Shuffling for understanding multifractality, application to asset price time series

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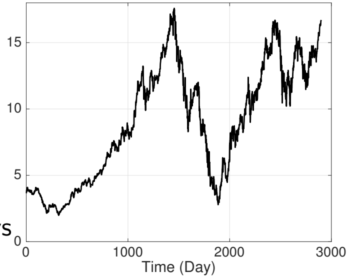
(4) Vivienne Investissement, Lyon, France

Finance time series: stylized facts



- Randomness:
  Random walk like
- Market Efficiency:
  No temporal correlation in returns
- Scale-free dynamics:
  From tens of minutes to tens of days
  Self-similarity with $H = 0.5$
  Long term memory in volatility $H > 0.5$

$\Longrightarrow$ Multifractality
  Maturity of the market
  Volatility clustering

- Theory: Gaussian self-similarity encodes signs dynamics !
- What does multifractality encode in temporal dynamics ?
  Signs of returns ?
  Modulus of returns ?

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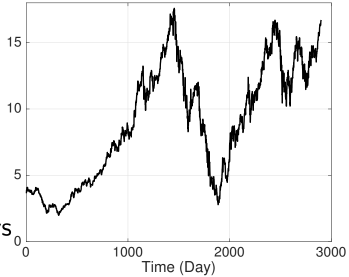

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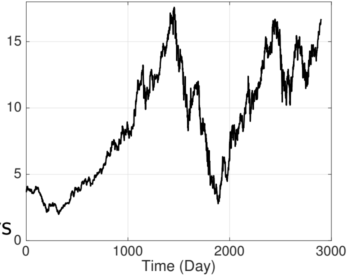

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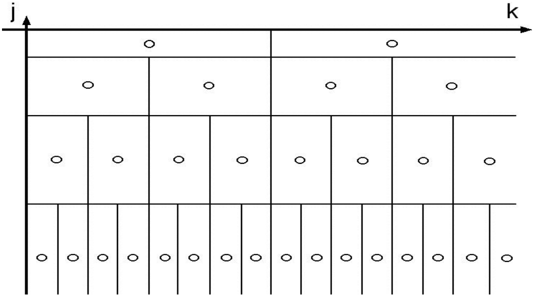
Scaling analysis: Discrete Wavelet Transform

- Fourier Transform: $X(t) \Longrightarrow \tilde{X}(\nu) = \langle X, e_\nu \rangle$.
  Fourier Basis: $e_\nu(t) = \exp(\imath 2\pi\nu t)$
  Interpretation: ever lasting pure tone

- Discrete Wavelet Transform:

$$d_X(j,k) = \left\langle \frac{1}{2^j}\psi\left(\frac{t-2^jk}{2^j}\right) \middle| X(t) \right\rangle$$

Interpretation: Joint time and frequency energy content



Scaling analysis: Logscale Diagrams

- Principle:

$$\mathbf{E}|d_X(j,k)|^q = |d_X(0,0)|^q 2^{jqH} \Rightarrow \text{log-log plots}$$

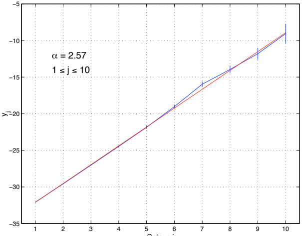
- Estimation: short-range dependence $\Rightarrow$

Ensemble averages $\rightarrow$ Time Averages

$$\mathbf{E}|d_X(j,k)|^q \quad \Rightarrow \quad S(2^j, q) = \frac{1}{n_j} \sum_k |d_X(j,k)|^q$$

- Logscale Diagrams:

$$\log_2 S(2^j, q) \text{ versus } \log_2 2^j = j \Rightarrow qH$$



Beyond self-similarity. . . ?

- Self-Similarity:

  Power Laws: $\mathbb{E}|d_X(j,k)|^q = C_q(2)^{jqH}$
  For all scales: $\forall a = 2^j$,
  For all orders: $q > -1$,
  A single parameter H.

