# On multivariate non-Gaussian scale invariance: fractional Lévy processes and wavelet estimation

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# Context

- scale invariant dynamics: wide range of physical systems
  - temporal dynamics lack a characteristic scale
  - → identification of mechanisms that relate the scales
- many sensors jointly record data → multivariate
  - scale invariance mostly restricted to univariate analysis
  - model for multivariate Gaussian self-similarity
- non Gaussian multivariate scale invariance
  - → operator fractional Lévy motion
  - → joint wavelet eigenanalysis estimation

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  - $\rightarrow$ operator fractional Lévy motion
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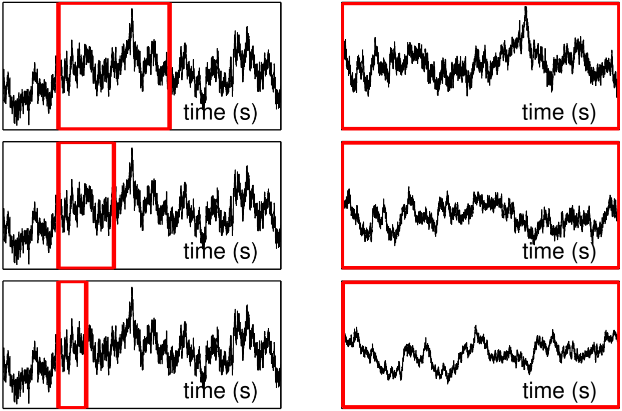
# Scale invariance and self-similarity

- Intuition: temporal dynamics without characteristic scale




Active Connections (WAND, Auckland, 1998) [Abry et al.]

## Scale invariance and self-similarity

- Intuition: temporal dynamics without characteristic scale

- Definition: $X$ is self-similar if its finite-dimensional distributions (fdd) are invariant w.r.t. change of time scale

$$\{X(t)\}_{t\in\mathcal{R}} \stackrel{\text{fdd}}{=} \{a^H X(t/a)\}_{t\in\mathcal{R}}, \qquad \forall a > 0$$

– $H$: Hurst parameter
– re-scaling power law factor $a^H$

- Example: fractional Brownian motion (fBm) [Mandelbrot68]
– only Gaussian, self-similar process with stationary increments

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## Multivariate covariance self-similarity for $Y = (Y_1, \ldots, Y_M)$

$$\mathbb{E}[Y(s)Y(t)^*] = a^{\underline{\underline{H}}}\, \mathbb{E}[Y(s/a)Y(t/a)^*]\, a^{\underline{\underline{H}}^*} \qquad \forall a > 0$$

- Hurst exponent vector
  $$\underline{H} = (H_1, \ldots, H_M)$$
- Hurst matrix parameter
  $$\underline{\underline{H}} = P\mathrm{diag}(\underline{H})P^{-1}$$
- $a^{\underline{\underline{H}}} := \sum_{k=0}^{+\infty} \log^k(a)\underline{\underline{H}}^k/k!$ (matrix exponentiation)
  $\longrightarrow$ mixtures of power laws
- Example: operator fractional Brownian motion (ofBm) [Didier11,Abry18]
- Only if mixing matrix $P$ is diagonal:
  $\longrightarrow$ component-wise covariance self-similarity relations $\sim$ fBm

$$\mathbb{E}[Y_\ell(s)Y_{\ell'}(t)] = a^{H_\ell + H_{\ell'}}\mathbb{E}[Y_\ell(s/a)Y_{\ell'}(t/a)], \quad \ell, \ell' = 1, 2, \ldots, M \qquad \forall a > 0$$

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# Operator fractional Lévy motion: definition (bi-variate)

new class of non-Gaussian multivariate fractional processes
with same covariance structure of ofBm

1. $\{L(t) = (L_1(t), L_2(t)\}_{t \in \mathcal{R}}$
   – two-sided symmetric Lévy process in $\mathcal{R}^2$
   – $\mathbf{E}L(1)L(1)^* =: \Sigma_L, |\Sigma_L| < \infty$

