathcal{H}_{p,H}$ and $L_\alpha$ (omitted here for space reasons) suggest the validity of Eq. (7) for all of these $H$-sssi process.

**Conclusions.** The experimental investigations of the multifractal properties of a large variety of $H$-sssi processes (Gaussian and non Gaussian, with and without finite variance) conducted in this contribution enable us to clearly validate that $H$-sssi does, in contrast to widely used heuristics, neither imply monofractality, nor a scaling function that is linear in $q$ for all $q$s. Note that in practice, $c_2 \equiv 0$ has often been considered as the signature of monofractality and of a scaling function that is linear for all $q$s. Results reported here demonstrate that this is incorrect: As it is designed for, $c_2$ only measures the second derivative of $\zeta(q)$ around $q = 0$ (hence, a local behavior of $\zeta(q)$) which excludes neither multifractality nor departure from linearity of $\zeta(q)$. Furthermore, the results obtained here suggest the following conjectures for $H$-sssi processes:

i) The scaling function is piecewise linear in $q$ and is necessarily controlled by $H$ in a (possibly narrow) neighborhood including $q = 0$, $\zeta(q) = qH$. As a consequence, $c_2 \equiv 0$.

ii) Furthermore, Eq. (7) holds.

This second conjecture contributes to suggesting that *oscillating singularities* are very unlikely to be present in the $H$-sssi processes studied here (cf. [14]). This contribution paves the way towards a theoretical study of the multifractal properties of non-Gaussian $H$-sssi processes. These tracks are under current investigations.

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