- Beyond $\Rightarrow$ Multifractal

  Power Laws: $\mathbb{E}|d_X(j,k)|^q = C_q(2)^{j\zeta(q)}$
  $\zeta(q)$ non linear concave function of q,
  For a limited range of scales: $a_m \leq a \leq a_M$,
  For a limited range of orders: $q_m \leq q \leq q_M$,
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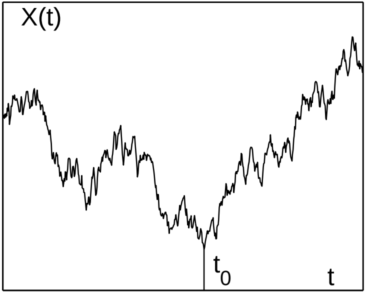
Multifractal Analysis

- **Local regularity** of $X(t)$ at t_0 : $0 < \alpha < 1$

  Compare: $|X(t) - X(t_0)| < C|t - t_0|^\alpha$



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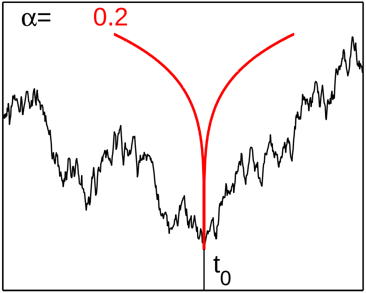

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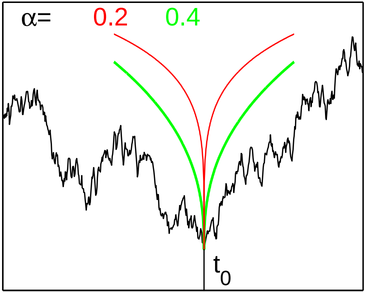

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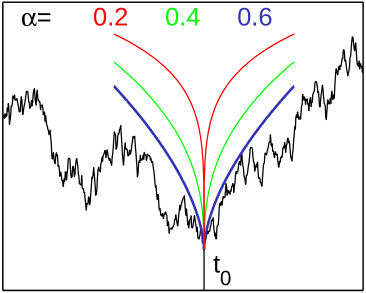

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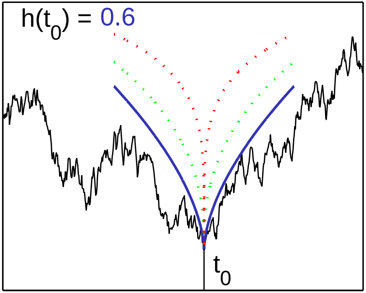

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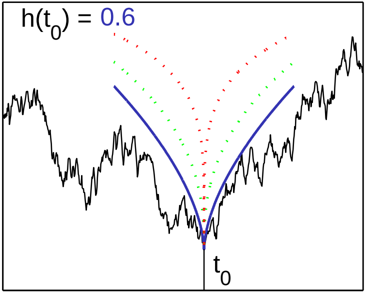
  $h(t_0) \to 1 \Rightarrow$, smooth, very regular,
  $h(t_0) \to 0 \Rightarrow$, rough, very irregular



Multifractal (or singularity) spectrum

- Data: a collection of singularities
  $|X(\mathbf{t}) - X(\mathbf{t_0})| \leq C|\mathbf{t} - \mathbf{t_0}|^{h(\mathbf{t_0})}$
- Fluctuations of local regularity: $h(\mathbf{t})$?

  $\rightarrow$ not interested in h for each $(\mathbf{t})$!
- Instead, set $E(h)$ of points $\mathbf{t}$ with same h: $h(\mathbf{t}) = h$,

  $\rightarrow$ fractal dimension of $E(h)$,

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- Multifractal spectrum

$$D(h) = \dim_{\text{Haussdorf}}(E(h)).$$

$$0 \leq D(h) \leq d, \; D(h) = -\infty \text{ if } E(h) = \{\emptyset\}$$

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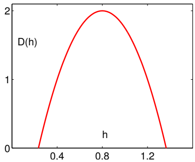


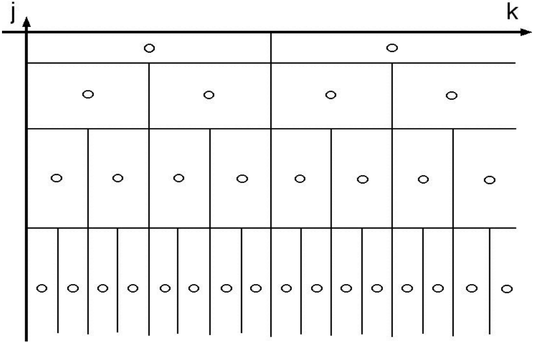
$\Rightarrow$ Global (geometric) description of local regularity fluctuations

- How to measure $D(h)$ from a single finite length observation?