2. (pre-mixed) process $X$

$$X(t) = (g_t * \dot{L})(t)$$

   – fractional kernel $g_t(u) := u_+^D - (u-t)_+^D$, $D = \mathrm{diag}(\underline{H}) - \frac{1}{2}I$,
   $0 < H_1 \leq H_2 < 1$

3. (bivariate) ofLm $Y^{\underline{H},L,P} = PX$

$$\{Y_1^{\underline{H},L,P}(t), Y_2^{\underline{H},L,P}(t)\}_{t \in \mathcal{R}} = P\{X_{H_1}(t), X_{H_2}(t)\}_{t \in \mathcal{R}}$$

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# Operator fractional Lévy motion: properties

- (pre-mixed) process $X$:
  – stationary increments
  – covariance function identical to that of fBm

$$\mathbb{E}X_{H_\ell}(t)X_{H_\ell}(s) = \{|t|^{2H_\ell} + |s|^{2H_\ell} - |t-s|^{2H_\ell}\}\sigma_\ell^2/2. \quad (1)$$

- entrywise processes $X_{H_\ell}$: correlated fractional Lévy processes with Hurst parameters $H_\ell \in (0,1)$

- ofLm $Y^{\underline{H},L,P}$: multivariate covariance self-similarity relation identical to ofBm

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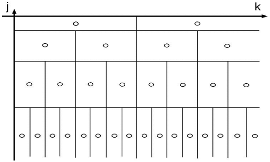
# Multivariate discrete wavelet transform




- Discrete Wavelet Transform (DWT):
  - oscillating reference pattern $\psi_0$
    – number of vanishing moments $N_\psi$:

$$\int_{\mathcal{R}} t^n \, \psi_0(t) dt \begin{cases} \equiv 0 & \forall n = 0, \ldots, N_\psi - 1 \\ \neq 0 & n \geq N_\psi \end{cases}$$

    – $\left\{ \psi_{j,k}(t) = \frac{1}{2^{j/2}} \psi_0 \left( \frac{t - 2^j k}{2^j} \right) \right\}_{(j,k)}$ orthonormal basis of $\mathcal{L}^2(\mathcal{R})$

- wavelet coefficients of single time series $X$:

$$d_X(2^j, k) = \langle \psi_{j,k}(t) | X(t) \rangle$$

- multivariate DWT of $Y = (Y_1, \ldots, Y_M)$:

$$(D(2^j, k)) \equiv D_Y(2^j, k) = (d_{Y_1}(2^j, k), \ldots, d_{Y_M}(2^j, k))$$

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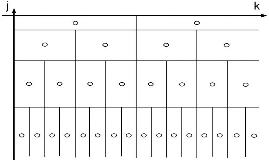


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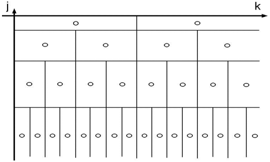


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# Joint estimation for $(H_1, \dots, H_M)$

- empirical wavelet spectrum (sample size $N$)

$$S(2^j) = \frac{1}{n_j} \sum_{k=1}^{n_j} D(2^j, k) D(2^j, k)^*, \quad n_j = \frac{N}{2^j},$$

- univariate estimation for $H_m$:
  $\rightarrow$ log-regressions of entries of $S(2^j)$ across scales

$$\tilde{H}_{mm'} = \left( \sum_{j=j_1}^{j_2} w_j \log_2 S_{mm'}(2^j) \right) /2 - \frac{1}{2}, \quad \forall m, m' = 1, \dots, M.$$

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  - consistent
  - asymptotically joint normal
  - covariance decrease as $N^{-1}$
- Empirically, very satisfactory performance for finite sample

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# Numerical synthesis: Quantifying ofLm tail behavior

- non-Gaussian part of $L_1$ chosen symmetric tempered stable:
  $\rightarrow$ marginals of $X_{H_1}$ have non-Gaussian tails
  $\rightarrow$ but have finite moments of all orders