Wavelet Leaders

- Discrete Wavelet Transform: $\lambda_{j,k} = [k2^j, (k+1)2^j)$

$$d_X(j,k) = \langle \frac{1}{2^j}\psi\left(\frac{t-2^j k}{2^j}\right)|X(t)\rangle \,,$$



- Wavelet Leaders: $3\lambda_{j,k} = \lambda_{j,k-1} \cup \lambda_{j,k} \cup \lambda_{j,k+1}$

$$L_X(j,k) = \sup_{\lambda' \subset 3\lambda_{j,k}} |d_{X,\lambda'}|$$

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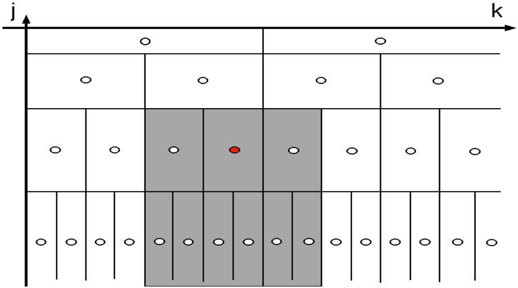
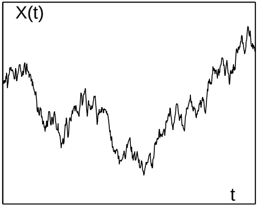


Multifractal Formalism

$X(t) \to d_X(a, t) \to L_X(a, t)$



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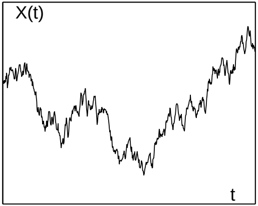
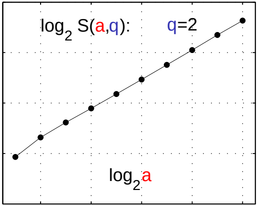


$\Downarrow$

$$S(a,q) = \frac{1}{n_a}\sum_{k=1}^{n_a} L_X(a,k)^q$$



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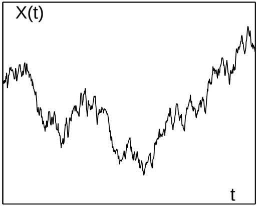

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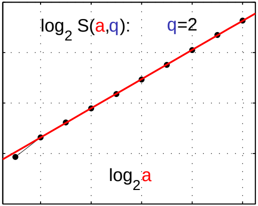
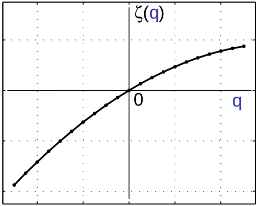


$\Longrightarrow$

$S(a,q) \simeq c_q a^{\zeta(q)}, \quad a \to 0$



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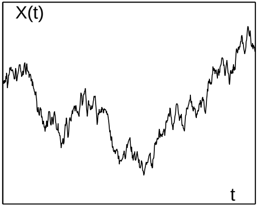

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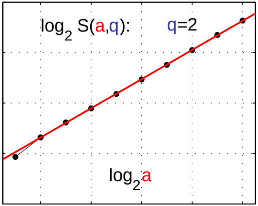
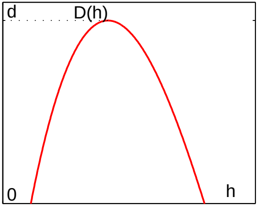


$$\Longrightarrow$$

$$D(h) = \min_{q \neq 0} \left(d + qh - \zeta(q)\right)$$



$$\Uparrow$$

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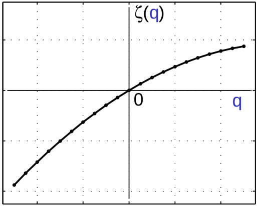
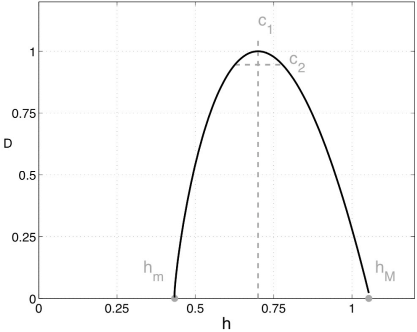


Multifractal Spectrum

- Multifractal Formalism:
  - $\frac{1}{n_j}\sum_{k=1}^{n_j} L_X(j,k)^q \sim K_q 2^{j\zeta(q)},\ 2^j \to 0$
  - $\mathcal{L}(h) = \inf_q(1 + qh - \zeta(q)) \geq \mathcal{D}(h)$
  - $\zeta(q) = \sum_{p\geq 1} c_p q^p/q! = c_1 q + c_2 q^2/2 + c_3 q^3/6 + \ldots$
  - $C_p(j) \triangleq \mathrm{Cum}_p \ln L_X(j,k) = c_p^0 + c_p j$

- Multifractal features:
  - c_1: Location of max,
  - $c_2 < 0$: width
  - c_3: asymmetry
  - c_4: flatness
  - ...
  - $h_{\min}$ Minimun regularity,
  - $h_{\max}$ Maximum regularity
  - $D(h) \simeq 1 + \frac{c_2}{2}\left(\frac{h-c_1}{c_2}\right)^2$



Random Shuffling

- Data:
    - $X(t)$: random walk
    - $Y(t) = X(t+\tau) - X(t)$: increments assumed stationary
    - Rewrite: $Y(t) = \text{sgn}(Y(t)) \times |Y(t)|$

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- Sign-Shuffling:
  - $Y^{(S)}(t) \triangleq \text{shuffle}(\text{sgn}(Y(t)) \times |Y(t)|$

$$X^{(\bullet)}(t) = \sum_{k=0}^{t} Y^{(\bullet)}(k)$$

- Are Scale Free dynamics altered by Shuffling?

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  - $Y^{(M)}(t) \triangleq \text{sgn}(Y(t)) \times \text{shuffle}(|Y(t)|)$
- Full-Shuffling:
  - $Y^{(F)}(t) \triangleq \text{shuffle}(\text{sgn}(Y(t)) \times \text{shuffle}(|Y(t)|)$

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Fractional Brownian motion in Multifractal time

- Synthetic Data:
  - Fractional Brownian motion (fBm) $B_H(t)$
    Gaussian, exactly selfsimilar.
    $\mathbf{E}L_X(j,k)^q = 2^{jqH}$
  - Multifractal Measure A(t)
    $\mathbf{E}A(t)^q = t^{1+c_2q^2+2+c_3q^3/3+\dots}$
  - MF-fBm $\mathcal{B}_{H,A}(t) \triangleq B_H(A(t))$
    $\mathbf{E}L_X(j,k)^q = 2^{j\zeta(q)}$
    $\zeta(q) = H + c_2q^2/2 + c_3q^3/3 + \dots$

- Simulation Set-Up:
  - 1000 independent copies, Sample size: $n = 2^{18}$.
  - Daubechies 3 (least asymmetric) orthogonal wavelet.
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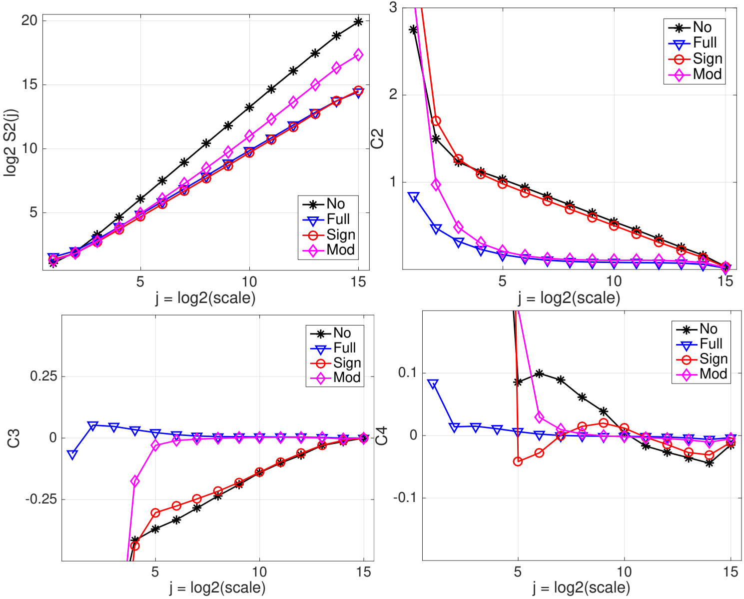


Scale-free and shuffling?

	H	c_2	c_3	c_4
Data	0.70 (0.01)	-0.140 (0.002)	0.054 (0.003)	-0.026 (0.006)
S	0.45 (0.01)	-0.135 (0.002)	0.052 (0.003)	-0.013 (0.004)
M	0.66 (0.01)	-0.006 (0.001)	-0.000 (0.002)	-0.003 (0.001)
F	0.49 (0.01)	-0.007 (0.001)	-0.001 (0.002)	-0.001 (0.001)