[Baeumer2010, Stoev2004]

- stability index $\alpha \in (0, 2)$
- tempering parameter $\gamma > 0$
  $\rightarrow$ exponentially tempered density

$$p_\gamma(x, t) \propto e^{-\gamma|x|} p(x, t)$$

  $\rightarrow$ $\gamma$ small $\Rightarrow$ heavier tails

- simulation: accept-reject procedure

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## Monte Carlo simulation

- $L_1$: Lévy with
  – tempered component $\alpha = 1$ and $\gamma = 10^{-7}, \ldots, \gamma = 10^{-1}$
  – Gaussian component
  – $H_1 = 0.35$
- $L_2 \equiv$ Gaussian component of $L_1$
  $\longrightarrow$ non-diagonal $\Sigma_L$, correlated $X_{H_1}, X_{H_2}$
  – $H_2 = 0.75$
- mixing matrix $P = ((1,0)^T; (1,1)^T)$
  $\rightarrow Y_1^{\underline{H},L,P}$ is sum of Gaussian and non-Gaussian components
  $\rightarrow Y_2^{\underline{H},L,P}$ is Gaussian.
- $N = 2^{15}$, 1000 independent realizations
- $N_\psi = 2$, $(j_1, j_2) = (4, 11)$

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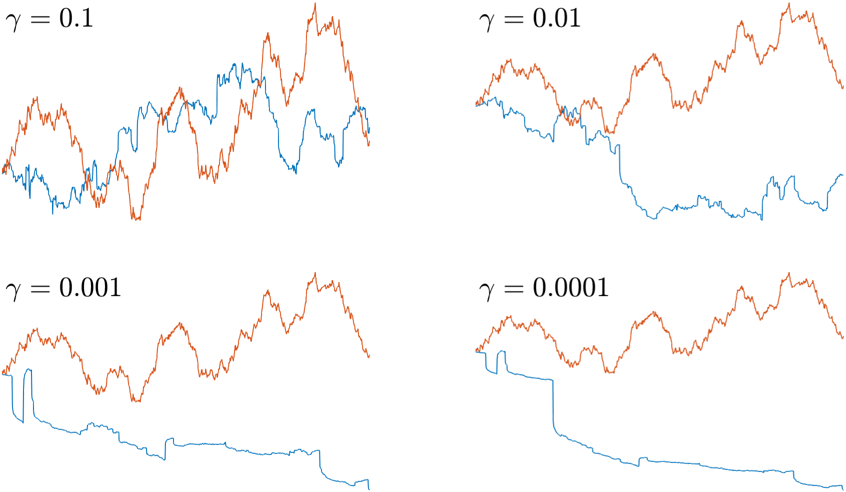
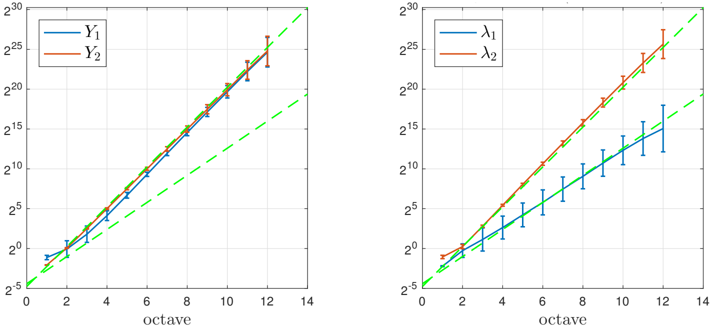
## Illustration