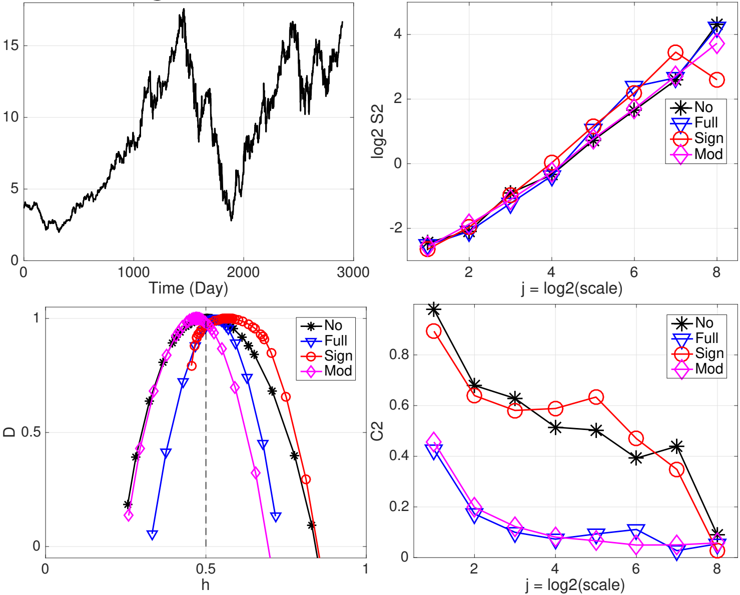
- Any Shuffling preserve some scale-free behavior
- Sign-Shuffling:
  - Alters self-similarity and destroys correlation
  - Leaves multifractality unchanged
- Modulus-Shuffling:
  - Leaves self-similarity and correlation unchanged
  - Destroys multifractality
- Full-Shuffling:
  - Alters self-similarity and destroys correlation
  - Destroys multifractality

Finance data sets: Eurostoxx600

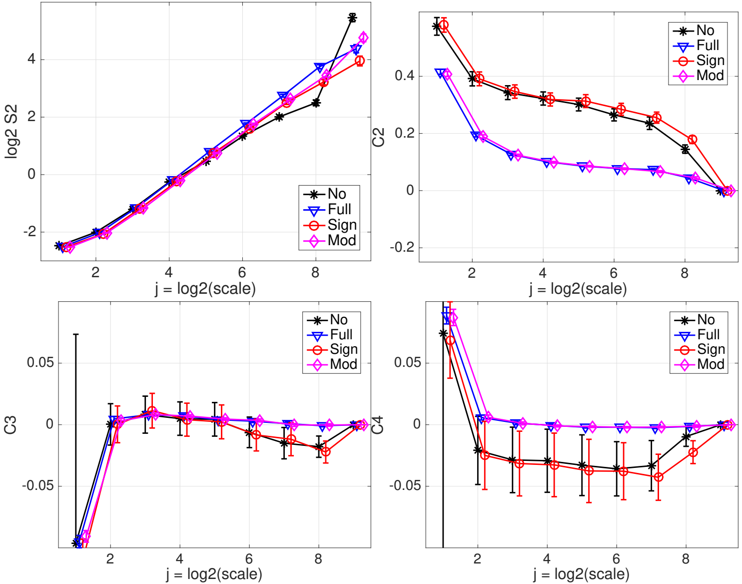
- Dataset:
  - $p = 211$ assets entering the Eurostoxx600 index
  - 12 years:
    from December, the 14th, 2001 to January, the 24th, 2013
  - Daily return: $n = 2900$ samples per time series

- Analysis setup:
  - same as synthetic data
  - averages across assets
    (as if realizations of one same (financial/stochastic) process)

Shuffling, scale-free in finance ? - One asset



Shuffling, scale-free in finance ? - Average across assets



Shuffling, scale-free and financial time series ?

	H	c_2	c_3	c_4
Data	0.47 (0.01)	-0.095 (0.008)	-0.003 (0.010)	0.030 (0.014)
S	0.43 (0.01)	-0.092 (0.008)	-0.005 (0.008)	0.028 (0.014)
M	0.48 (0.01)	-0.029 (0.002)	-0.003 (0.001)	-0.002 (0.001)
F	0.46 (0.01)	-0.029 (0.002)	-0.003 (0.001)	-0.002 (0.001)

- Overall conclusions:
  - Essentially consistent with simulated data.
- Any Shuffling preserve scale-free
- Sign-Shuffling:
  - Does not create correlation
  - Leaves multifractality unchanged
- Modulus-Shuffling:
  - Does not create correlation
  - Destroys multifractality

Conclusions and perspectives

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  - Synthetic data
    - * self-similarity $\longleftrightarrow$ temporal dynamics of increment signs
    - * multifractality $\longleftrightarrow$ temporal dynamics of increment moduli
  - Financial data
    - * multifractality not related to returns (increment signs)
      $\longrightarrow$ cannot be used to predict returns
    - * multifractality $\longleftrightarrow$ temporal dynamics of increment moduli
      $\longrightarrow$ associated with "volatility clustering"
      (long memory in squared returns)
      $\longrightarrow$ conveys information for volatility prediction
- Perspectives
  - Multivariate scale-free dynamics ?
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