# Univariate vs. joint estimation for $H_1, H_2$

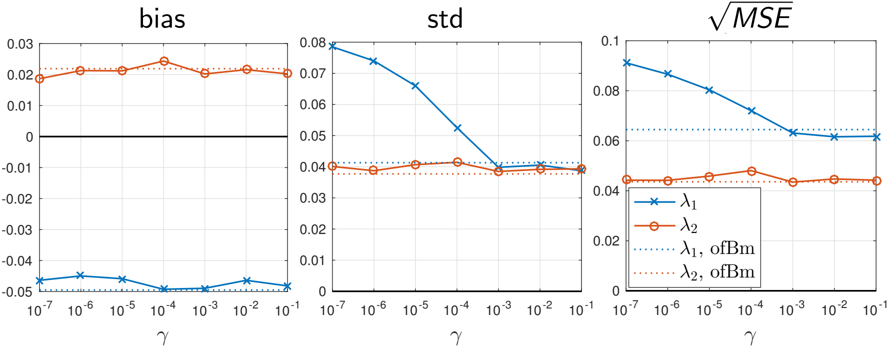
$(\gamma = 10^{-5})$

univariate — multivariate/eigenanalysis




| univariate | multivariate |
|---|---|
| only dominant exponent $H_2$ | both exponents $H_1, H_2$ |
| no evidence of non-Gaussian tails | CI for $\lambda_1$ differ from Gaussian case |

# Performance with respect to non-Gaussianity




- bias constant w.r.t. $\gamma$
- $\gamma$ small / heavier tails $\rightarrow$ std increase

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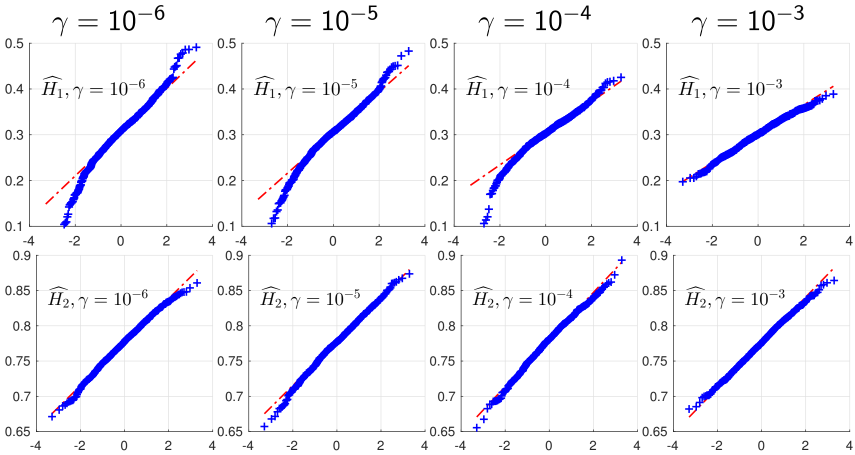
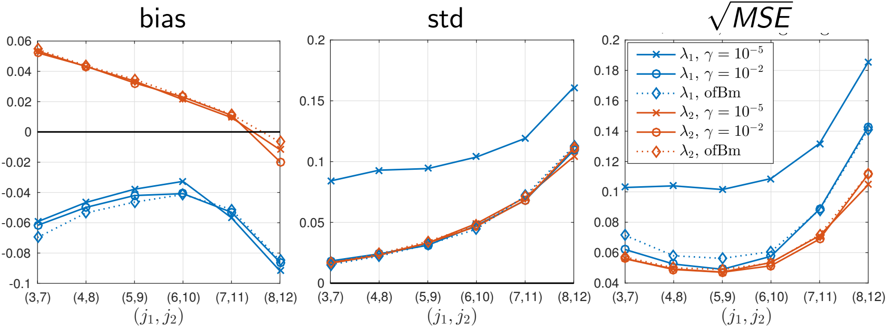



- $\hat{H}_2$: (asymptotic) Gaussanity achieved for all instances
- $\hat{H}_1$: $\gamma$ small / heavier tails $\rightarrow$ further departure from Gaussian

# Performance with respect to scale choices

fixed number of scales $(j_1, j_2)$ from fine $(3,7)$ to coarse $(8,12)$ scales




- bias (fine scales) - variance (coarse scales) trade-off
- similar for Gaussian and non-Gaussian

# Conclusions and Perspectives

- ofLm: a new model for non-Gaussian scaling dynamics
  - given non-Gaussian driving noise, ofLm is parametrized by
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